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ON INVARIANT REGULARIZATION

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Conditions that ensure universal invariance of the procedure of regularized integration with respect to internal momenta of diagrams are obtained. The only regularization scheme satisfying these conditions is dimensional regularization. It is shown that despite the invariance of the integration with respect to the momenta in the presence of anomalies the regularization scheme as a whole may be noninvariant.

1. Introduction

The renormalization problem in quantum field theory with a Lagrangian that is invariant under a group is simplified considerably by using a regularization that is invariant, i.e., does not destroy the symmetry of the original problem. For various symmetry groups, such regularizations have frequently been proposed. For a very large group of quantum-field models, dimensional regularization [1] has proved very convenient; its gauge invariance was proved by different methods in [2, 3, 4].

In the approach developed below, dimensional regularization is invariant by construction. The aim of the paper is to investigate the restrictions imposed on the regularization by the requirement that it be invariant, and to construct a regularization procedure satisfying these restrictions for the maximally large class of symmetry transformations (theories containing anomalies are an exception). The regularization scheme then constructed turns out to be identical with dimensional regularization.

With regard to the renormalization procedure, it should be noted that on the basis of invariant regularization it can be implemented, for example, by the method proposed by 't Hooft [5]. The invariance of this method becomes obvious if one uses the background-field formalism developed by 't Hooft as well [6].

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2. Invariant Integration

In quantum field theory, the symmetry properties of the Lagrangian have as a consequence definite relations for the Green's functions, which are called Ward identities. A convenient method for deriving these identities, which uses the formalism of generating functionals, was proposed by Slavnov [7]. Formal application of this technique (i.e., one that ignores the problem of divergences) leads to relations that have meaning only for the integrands in the corresponding Green's functions, and to give a meaning to the divergent integrals one must regularize them.

We define a universal invariant regularization as a procedure that leaves all the Ward identities derived formally by the above method true for the regularized Green's functions as well. Therefore, the

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invariantly regularized integration procedure must have properties that ensure validity of all the necessary transformations in the process of the derivation of the Ward identities. Slavnov's papers [7, 8] contain a detailed analysis of manipulations of this kind with generating functionals and indicate that the required properties of universally invariant integration in x space are: 1) uniqueness (the result of the integration must not change under identity transformations of the integrand), 2) linearity, and 3) the possibility of integration by parts with neglect of boundary terms. The third requirement will be automatically satisfied if all expressions in the x representation are understood only in the sense of the Fourier transition from the p representation, in which we shall work in the following.

We shall assume that the integrands corresponding to the Feynman diagrams are composed of symbols $(p_\mu, g_{\mu\nu}, \dots)$ that are Lorentz covariant as regards their form and properties $(p_\mu g_{\mu\nu}=p_\nu,\ p_\mu p_\mu=p^2,\ g_{\mu\mu}=n, \dots)$ and are a generalization of the four-dimensional Lorentz algebra (in particular, n is not necessarily equal to four). The ellipsis reflects the possibility of using other symbols as well, for example, symbols that generalize the matrices γ_μ, γ_5, etc, from which we require only internal consistency of their complete set of properties. One of the possible variants of such a system of symbols is given in [4]. Functions for which some of the numerical arguments are replaced by symbolic arguments are assumed to retain their ordinary properties with respect to these arguments, for example, $e^{\alpha p^2}e^{\beta q^2}=e^{\alpha p^2+\beta q^2}$, etc.

We now reformulate in the language of the p representation the restrictions imposed on the universal invariant procedure of regularized integration:

$$\int dpf(p+k,\dots)=\int dpf(p,\dots), \tag{1}$$

$$\int dpf(-p,\dots)=\int dpf(p,\dots), \tag{2}$$

$$\int dp\int dqf(p,q,\dots)=\int dq\int dpf(p,q,\dots), \tag{3}$$

these relations guaranteeing that the arbitrariness in the choice of the independent internal momenta of integration in the diagrams does not affect the result;

$$\int dp\sum_i a_i f_i(p,\dots)=\sum_i a_i\int dpf_i(p,\dots). \tag{4a}$$

In this relation, which expresses the linearity of the invariant integral, the sum can also be infinite, which gives the possibility of transferring part of the free Lagrangian to the interaction and vice versa, thereby ensuring uniqueness of the representation of the Green's functions in the form of functional integrals. Note that infinite sums can arise only as a result of expansion in power series; at the same time, the equality of the two sides of (4a) is understood in the sense of equality of the coefficients of equal powers of the expansion parameter.

Expanding in a series with respect to the parameter α the relation

$$\int dpf(\alpha,p,\dots)=g(\alpha,\dots)$$

and using (4a), we arrive at

$$\int dp\frac{\partial}{\partial\alpha}f(\alpha,p,\dots)=\frac{\partial}{\partial\alpha}\int dpf(\alpha,p,\dots), \tag{4b}$$

i.e., it is permissible to differentiate with respect to the parameter inside the invariant integral. From this there immediately follows the possibility of the inverse operation — integration with respect to the parameter:

$$\int d\alpha\int dpf(\alpha,p,\dots)=\int dp\int d\alpha f(\alpha,p,\dots)+\text{quantity independent of }\alpha. \tag{4c}$$

The universally invariant integration procedure must, in particular, be Lorentz invariant, i.e., preserve the tensor structure of the integrand

$$\int dpf_{\mu\nu\dots\rho}(p,\dots)=g_{\mu\nu\dots\rho}(\dots). \tag{5}$$

Finally, as in the x representation, we must require that identity transformations of the integrand do not affect the result of the regularized integration with respect to the momenta.

Thus, we obtain a set of conditions characterizing the integration procedure that ensure uniqueness of the representation of the Green's functions in terms of a functional integral and validity of all the necessary manipulations with generating functionals in the derivation of the Ward identities, the proof of the equivalence theorem, etc. Leaving aside the question of whether these conditions are necessary for universal invariant integration with respect to the momenta and of the class of integral functions for which an integration scheme satisfying these conditions exists, we shall show in the following section that for the integrands corresponding to the diagrams of local Lagrangian field theory the properties listed above enable one to construct explicitly (and uniquely) an invariant integration procedure.

3. Dimensional Regularization

The relation (4c) makes it possible to use the well-known parametric representation for the propagators (the α representation):

$$\frac{1}{(p^2-m^2+i\varepsilon)^\lambda}=\frac{i^{-\lambda}}{\Gamma(\lambda)}\int_0^\infty d\alpha\alpha^{\lambda-1}e^{i\alpha(p^2-m^2+i\varepsilon)}.$$

The imaginary correction $i\varepsilon$ ($\varepsilon>0$) here plays its usual role of a cutoff at the upper limit since the symbols p^2 and m^2 can be regarded as real quantities. After the transition to the α representation, we face the problem of calculating integrals of the form

$$\int dpp_{\mu_1}\dots p_{\mu_r}e^{i(\alpha p^2+2\beta kp)}. \tag{6}$$

We begin with the simpler ancillary integral

$$I(\alpha,\beta k)=\int dpe^{i(\alpha p^2+2\beta kp)}, \tag{7}$$

where α and β are parameters and k is an external momentum. By a shift $p\to p-\frac{\beta}{\alpha}k$ we obtain in accordance with (1)

$$I(\alpha,\beta k)=e^{-ik^2\beta^2/\alpha}I(\alpha),\quad I(\alpha)=\int dpe^{i\alpha p^2}.$$

To find $I(\alpha)$, we are justified in using only the properties (1)-(5), but we must not use dimensional arguments [9]. Therefore, we consider the integral

$$\int dpp_\mu p_\nu e^{i\alpha p^2}, \tag{8}$$

which must, by the Lorentz invariance of the integration procedure, be equal to $A(\alpha)g_{\mu\nu}$. We find $A(\alpha)$, multiplying (8) by $g_{\mu\nu}$:

$$nA(\alpha)=\int dpp^2e^{i\alpha p^2}=-i\frac{\partial}{\partial\alpha}I(\alpha).$$

Therefore

$$\int dpp_\mu p_\nu e^{i\alpha p^2}=-\frac{i}{n}g_{\mu\nu}\frac{\partial}{\partial\alpha}I(\alpha). \tag{9}$$

Using (9) and the properties (1) and (2), we obtain

$$\int dpkpp_\mu e^{i(\alpha p^2+2\beta kp)}=e^{-ik^2\beta^2/\alpha}k_\mu\left(\frac{\beta^2}{\alpha^2}k^2-\frac{i}{n}\frac{\partial}{\partial\alpha}\right)I(\alpha). \tag{10}$$

But the integral (10) can also be calculated differently by differentiation with respect to β:

$$\int dpkpp_\mu e^{i(\alpha p^2+2\beta kp)}=-\frac{i}{2}\frac{\partial}{\partial\beta}\int dpp_\mu e^{i(\alpha p^2+2\beta kp)}=-\frac{i}{2}\frac{\partial}{\partial\beta}\left(-\frac{\beta}{\alpha}k_\mu I(\alpha)e^{-ik^2\beta^2/\alpha}\right)=e^{-ik^2\beta^2/\alpha}k_\mu\left(\frac{\beta^2}{\alpha^2}k^2+\frac{i}{2\alpha}\right)I(\alpha). \tag{11}$$

Comparison of the results (10) and (11) leads to

$$\frac{\partial}{\partial\alpha}I(\alpha)=-\frac{n}{2\alpha}I(\alpha). \tag{12}$$

Solving the equation, we find $I(\alpha)$, and, therefore, $I(\alpha,\beta k)$ as well:

$$I(\alpha)=N\alpha^{-n/2}, \tag{13}$$

$$I(\alpha,\beta k)\equiv\int dpe^{i(\alpha p^2+2\beta kp)}=N\alpha^{-n/2}e^{-ik^2\beta^2/\alpha}, \tag{14}$$

where N is an arbitrary constant. We can show [10] that the lack of uniqueness manifested here in the regularization procedure is entirely due to the arbitrariness in the choice of the normalization points and is therefore immaterial.

We now consider integrals of the form

$$\int dpp_{\mu_1}\ldots p_{\mu_r}e^{i\alpha p^2}. \tag{15}$$

For odd r, this is zero in accordance with (2); for even r, such an integral is proportional to the corresponding symmetric combination of symbols $g_{\mu\nu}$; the coefficient of proportionality can be found by contracting (15) with $g_{\mu\nu}$, as we did above to find $A(\alpha)$.

Finally, the main integral (6) is reduced by the shift $p\to p-\frac{\beta}{\alpha}k$ to the form (15). Thus, the formulas obtained in this section enable us to carry out all the necessary integrations with respect to the momenta.

We now consider the integrals with respect to the α parameters. They may diverge at the lower limit and must be regularized. We make a transformation

$$\int\frac{dpp^2f(p,\ldots)}{(p^2+i\varepsilon)^{\lambda+1}}=\frac{i^{-\lambda-1}}{\Gamma(\lambda+1)}\int_0^\infty d\alpha\alpha^\lambda\int dpf(p,\ldots)p^2e^{i\alpha(p^2+i\varepsilon)}=\frac{i^{-\lambda-2}}{\Gamma(\lambda+1)}\int_0^\infty d\alpha\alpha^\lambda\frac{\partial}{\partial\alpha}\int dpf(p,\ldots)e^{i\alpha(p^2+i\varepsilon)}=$$

$$\int\frac{dpf(p,\ldots)}{(p^2+i\varepsilon)^\lambda}+\frac{i^{-\lambda-2}}{\Gamma(\lambda+1)}\int dpf(p,\ldots)e^{i\alpha(p^2+i\varepsilon)}\alpha^\lambda\Big|_0^\infty.$$

The requirement of uniqueness of the procedure of integration with respect to the momenta will be satisfied only if the boundary contribution from the lower limit vanishes, i.e., under the condition

$$0^\lambda=0 \text{ for any } \lambda. \tag{16}$$

We now integrate by parts the relation that determines the Γ function, taking into account (16):

$$\Gamma(z)\equiv\int_0^\infty dxx^{z-1}e^{-x}=\frac{1}{z}\int_0^\infty dxx^ze^{-x}+\frac{1}{z}x^ze^{-x}\Big|_0^\infty=\frac{\Gamma(z+1)}{z}.$$

The condition (16) in the given case enables us to continue the Γ function analytically into the region of negative z. But by appropriate changes of the variables the singularities in the integrals with respect to the α parameters can be reduced to the form $\int_0^\infty d\alpha\alpha^{\lambda-1}e^{i\alpha(a+i\varepsilon)}$. Therefore, the condition (16) regularizes the α integrals by means of analytic continuation with respect to λ:

$$\int_0^\infty d\alpha\alpha^{\lambda-1}e^{i\alpha(a+i\varepsilon)}=i^\lambda a^{-\lambda}\Gamma(\lambda). \tag{17}$$

Hence and from (14) we conclude that the regularization procedure constructed in this section on the basis of the requirements imposed by the invariance condition is precisely dimensional regularization. The consistency of the complete approach, i.e., the fulfillment in this regularization of the original conditions (1)-(5), has been verified by Collins [3]. It follows from all that we have said that the procedure of integration in dimensional regularization is universally invariant by construction since the relations (1)-(5) are in reality the definition of it. Despite this, as we shall see below, neither dimensional nor any other regularization, taken as a whole, is universally invariant.

4. Anomalies

Since the usual four-dimensional integration, although invariant, leads to divergences, we must use formulas of invariant (dimensional) integration with $n \neq 4$. Thus, besides the stage considered above associated with the integrations with respect to the momenta, the regularization procedure, considered as a whole, contains one further stage of no small importance due to the requirement that one must go over from $n = 4$ in the original theory to $n \neq 4$ in the regularized theory. If the regularization is to be invariant, it is therefore necessary that all the symmetry properties of the Lagrangian be preserved in this stage. This is the case when the symmetry relations of the original Lagrangian are valid for all n and do not depend explicitly on n; in the opposite (anomalous) case the original symmetry is lost on the transition to $n \neq 4$ [11]. The symmetry relations, which are distorted in the transition, naturally cannot be recovered in their previous form after integration performed in the invariant manner. Therefore, it is only when anomalies are absent that dimensional regularization is invariant.

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SPONTANEOUS BREAKING OF CONFORMAL INVARIANCE AND THE HIGGS MECHANISM

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A possibility of covariant generalization of the Higgs mechanism is pointed out in which all the dimensional constants – masses of particles, and the constants of the gravitational and weak interaction – appear simultaneously as a result of the spontaneous breaking of conformal invariance.

The successes of field theory models with spontaneous symmetry breaking suggest the attractive idea that the asymmetry of the observed world must be transferred from the interaction to the ground state (vacuum). In the spirit of this approach, we demonstrate the possibility of introducing dimensional constants into the theory as a result of spontaneous breaking of conformal invariance. Let us make the following remark.

It is well known that the existence of dimensional constants such as the velocity of light and Planck's constant $\hbar$, corresponds in mathematical language to a pseudo-Euclidean nature of space and noncommutativity of the operators of physical quantities. In the units $c = \hbar \doteq 1$, the experimentally known dimensional constants are the masses m_i of the elementary particles, and the constants of the weak interaction G_F and gravitation G_N. Suppose that in nature there exists only a single fundamental length, in terms of which all dimensional constants are expressed and in units of which all physical quantities are measured. Then worlds having different values of this parameter can be related by means of a scale transformation. The infinitefold degeneracy of the vacuum with respect to the fundamental length makes it possible to introduce this quantity as a result of spontaneous breaking of scale or conformal invariance. The latter may, as is well known (see, for example, [1, 2]), be important for the renormalizability of field models that include a gravitational interaction.

As a concrete realization of this scheme, we can take a generalization of the Higgs mechanism. The Higgs Lagrangian has the form [3]

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