ON THE SPECTRUM OF RANDOM MATRICES

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A study is made of the distribution of eigenvalues in a certain ensemble of random particles that contains as a special case the ensemble used by Wigner to give a statistical description of the energy levels of heavy nuclei. It is shown that the distribution function of the eigenvalues divided by the factor N (the order of the matrices) becomes nonrandom in the limit $N \to \infty$ and can be found by solving a definite functional equation.

We shall study the distribution of eigenvalues in an ensemble of random matrices of sufficiently high order. Problems of this kind arise, for example, in nuclear physics in connection with the desire to describe statistically the very complicated structure of the energy spectra of heavy nuclei [1, 2] and in statistical physics in the study of the thermodynamic properties of disordered condensed systems [3].

The problem considered below is similar to problems studied by Wigner on a number of occasions (see [1, 2]) in which he showed that the distribution density of the eigenvalues of N-dimensional symmetric matrices with random, statistically independent elements of a definite form has a fairly simple nature in the limit $N \to \infty$. Wigner obtained this result by making an asymptotic ($N \to \infty$) study of the traces of the powers of the random matrices, i.e., essentially by perturbation theory. This circumstance, in particular, led to fairly stringent restrictions on the probability properties of the matrix elements [see the conditions (1.10) below] and necessitated very subtle combinatorial arguments. On the other hand, Marchenko and the author of [4] have proposed a method of studying the spectra of random operators which are, putting it imprecisely, the sum of a large number of one-dimensional random operators. In the present paper we shall show how this method, with some technical modifications, can be applied to the study of the distribution of the eigenvalues in a certain ensemble of random matrices containing Wigner's ensemble as a special case. Moreover, since our method differs significantly from that of Wigner, we do not need to impose nearly such stringent conditions on the distribution functions of the matrix elements [see the conditions (1.5) below].

1. Statement of the Problem and the Results. Examples

Let H_N be a symmetric random matrix of order N having the form

$$H_N = h_N + \frac{V_N}{\sqrt{N}}, \tag{1.1}$$

where the matrix h_N is diagonal and the numbers h_i, $i = 1, 2, \ldots, N$, on the diagonal are real, identically distributed random variables with distribution function $\sigma(h)$, and the matrix V_N is real and symmetric and its elements v_{ik} are independent – except for their symmetry – random variables that satisfy the conditions (1.5) below.

We shall be interested in the function $\nu(\lambda; H_N)$, – the number of eigenvalues, divided by N, of H_N to the left of λ; we shall call this the normalized spectral function of the matrix H_N (in the physical literature this function is frequently called the number of states). For any H_N, the function $\nu(\lambda; H_N)$ is obviously a nondecreasing, piecewise-constant function of λ and $0 \le \nu(\lambda; H_N) \le 1$. The main burden of the present paper is to find this function. Particular interest attaches to the case of large N, since it frequently happens that $\nu(\lambda; H_N)$ tends in the limit $N \to \infty$ in some sense to a nonrandom function $\nu(\lambda)$. Expressed more precisely,

Physicotechnical Institute of Low Temperatures, Academy of Sciences of the Ukrainian SSR. Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 10, No. 1, pp. 102–112, January, 1972. Original article submitted January 11, 1971.

the problem is to establish, above all, which probability properties of the matrices H_N of the form (1.1) ensure the existence of a nondecreasing and nonrandom function $\nu(\lambda)$ such that at all its points of continuity

$$\lim_{N\to\infty} \Pr\{|\nu(\lambda; H_N) - \nu(\lambda)| > \varepsilon\} = 0 \quad (1.2)$$

for any $\varepsilon > 0$. The principal problem consists, of course, of finding the limit function $\nu(\lambda)$ itself.

We begin by remarking that both these problems can be easily solved for the diagonal matrix h_N in (1.1); for the normalized spectral function of h_N is, by definition,

$$\nu(\lambda; h_N) = \frac{1}{N}\sum_{h_i<\lambda} 1 = \frac{1}{N}\sum_{i=1}^{N}\theta(\lambda - h_i), \quad (1.3)$$

where $\theta(\lambda)$ is the Heaviside function, i.e.,

$$\theta(\lambda) = \begin{cases} 1, & \lambda > 0, \\ 0, & \lambda \leq 0. \end{cases}$$

Since the random variables h_i are independent, it follows from the law of large numbers that for each λ and any $\varepsilon > 0$

$$\lim_{N\to\infty} \Pr\{|\nu(\lambda; h_N) - \sigma(\lambda)| > \varepsilon\} = 0. \quad (1.4)$$

Thus, the normalized spectral function of the "unperturbed" matrix in (1.1) possesses the property (1.2) and the role of the limit function $\nu(\lambda)$ is simply played in this case by the distribution function $\sigma(h)$ of the random variables h_i.

We now turn to the matrix H_N. Suppose the random variables v_{ik} – the elements of the "perturbing" matrix V_N – which are independent for $i \leq k$ satisfy the further conditions:*

a) $Mv_{ik} = 0$, $i, k = 1, 2, \ldots, N$;

b) $Mv_{ik}^2 = v^2$, $i, k = 1, 2, \ldots, N$;

c) for any $\tau > 0$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \int_{|x|>\tau\sqrt{n}} x^2 dF_{in}(x) = 0, \quad (1.5)$$

where $F_{ik}(x)$ is the distribution function of the random variable v_{ik}.

Note that the condition c) is none other than the well-known Lindeberg condition of probability theory [5] for the validity of the central limit theorem. For this condition to be satisfied, it is sufficient, for example, to require that the variables v_{ik} have a bounded moment of order $2+\delta$, $\delta > 0$, uniformly in i and k, i.e.,

$$M|v_{ik}|^{2+\delta} \leq C.$$

If all the v_{ik} are distributed in the same manner, the condition c) is a consequence of b).

As we have already mentioned, our problem is to prove the existence and, above all, actually find the function $\nu(\lambda)$ defined by formula (1.2). However, rather than $\nu(\lambda)$ it is more convenient to find its Stieltjes transform

$$f(z) = \int_{-\infty}^{\infty}\frac{d\nu(\lambda)}{\lambda - z} \quad (\operatorname{Im} z \neq 0),$$

from which, if it is known, one can find $\nu(\lambda)$ at all points of continuity from the well-known inversion formulas.

The main results of the paper are contained in the following theorem.

THEOREM. Suppose H_N is a random matrix of the form (1.1) that satisfies the conditions (1.5). Then 1) the sequence of normalized spectral functions $\nu(\lambda; H_N)$ of the random matrices H_N as $N \to \infty$ converges in probability to some nonrandom, nondecreasing function $\nu(\lambda)$ at all of its points of continuity and

$$\nu(\pm\infty) = \sigma(\pm\infty),$$

*We shall use the symbol M{. . .} to denote averaging over the realizations of the corresponding random variables.

where $\sigma(h)$ is the distribution function of the variables h_i on the diagonal of the matrix h_N in (1.1);

2) the Stieltjes transform $f(z)$ of $\nu(\lambda)$ is the solution of the functional equation

$$f(z)=\int_{-\infty}^{\infty}\frac{d\sigma(\lambda)}{\lambda-z-v^2f(z)} \tag{1.6}$$

in the class of functions that are analytic in z for $\operatorname{Im} z \neq 0$ and such that $\operatorname{Im} f \times \operatorname{Im} z > 0$ for $\operatorname{Im} z \neq 0$. The solution of this equation exists in the given class, is unique, and can always be found by the method of successive approximations.

Recalling that the limit normalized spectral function of h_N is, in accordance with (1.3), simply $\sigma(\lambda)$, the main assertion of the theorem expressed by formula (1.6) can also be written in the form

$$f(z)=f_0(z+v^2f(z)),$$

where

$$f_0(z)=\int_{-\infty}^{\infty}\frac{d\sigma(\lambda)}{\lambda-z} \tag{1.7}$$

is the Stieltjes transform of the limit normalized spectral function of the matrix h_N. In other words, the function $f(z)$ of the "perturbed" operator H_N is obtained from the function $f_0(z)$ corresponding to the unperturbed operator h_N by a shift of its argument by the amount $v^2f(z)$.

Before we turn to the proof of the theorem, let us consider some examples.

1. $h_N \equiv 0$. In this case $\sigma(\lambda) = \theta(\lambda)$ and Eq. (1.6) therefore takes the form

$$f=-\frac{1}{z+v^2f}.$$

Solving this equation for f, we obtain

$$f(z)=\frac{-z+\sqrt{z^2-4v^2}}{2v^2}, \tag{1.8}$$

where for $\operatorname{Im} z > 0$ one must take the branch of the root such that $\operatorname{Im} f > 0$. Applying the inversion formulas to (1.8), we find that

$$\frac{d\nu(\lambda)}{d\lambda}=\begin{cases}\frac{1}{2\pi v^2}\sqrt{4v^2-\lambda^2}, & \lambda^2\leqslant 4v^2,\\ 0, & \lambda^2\geqslant 4v^2.\end{cases} \tag{1.9}$$

This is the so-called "semicircle law." It was established by Wigner [2] under the following restrictions on the distribution function of the variables v_{ik} (the elements of V_N), which are independent for $i \leq k$:

a) $Mv_{ik}^{2l+1} = 0$, i, k = 1, 2, ..., N, l = 0,1, . . . ;

b) $Mv_{ik} = v^2$, i, k = 1, 2, . . . N;

c) uniformly in i and k

$$Mv_{ik}^{2l}\leqslant C_l,\quad l=1,2,\ldots \tag{1.10}$$

Thus, a corollary of the theorem is that the semicircle law (1.9) of the distribution of the eigenvalues of the random symmetric matrix $V_N\sqrt{N}$ with elements that are independent for $i \leq k$ is valid under the much less stringent conditions (1.5), except the condition $Mv_{ik} = 0$, require essentially only that the variables v_{ik} have the same variance, which is all that figures in the final result.

2. The random variables h_i have a distribution function $\sigma(\lambda)$ of the form (1.9), i.e.,

$$\frac{d\sigma}{d\lambda}=\begin{cases}\frac{1}{2\pi a^2}\sqrt{4a^2-\lambda^2}, & \lambda^2\leqslant 4a^2,\\ 0, & \lambda^2\geqslant 4a^2.\end{cases} \tag{1.11}$$

In this case $f_0(z)$ has the form (1.8) with a^2 instead of v^2. Substituting $z + v^2f$ instead of z in accordance with (1.7) into (1.8), and making some simple calculations, we find that $d\nu/d\lambda$ also has the form (1.9) with v^2 replaced by $v^2 + a^2$. Since the normalized spectral function of the unperturbed matrix is (1.11) in accordance with (1.3), the result we have obtained can be formulated as follows: if a diagonal matrix that already satisfies the semicircle law with the parameter a^2 is perturbed by a random matrix of the form $V_N/\sqrt{N}$ we again obtain a semicircle law whose parameter is the sum of the unperturbed parameter and the variance v^2 of the elements of the perturbing matrix V_N.

3.

$$d\sigma(\lambda)=\frac{b\,d\lambda}{\pi(b^2+\lambda^2)},\quad b>0.$$

In this case

$$f_0=-\frac{1}{z+ib}$$

and therefore

$$\frac{d\nu}{d\lambda}=\frac{1}{2\pi v^2}\left\{\sqrt{\frac{[(\lambda_1^2+b^2)(\lambda_2^2+b^2)]^{1/2}-\lambda^2+4v^2+b^2}{2}}-b\right\},$$

$$\lambda_{1,2}=\lambda\pm 2v.$$

It can be seen that in contrast to the first two cases the limit spectrum occupies the entire axis – not surprisingly, because the unperturbed matrix has this property.

Finally, we should like to point out that the determination of f(z) and a fortiori $\nu(\lambda)$ in an explicit form is, as a rule, not at all easy, since Eq. (1.7) for f(z) cannot in general be solved explictly. In this sense the examples given above are exceptions. Nevertheless, one can frequently study the qualitative nature of the spectrum – the number and disposition of its connected components and also the behavior of $\nu(\lambda)$ near the boundaries of the spectrum. However, we shall not dwell on this further – the paper [4] shows how this must be done in a situation very similar to ours.

2. Proof of the Theorem

The gist of the proof is as follows. For every realization of the random matrix H_N a chain of matrices $H_N(n)$ is introduced in a special manner [see (2.3)] such that $H_N(0) = 0$ and $H_N(N) = H_N$. We then consider a function $u(t, z; H_N)$, which for every z is a broken line for $t \in [0, 1]$, its vertices having the abscissas $t = n/N$ and the values $N^{-1}\mathrm{Sp}R_z(n)$, where $R_z(n) = (H_N(n) - z)^{-1}$. It can be shown that $u(t, z: H_N)$ in the limit $N \to \infty$ satisfies a first-order differential equation in z and t [see (2.10)]. This equation can be solved and u(t, z) found, admittedly only implicitly. Then, remembering that u(1, z) is none other than the Stieltjes transform of the normalized spectral function of H_N, we can readily prove all the assertions of the theorem.

Now the details. Note first that the final result does not depend on whether the diagonal elements of V_N vanish or not; for set

$$H_N=\tilde{H}_N+D_N,$$

where D_N is a diagonal matrix with the variables $v_{ii}/\sqrt{N}$ on the diagonal. Then

$$R_N-\tilde{R}_N=-R_N D_N \tilde{R}_N, \tag{2.1}$$

where $R_N = (H_N - z)^{-1}$, $\tilde{R}_N = (\tilde{H}_N - z)^{-1}$, $\mathrm{Im}\, z \neq 0$. It follows from (2.1) and (1.5) that with probability 1

$$\lim_{N\to\infty} N^{-1}\,\mathrm{Sp}(R_N-\tilde{R}_N)=0,$$

from which it is readily concluded that the limit normalized spectral functions of H_N and $\tilde{H}_N$ are identical. We may therefore regard only matrices V_N with zeros on the diagonal.

We shall also first assume that the random variables h_i are bounded by a certain constant C, i.e.,

$$|h_i|\leqslant C,\quad i=1,2,\ldots,N. \tag{2.2}$$

For what follows it is convenient to introduce the operator terminology, i.e., to assume that we are given an N-dimensional Euclidean space E_N with orthonormalized basis $e_1, e_2, \ldots, e_N$. Then the matrices n in (1.1) will correspond to linear selfadjoint operators in E_N. We shall denote them by the same symbols as the matrices.

Let $L(a, b)$ ($a, b \in E_N$) be an operator in E_N that acts on the vectors $x \in E_N$ in accordance with the formula

$$L(a,b)x=(x,a)b,$$

where (x, a) is the scalar product in E_N. The matrix of this operator is obviously $b_i a_k$.

Now consider some realization of the random matrix (1.1) and rearrange simultaneously the columns and rows in it in such a manner that the h_i are labeled in nondecreasing order: $h_1 \le h_2 \le \ldots$ (this does not affect the spectrum!). Introduce a chain of selfadjoint operators $H_N(n)$, setting $H_N(0) = 0$,

$$H_N(n) = H_N(n-1) + L(v_{n-1}, e_n) + L(e_n, v_{n-1}) + h_n L(e_n, e_n), \tag{2.3}$$

where

$$v_{n-1} = N^{-1/2}\{v_{1n}, v_{2n}, \ldots, v_{n-1,n}, 0, \ldots, 0\}.$$

Obviously, $H_N(n)$ is obtained from H_N by replacing by zeros the elements for which at least one of the subscripts exceeds n.

Let $R_z(n)$ be the resolvent (Green's function) of the operator $H_N(n)$, i.e., the operator $(H_N(n)-z)^{-1}$ ($\operatorname{Im} z \ne 0$). It is readily seen that $R_z(n)$ carries vectors for which the components with label greater than n vanish into vectors of the same form. If P_n is the operator of orthogonal projection onto the first n unit vectors $e_1, e_2, \ldots, e_n$, this fact can be expressed in the form

$$R_z(n) = P_n R_z(n) P_n - \frac{1}{z} Q_n, \tag{2.4}$$

where $Q_n = E - P_n$ is the projector onto the last N−n unit vectors.

With each H_N we now associate a function $u(t, z; H_N)$ defined for nonreal z and $t \in [0, 1]$ by the equations

$$u(t, z; H_N) = N^{-1}\operatorname{Sp} R_z(n) + \operatorname{Sp}\{R_z(n+1) - R_z(n)\}\left(t - \frac{n}{N}\right) \tag{2.5}$$

for $t \in [n/N, n+1/N]$, $n = 0, \ldots, N-1$.

Note that $u(0, z; H_N) = -z^{-1}$, and $u(1, z; H_N)$ is the Stieltjes transform of the normalized spectral function of H_N:

$$u(1, z; H_N) = N^{-1}\operatorname{Sp} R_z(N) = \int_{-\infty}^{\infty} \frac{d\nu(\lambda; H_N)}{\lambda - z}. \tag{2.6}$$

In addition, $u(t, z; H_N)$ is continuous in both variables in the region $\operatorname{Im} z \ne 0$, $t \in [0, 1]$, and it is analytic in z and piecewise linear in t.

To derive the equations mentioned above we shall above all use the following formula for the difference of the traces of the resolvents $\tilde{R}_z$ and R_z of two selfadjoint operators $\tilde{A}$ and A in E_N (see, for example, [3, 4]):

$$\operatorname{Sp}\tilde{R}_z - \operatorname{Sp} R_z = -\frac{\partial}{\partial z}\ln\det[E + R_z(\tilde{A} - A)].$$

Substituting $H_N(n)$ and $H_N(n-1)$ as $\tilde{A}$ and A into this equation and taking into account (2.3)-(2.5), we obtain

$$u\left(\frac{n}{N}, z; H_N\right) - u\left(\frac{n-1}{N}, z; H_N\right) = -\frac{1}{N}\frac{\partial}{\partial z}\ln \Delta_N(n, z),$$

$$\Delta_N(n, z) = 1 - \frac{h_n}{z} + \frac{r_n}{z}, \quad r_n = \frac{1}{N}\sum_{i,k=1}^{n} R_{ik}(n) v_{i,n-1} v_{k,n-1}. \tag{2.7}$$

The right-hand side of (2.7) can be represented in the form

$$-\frac{1}{N}\frac{\partial}{\partial z}\ln\left\{1 - \frac{h_n}{z} + \frac{v_2}{z}\left(N^{-1}\operatorname{Sp} R_z(n-1) + \frac{1}{z}\left(1 - \frac{n-1}{N}\right)\right)\right\} + \frac{1}{N}\delta_N(n, z), \tag{2.8}$$

where $M\delta_N(n, z) \to 0$ as $n, N \to \infty$ and $|\operatorname{Im} z| \ge y_0 > 0$. This representation can be proved in the same way as Lemma 2 in [4] and uses the following assertion, which is proved in the appendix.

LEMMA 1. Suppose $x_n \in E_N$ has the form $N^{-1/2}\{\xi_1, \xi_2, \ldots, \xi_n, 0, \ldots, 0\}$, where the random variables ξ_i are independent and satisfy conditions of the type (1.5), i.e., $M\xi_i = 0$, $M\xi_i^2 = v^2 > 0$, and for any $\tau > 0$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\int_{|x|>\tau\sqrt{n}} x^2 dF_i(x) = 0.$$

Further, suppose R is some operator in E_N with matrix R_{ik}.

Then

$$M\left|(Rx_n, x_n)-\frac{v^2}{n}\sum_{i=1}^{n}R_{ii}\right|\leq\|R\|\cdot\varepsilon(n),$$

where $\|R\|$ is the norm of R, $\varepsilon(n)$ does not depend on R, and

$$\lim_{n\to\infty}\varepsilon(n)=0.$$

Now, using the representation (2.8), and proceeding essentially as in the proof of Lemma 4 in [4], we can show that the mathematical expectation of

$$\varphi_N=\sup_{\substack{t\in[0,1]\\ z\in G}}\left|u(t,z;H_N)+\frac{1}{z}+\int_0^t\frac{\partial}{\partial t}\ln\left\{1-\frac{h(\tau)}{z}+\frac{v^2}{z}\left[u(\tau,z;H_N)+\frac{1-\tau}{z}\right]\right\}d\tau\right| \tag{2.9}$$

tends to zero as $N\to\infty$:

$$\lim_{N\to\infty}M\varphi_N=0.$$

Here G is any bounded set in the z plane that does not contain the real axis and h(t) is the function that is the inverse of $\sigma(x)$ – the distribution function of the random variables h_i.* The function h(t) possesses the following property [4], which is used in the proof of (2.9): with probability 1

$$\lim_{N\to\infty}\int_0^1|h(t)-h_N(t)|\,dt=0.$$

Here $h_N(t)=h_{i+1}$ for $t\in[i/N, (i+1)/N]$, $i=0, 1, \ldots, N-1$, and the numbers $h_1\leq h_2\leq\ldots$ form a realization of the random variables h_i labeled in nondecreasing order.

Since (2.9) holds and the families $\{u(t, z; H_N)\}$ and $\{\partial/\partial z\, u(t, z; H_N)\}$ are compact (see [4] for the details) the family $\{u(t, z; H_N)\}$ contains a subsequence which converges uniformly in $z\in G$ and $t\in[0, 1]$ in the limit $N\to\infty$ to a function u(t, z) that satisfies the equation

$$u(t,z)=-\frac{1}{z}-\int_0^t\frac{\partial}{\partial z}\ln w(\tau,z)\,d\tau,$$
$$w(t,z)=1-\frac{h(t)}{z}+\frac{v^2}{z}\left[u(t,z)+\frac{1-t}{z}\right], \tag{2.10}$$

or

$$\frac{\partial u}{\partial z}+\frac{\partial}{\partial z}\ln w=0,\quad u|_{t=0}=-\frac{1}{z}. \tag{2.11}$$

Making a minor modification of Haar's method [6], one can show that this equation has a unique solution in the considered class of functions that are continuous in (t, z) ($t\in[0, 1]$, Im $z>0$) and analytic in z(Im $z>0$) for fixed t. Then, after the manner of [4], one can prove that the first assertion of our theorem holds if the h_i are bounded. We shall show that the second assertion also holds for such h_i.

We introduce a function g(t, z):

$$g(t,z)=1+\frac{v^2}{z}\left[u(t,z)+\frac{1-t}{z}\right]. \tag{2.12}$$

Direct verification shows that we obtain the general solution of the differential equation (2.10) if we define g(t, z) implicitly as follows:

$$g(t,z)=\frac{v^2}{z}F\left(z+v^2\int_0^t\frac{d\tau}{h(\tau)-zg(t,z)}\right), \tag{2.13}$$

where F(z) is an arbitrary function. As regards the solution of the Cauchy problem (2.10)-(2.11), it can be obtained by virtue of the uniqueness theorem mentioned above by taking $F(z)=v^{-2}z$ ($g(0, z)=1$). Since the Stieltjes transform f(z) of the limit normalized spectral function $\nu(\lambda)$ is, in accordance with (2.6) and (2.12), related to g(1, z) by the equation

$$g(1,z)=1+\frac{v^2}{z}f(z),$$

it follows from (2.13) with $F=v^{-2}z$ that

*On the sections of strict increase, h(t) and $\sigma(x)$ are functions that are the inverses of each other. To constancy intervals of $\sigma(x)$ there correspond discontinuities of h(z); to discontinuities of $\sigma(x)$, constancy intervals of h(t).

$$f(z) = \int_0^1 \frac{d\tau}{h(\tau) - z - v^2 f(z)}.$$

This equation can, in view of the definition of $h(\tau)$, be now written in the form (1.6), i.e.,

$$f(z) = \int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{\lambda - z - v^2 f(z)}. \tag{2.14}$$

Equation (2.14) can be solved by the method of successive approximations. To this end, we introduce a distance in the set $\mathfrak{M}$ of functions that are analytic in the upper half-plane and have there a positive imaginary part:

$$\rho(f_1, f_2) = \sup_{y > y_0 > 0} |f_1(iy) - f_2(iy)|.$$

In this metric $\mathfrak{M}$ becomes a complete metric space. A simple estimate shows that the right-hand side of (2.14) defines for $y_0 > v$ a contractive operator in $\mathfrak{M}$, this result being true even without the restriction (2.2) (in estimating the norm of the operator we used only the inequality $\int_{-\infty}^{\infty} d\sigma(\lambda) < 1$). This remark enables us to prove our theorem in complete generality, i.e., without the restriction (2.2). The arguments that are used are essentially the same as in §5 of [4].

In conclusion, let us point out some possible generalizations. For example, the condition (1.5) could be imposed on only the "majority" of the elements, since, as can be shown by means of Eq. (1.5), a perturbation of V_N by a matrix for which the number of nonvanishing elements is in order of magnitude less than $N^{3/2}$ does not affect the final result. But then for every n a part of v_{in} that has order less than $N^{1/2}$ need not satisfy the condition (1.5). This suggests that the condition of independence of the v_{in} in one column, i.e., for the same n, could be replaced by a different condition which could be satisfied by dependent random variables. Examples of variables of this kind are given in [4], which also contains formulations of general sufficient conditions for weakly dependant familities of random variables of this kind. In [4] an analog of Lemma 1 for random vectors of this kind is also proved.

Finally, one need not assume that the operator h_N in (1.1) is random; rather one can require that the normalized spectral function of this operator, i.e., an expression of the form (1.3), converges sufficiently rapidly in the limit $N \to \infty$ to some limit distribution function.

I should like to thank V. A. Marchenko and I. M. Slivnyak for their interest.

Appendix

Proof of Lemma 1. For brevity we denote by r the quantity

$$(Rx_n, x_n) = \frac{1}{N} \sum_{i,k=1}^{n} R_{ik} \xi_i \xi_k,$$

and we then have

$$Mr = \frac{v^2}{N} \sum_{i=1}^{n} R_{ii}.$$

We now introduce variables η_i and ζ_i that are independent for different i by setting

$$\eta_i = \begin{cases} \xi_i, & |\xi_i| \leqslant a, \\ 0, & |\xi_i| > a, \end{cases} \qquad \zeta_i = \xi_i - \eta_i,$$

and we represent r in the form

$$r = r_1 + r_2 + r_3,$$

where

$$r_1 = \frac{1}{N} \sum_{i=1}^{n} R_{ii} \zeta_i^2, \quad r_2 = \frac{1}{N} \sum_{i \neq k} R_{ik} \xi_i \xi_k, \quad r_3 = \frac{1}{N} \sum_{i=1}^{n} R_{ii} \eta_i^2.$$

Then

$$M \left| (Rx_n, x_n) - \frac{v^2}{N} \sum_{i=1}^{n} R_{ii} \right| = M|r - Mr| \leqslant M|r_1| + \sqrt{Mr_2^2} + \sqrt{M|r_3 - Mr|^2}.$$

Estimating each term on the right-hand side of this inequality by means of the relations

$$M\eta_i^4 \leqslant \alpha^2 v^2, \quad \frac{1}{n}\sum_{i=1}^{n} M\zeta_i^2 = \varphi(n,\alpha), \quad |R_{ii}| \leqslant \|R\|, \quad \sum_{i=1}^{n} |R_{ik}|^2 \leqslant n\|R\|^2,$$

where

$$\varphi(n,\alpha) = \frac{1}{n}\sum_{i=1}^{n} \int\limits_{|x|>\alpha} x^2 dF_i(x),$$

we find that

$$M|r - Mr| \leqslant \|R\| \left\{ \varphi(n,\alpha) + \frac{2v^2}{\sqrt{n}} + \left[\frac{v^4}{n} + \frac{2v^2}{n}\varphi(n,\alpha) + \varphi^2(n,\alpha) + \frac{v^2\alpha^2}{n} \right]^{1/2} \right\}.$$

Setting $\alpha = \tau\sqrt{n}$ in this inequality, we see that the lemma holds by virtue of (1.5).

LITERATURE CITED

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