

## ADELIC MODEL OF HARMONIC OSCILLATOR

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*Adelic quantum mechanics is formulated. The corresponding model of the harmonic oscillator is considered. The adelic harmonic oscillator exhibits many interesting features. One of them is a softening of the uncertainty relation.*

### 1. INTRODUCTION

Since 1987,  $p$ -adic numbers have been applied in string theory [1–4], quantum mechanics [5–12], and in some other parts of theoretical [13] and mathematical [14–16] physics. The obtained models are  $p$ -adic analogues of some standard models constructed over real numbers. In particular,  $p$ -adic strings are very attractive because of their relevance to Planck scale physics and the product (adelic) formula for string amplitudes [2].

Much attention has been paid to constructing  $p$ -adic quantum mechanics, which has complex-valued wave functions and  $p$ -adic canonical variables. It is significant that such theory does exist and that it allows an exact solution of the harmonic oscillator. What is so far unclear is the connection between  $p$ -adic quantum mechanics and the standard one. This is not only the problem of  $p$ -adic quantum mechanics but also of other  $p$ -adic models.

The most natural framework which offer mathematics to unify standard and  $p$ -adic models is the analysis of adeles. So, according to adelic formula in string theory the product of standard four-point amplitude and all  $p$ -adic analogues is equal to a constant. Some aspects of an adelic approach in quantum field theory are considered in [17].

This article is devoted to a formulation of adelic quantum mechanics and its illustration by the harmonic oscillator. Adelic concepts are more fundamental than those of standard or  $p$ -adic quantum mechanics. The latter (standard and  $p$ -adic) are building blocks of adelic quantum theory as a whole. The problem of the connection between  $p$ -adic quantum mechanics and the standard one solves in this approach is as follows: they are independent components of adelic quantum mechanics. Standard quantum mechanics may be considered as an approximation of the adelic one when  $p$ -adic effects can be neglected.

In Sec. 2 we present some of the main properties of  $p$ -adic numbers, adeles, and their analysis. Section 3 contains a necessary review of the harmonic oscillator in standard and  $p$ -adic quantum mechanics. Adelic quantum mechanics and adelic harmonic oscillator are presented in Sec. 4. In the last section we discuss the results obtained and make some conclusions.

### 2. ADELES

The set of rational numbers  $Q$  is the simplest infinite number field. Completion of  $Q$  with respect to the usual absolute value gives the field of real numbers  $R$ . An analogous completion with respect to the  $p$ -adic norms (valuations) yields the fields of  $p$ -adic numbers  $Q_p$  ( $p =$  a prime number). According to the Ostrowski theorem,  $R \equiv Q_\infty$  and  $Q_p$  (for every  $p$ ) exhausts all number fields which can be obtained by completions of  $Q$ .

Recall that a series

$$\varepsilon \sum_{k=-\infty}^{+\infty} a_k p^k, \quad a_k = \{0, 1, \dots, p-1\}, \quad (2.1)$$

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where  $\varepsilon = \pm 1$  and  $a_k = 0$  for  $k \geq k_0$ , represents a real number. If  $\varepsilon = 1$  and  $a_k = 0$  for  $k \leq k_0$ , the series (2.1) represents a  $p$ -adic number in  $Q_p$ . The ring of  $p$ -adic integers  $Z_p$  consists of  $x = \sum_{k \geq 0} a_k p^k$  or, in other words,  $Z_p = \{x \in Q_p : |x|_p \leq 1\}$ , where  $|x|_p$  denotes the  $p$ -adic norm of  $x$ .

On the additive group  $Q_p^+$  there exists the Haar measure  $dx_p$  which is invariant under translation, i.e.,  $d(x+a)_p = dx_p, a \in Q_p$ . Also on the multiplicative group  $Q_p^* = Q_p \setminus \{0\}$  there is the Haar measure  $d^*x_p$  invariant under multiplication:  $d^*(bx)_p = |b|_p d^*x_p, b \in Q_p^*$ . These measures are connected by the equality

$$d^*x_p = \frac{1}{1-p^{-1}} \frac{dx_p}{|x|_p} \quad (2.2)$$

An adèle is an infinite sequence

$$a = (a_\infty, a_2, \dots, a_p, \dots), \quad (2.3)$$

where  $a_\infty \in R, a_p \in Q_p$  with the restriction that all but a finite number of  $a_p \in Z_p$ . Let  $\mathcal{A}$  be the set of all adèles,  $\mathcal{A}$  a ring under componentwise addition and componentwise multiplication, and  $\mathcal{A}^+$  an additive group with respect to addition. A multiplicative group of ideles  $\mathcal{A}^*$  is a subset of  $\mathcal{A}$  with elements  $b = (b_\infty, b_2, \dots, b_p, \dots)$  such that  $b_\infty \neq 0$  and  $b_p \neq 0$  for every  $p$ , and  $|b_p|_p = 1$  for all except a finite number of  $p$ . A principal adèle (idele) is a sequence  $(r, r, \dots, r, \dots) \in \mathcal{A}$ , where  $r \in Q$  ( $r \in Q^* = Q \setminus \{0\}$ ). One can define a module of the ideles,

$$|b| = |b_\infty|_\infty \prod_p |b_p|_p \quad (2.4)$$

which for a principal idele is

$$|r| = |r|_\infty \prod_p |r|_p = 1. \quad (2.5)$$

An additive character on  $\mathcal{A}^+$  is

$$\chi(xy) = \chi_\infty(x_\infty y_\infty) \prod_p \chi_p(x_p y_p) = \exp(-2\pi i x_\infty y_\infty) \prod_p \exp 2\pi i \{x_p y_p\}_p \quad (2.6)$$

where  $x, y \in \mathcal{A}^+$  and  $\{x_p y_p\}_p$  is the fractional part of  $x_p y_p$ . A multiplicative character on  $\mathcal{A}^*$  can be defined as

$$\pi(b) = \pi_\infty(b_\infty) \pi_2(b_2) \dots \pi_p(b_p) \dots = |b_\infty|_\infty^s \prod_p |b_p|_p^s = |b|_p^s, \quad (2.7)$$

where  $b$  is an idele and  $s \in C$  (the field of complex numbers). It is evident that only finitely many factors in (2.6) and (2.7) are different from unity. One can easily see that  $\chi(r) = 1$  when  $r$  is a principal adèle, and  $\pi(r) = 1$  if  $r$  is a principal idele.

An elementary function on the group of adèles  $\mathcal{A}^+$  is

$$\varphi(x) = \varphi_\infty(x_\infty) \prod_p \varphi_p(x_p), \quad (2.8)$$

where  $x \in \mathcal{A}^+, \varphi_\infty(x_\infty) \in S(R), \varphi_p(x_p) \in S(Q_p)$ . Note that  $\varphi(x)$  is a complex-valued function and must satisfy the following conditions: (i)  $\varphi_\infty(x_\infty)$  is an analytic function on  $R$  and for any  $n \in N$  the expression  $|x_\infty|_\infty^n \varphi_\infty(x_\infty) \rightarrow 0$  as  $|x_\infty|_\infty \rightarrow \infty$ ; (ii)  $\varphi_p(x_p)$  is a finite and locally constant function, i.e.,  $\varphi_p$  has a compact support and  $\varphi_p(x_p + y_p) = \varphi_p(x_p)$  if  $|y_p|_p \leq p^{-n}, n = n(\varphi_p) \in N$ ; (iii) for all but a finite number of  $p, \varphi_p(x_p) = \Omega(|x_p|_p)$ , where

$$\Omega(a) = \begin{cases} 1 & \text{if } 0 \leq a \leq 1, \\ 0 & \text{if } a > 1. \end{cases}$$

All finite linear combinations of elementary functions  $\varphi(x)$  make a set of the Schwartz–Bruhat functions  $S(\mathcal{A})$ .

The Fourier transform of  $\varphi(x) \in S(\mathcal{A})$  is

$$\tilde{\varphi}(y) = \int_{\mathcal{A}^+} \varphi(x) \chi(xy) dx = \int_{-\infty}^{+\infty} \varphi_{\infty}(x_{\infty}) e^{-2\pi i x_{\infty} y_{\infty}} dx_{\infty} \prod_p \int_{Q_p} \varphi_p(x_p) e^{2\pi i \{x_p y_p\}_p} dx_p, \quad (2.9)$$

where  $dx = dx_{\infty} dx_2 \dots dx_p \dots$  is the Haar measure on the additive group  $\mathcal{A}^+$ . The Mellin transform of  $\varphi(x) \in S(\mathcal{A})$  is defined with respect to the multiplicative character  $\pi(x) = |x|^s$ , i.e.,

$$\begin{aligned} \Phi(s) &= \int_{\mathcal{A}^*} \varphi(x) |x|^s d^*x = \int_{-\infty}^{+\infty} \varphi_{\infty}(x_{\infty}) |x_{\infty}|_{\infty}^{s-1} dx_{\infty} \\ &\quad \times \prod_p \int_{Q_p} \varphi_p(x_p) |x_p|_p^{s-1} \frac{dx_p}{1-p^{-1}}, \quad \text{Re } s > 1, \end{aligned} \quad (2.10)$$

where  $d^*x = d^*x_{\infty} d^*x_2 \dots d^*x_p \dots$  is the Haar measure on the multiplicative group  $\mathcal{A}^*$ .

The function  $\Phi(s)$  can be continued analytically on the whole field of complex numbers, except  $s = 0$  and  $s = 1$ , where it has simple poles with residues  $-\varphi(0)$  and  $\tilde{\varphi}(0)$ , respectively.  $\Phi(s)$  satisfies the Tate formula

$$\Phi(s) = \tilde{\Phi}(1-s), \quad (2.11)$$

where  $\tilde{\Phi}$  is the Mellin transform of  $\tilde{\varphi}$ .

Let us also note that any other necessary information on  $p$ -adic numbers, adeles, and their analysis can be found in [18, 19, 14].

### 3. REAL AND $p$ -ADIC HARMONIC OSCILLATOR

The harmonic oscillator is a very attractive theoretical model because of its exact solvability and many applications. The corresponding Hamiltonian is

$$H = \frac{1}{2m} k^2 + \frac{m\omega^2}{2} q^2, \quad (3.1)$$

where  $q$  and  $k$  are position and momentum, respectively. The evolution of classical state can be presented in the form

$$\begin{pmatrix} q(t) \\ k(t) \end{pmatrix} = T_t \begin{pmatrix} q \\ k \end{pmatrix} = T_t z, \quad T_t = \begin{pmatrix} \cos \omega t & (m\omega)^{-1} \sin \omega t \\ -m\omega \sin \omega t & \cos \omega t \end{pmatrix}, \quad (3.2)$$

where  $q = q(0)$ ,  $k = k(0)$ . In the real case  $0 \neq m, \omega, q, k, t \in \mathbb{R}$  and in the  $p$ -adic one  $0 \neq m, \omega, q, k, t \in Q_p$  with conditions  $|\omega t|_p \leq p^{-1}$  for  $p \neq 2$  and  $|\omega t|_2 \leq 2^{-2}$ , which represent convergence domains for the  $p$ -adic expansions of  $\cos \omega t$  and  $\sin \omega t$  (we shall denote these domains by  $G_p$ ). In standard (over real numbers) quantum mechanics the harmonic oscillator is given by the Schrödinger equation

$$\frac{d^2 \psi}{dx^2} + \frac{2m}{\hbar^2} \left( E - \frac{m\omega^2}{2} x^2 \right) \psi = 0 \quad (3.3a)$$

or

$$\frac{d^2 \psi}{d\xi^2} + \left( \frac{2E}{\hbar\omega} - \xi^2 \right) \psi = 0, \quad (3.3b)$$

where

$$\xi = \left( \frac{m\omega}{\hbar} \right)^{\frac{1}{2}} x \sqrt{2\pi} \quad (3.4)$$

is a dimensionless position coordinate. (From now on we shall use  $m = \omega = h = 1$ .) As is well known, the physical solutions to (3.3b) are orthonormal eigenfunctions

$$\psi_n(x) = \frac{2^{\frac{1}{4}}}{(2^n n!)^{\frac{1}{2}}} e^{-\pi x^2} H_n(x\sqrt{2\pi}), \quad (3.5)$$

where  $H_n(x\sqrt{2\pi})$  ( $n = 0, 1, 2, \dots$ ) are the Hermite polynomials. One can easily show that  $\psi_n(x) \in S(\mathbb{R})$ .

In  $p$ -adic quantum mechanics, which we shall adopt here, canonical variables are  $p$ -adic numbers and wave functions are complex valued (for quantum mechanics of  $p$ -adic valued functions, see [5, 15, 20]). Since  $x_p \in Q_p$  and  $\psi^{(p)}(x_p) \in C$  the Hamiltonian quantization procedure does not work.

According to the Vladimirov–Volovich approach [5, 7, 8]  $p$ -adic quantum mechanics is given by a triple  $(L_2(Q_p), W_p(z), U_p(t))$ , where  $Q_p$  is the field of  $p$ -adic numbers,  $z = \begin{pmatrix} q \\ k \end{pmatrix}$  is a point of  $p$ -adic classical phase space, and  $t$  is a  $p$ -adic time.  $L_2(Q_p)$  is the Hilbert space of complex-valued square integrable functions with respect to the Haar measure on  $Q_p$ ,  $W_p(z)$  is a unitary representation of the Heisenberg–Weyl group on  $L_2(Q_p)$ , and  $U_p(t)$  (the evolution operator) is a unitary representation on  $L_2(Q_p)$  of a subgroup  $G_p$  of the additive group  $Q_p$ .

The operator  $W_p(z)$  realizes the Weyl representation of commutation relations and has the form

$$W_p(z)\psi^{(p)}(x) = \int_{Q_p} W_p(z; x, y)\psi^{(p)}(y)dy, \quad \psi_p \in L_2(Q_p), \quad (3.6)$$

with the kernel

$$W_p(z; x, y) = \chi_p(2kx + kq)\delta(x - y + q) \quad (3.7)$$

and gives

$$W_p(z)\psi^{(p)}(x) = \chi_p(2kx + kq)\psi^{(p)}(x + q). \quad (3.8)$$

The evolution operator in  $p$ -adic quantum mechanics is given by

$$U_p(t)\psi^{(p)}(x) = \int_{Q_p} \mathcal{K}_t^{(p)}(x, y)\psi^{(p)}(y)dy, \quad (3.9)$$

where the kernel for the harmonic oscillator is

$$\mathcal{K}_t^{(p)}(x, y) = \lambda_p(2t) \left| \frac{1}{t} \right|_p^{\frac{1}{2}} \chi_p \left( \frac{xy}{\sin t} - \frac{x^2 + y^2}{2 \tan t} \right), \quad t \in G_p \setminus \{0\}, \quad (3.10a)$$

$$\mathcal{K}_0^{(p)}(x, y) = \delta_p(x - y), \quad (3.10b)$$

where  $\delta_p(x - y)$  is a  $p$ -adic analogue of the Dirac  $\delta$ -function.

If  $t \in Q_p$  has the canonical expansion

$$t = p^\nu(t_0 + t_1p + t_2p^2 + \dots), \quad \nu \in Z, t_0 \neq 0, 0 \leq t_i \leq p - 1, \quad (3.11)$$

then the number-theoretic function  $\lambda_p(t)$  is

$$\lambda_p(t) = \begin{cases} 1, & \nu = 2k, \\ \left(\frac{t_0}{p}\right), & \nu = 2k + 1, \quad p \equiv 1(\text{mod } 4), \\ i\left(\frac{t_0}{p}\right), & \nu = 2k + 1, \quad p \equiv 3(\text{mod } 4); \end{cases} \quad (3.12a)$$

$$\lambda_2(t) = \begin{cases} \frac{1}{\sqrt{2}}[1 + (-1)^{t_1}i], & \nu = 2k, \\ \frac{1}{\sqrt{2}}(-1)^{t_1+t_2}[1 + (-1)^{t_1}i], & \nu = 2k + 1, \end{cases} \quad (3.12b)$$

where  $\left(\frac{t_0}{p}\right)$  is the Legendre symbol and  $k \in Z$ . The analogous kernel in the real case [21] is

$$\mathcal{K}_t^{(\infty)}(x, y) = \lambda_\infty(\sin t) \left| \frac{1}{\sin t} \right|_\infty^{\frac{1}{2}} \exp 2\pi i \left( \frac{x^2 + y^2}{2 \tan t} - \frac{xy}{\sin t} \right), \quad (3.13a)$$

$$\mathcal{K}_0^{(\infty)}(x, y) = \delta_\infty(x - y), \quad (3.13b)$$

where  $| \cdot |_\infty$  denotes the usual absolute value and

$$\lambda_\infty(t) = \begin{cases} \frac{1}{\sqrt{2}}(1 - i), & t > 0, \\ \frac{1}{\sqrt{2}}(1 + i), & t < 0. \end{cases} \quad (3.14)$$

The operator  $U_p(t)$  satisfies the relation

$$U_p(t)W_p(z)U_p^{-1}(t) = W_p(T_t z). \quad (3.15)$$

A character  $\chi_p(\alpha t)$  can be an eigenvalue of the operator  $U_p(t)$  for the harmonic oscillator if and only if  $\alpha \in I_p \subset Q_p$  takes the following forms:

$$\alpha = 0, \quad (3.16a)$$

$$\alpha = p^{-\nu}(\alpha_0 + \alpha_1 p + \dots + \alpha_{\nu-2} p^{\nu-2}), \quad \alpha_0 \neq 0, 0 \leq \alpha_i \leq p - 1, \quad (3.16b)$$

where (i)  $\nu \geq 2$  for  $p \equiv 1(\text{mod } 4)$ , (ii)  $\nu = 2n$  ( $n \in N$ ) for  $p \equiv 3(\text{mod } 4)$ , and (iii)  $\nu \geq 4, \alpha_0 = \alpha_1 = 1$  for  $p = 2$ . It means that  $\alpha$  has discrete values and may be considered as a  $p$ -adic energy of the harmonic oscillator.

The corresponding eigenfunctions satisfy the equation

$$U_p(t)\psi_\alpha^{(p)}(x) = \chi_p(\alpha t)\psi_\alpha^{(p)}(x). \quad (3.17)$$

The value  $\alpha = 0$  corresponds to a vacuum state which is invariant under  $U_p(t)$ , i.e.,

$$U_p(t)\psi_0^{(p)}(x) = \psi_0^{(p)}(x). \quad (3.18)$$

The Hilbert space  $L_2(Q_p)$  can be presented as a direct sum of mutually orthogonal subspaces, i.e.,

$$L_2(Q_p) = \bigoplus_{\alpha \in I_p} H_\alpha^{(p)}. \quad (3.19)$$

The dimensions of  $H_\alpha^{(p)}$  are as follows: (i) when  $p \equiv 1(\text{mod } 4)$ ,  $\dim H_\alpha^{(p)} = \infty$  for every possible  $\alpha$ ; (ii) when  $p \equiv 3(\text{mod } 4)$ ,  $\dim H_0^{(p)} = 1$  and  $H_\alpha^{(p)} = p + 1$  for  $|\alpha|_p \geq p^{2n}$  ( $n \in N$ ); and (iii) when  $p = 2$ ,  $\dim H_0^{(2)} = \dim H_\alpha^{(2)} = 2$  for  $|\alpha|_2 = 2^3$  and  $\dim H_\alpha^{(2)} = 4$  for  $|\alpha|_2 \geq 2^4$ . Any dimension determines the number of linearly independent eigenfunctions which correspond to the degenerate eigenvalue  $\chi_p(\alpha t)$ . So far eigenfunctions  $\psi_\alpha^{(p)}(x)$  are obtained [11, 12] in an explicit form for the vacuum state  $\psi_0^{(p)}(x)$  and for some higher states ( $\alpha \neq 0$ ). These eigenfunctions belong to  $S(Q_p)$ . From here we mainly restrict consideration to vacuum states.

The orthonormal vacuum eigenfunctions of the  $U_p(t)$  for the harmonic oscillator are: (i)  $\varphi_0^{(p)}(x) = \Omega(|x|_p)$ ,  $\varphi_\nu^{(p)}(x) = p^{-\frac{\nu}{2}}(1 - p^{-1})^{-\frac{1}{2}}\chi_p(\tau x^2)\delta(p^\nu - |x|_p)$ ,  $\nu \in N$ ,  $\tau^2 = -1$ , for  $p \equiv 1(\text{mod } 4)$ ; (ii)  $\varphi_0^{(p)}(x) = \Omega(|x|_p)$  for  $p \equiv 3(\text{mod } 4)$ ; and (iii)  $\varphi_0^{(2)}(x) = \Omega(|x|_2)$ ,  $\varphi_1^{(2)}(x) = 2\Omega(2|x|_2) - \Omega(|x|_2)$  for  $p = 2$ . Here,  $\delta(p^\nu - |x|_p)$  is an elementary function defined [21] as

$$\delta(p^\nu - |x|_p) = \begin{cases} 1, & |x|_p = p^\nu, \\ 0, & |x|_p \neq p^\nu. \end{cases} \quad (3.20)$$

#### 4. ADELIC HARMONIC OSCILLATOR

We shall consider adelic quantum mechanics as a triple  $(L_2(\mathcal{A}), W(z), U(t))$ , where  $\mathcal{A}$  is a ring of adeles,  $z = \begin{pmatrix} q \\ k \end{pmatrix}$  is an adelic point of a classical phase space, and  $t$  is an adelic time.  $L_2(\mathcal{A})$  is the Hilbert space of complex-valued square integrable functions with respect to the Haar measure on  $\mathcal{A}$ ,  $W(z)$  is a unitary representation of the Heisenberg–Weyl group on  $L_2(\mathcal{A})$ , and  $U(t)$  (the evolution operator) is a unitary representation on  $L_2(\mathcal{A})$  of a subgroup  $G$  of the additive group  $\mathcal{A}^+$ .

An orthonormal basis of the adelic Hilbert space for the harmonic oscillator is

$$\psi_{\alpha\beta}(x) = \psi_{n_0}^{(\infty)}(x_\infty) \prod_p \psi_{\alpha_p}^{(p)}(x_p), \quad (4.1)$$

where  $\psi_{n_0}^{(\infty)}(x_\infty) \equiv \psi_n(x_\infty)$  and  $\psi_{\alpha_p \beta_p}^{(p)}(x_p)$  are orthonormal eigenfunctions in real and  $p$ -adic cases, respectively. By  $\alpha = (n, \alpha_2, \dots, \alpha_p, \dots)$  and  $\beta = (0, \beta_2, \dots, \beta_p, \dots)$  we denote adelic indices, which characterize energy levels and their degeneration. According to [11] and the preceding section, for  $p \geq 3$  and  $|x_p|_p \leq 1$  all  $p$ -adic eigenfunctions are

$$\psi_{00}^{(p)}(x_p) \equiv \varphi_0^{(p)}(x_p) = \Omega(|x_p|_p). \quad (4.2)$$

Thus for any value of the adelic variable  $x$  one has

$$\psi_{\alpha_p \beta_p}^{(p)}(x_p) \neq \Omega(|x_p|_p) \quad (4.3)$$

only for a finite number of primes  $p$ . In other words, in (4.1) all but a finite number of  $\psi_{\alpha_p \beta_p}^{(p)}(x_p)$  are vacuum states  $\varphi_0^{(p)}(x_p) = \Omega(|x_p|_p)$ , i.e., all except a finite number of  $p$ -adic indices satisfy  $\alpha_p = \beta_p = 0$ . Any  $\psi(x) \in L_2(\mathcal{A})$  may be presented as

$$\psi(x) = \sum C_{\alpha\beta} \psi_{\alpha\beta}(x), \quad (4.4)$$

where  $C_{\alpha\beta} = (\psi_{\alpha\beta}, \psi)$ . It is worth noting that all finite superpositions in (4.4) belong to the set of the Schwartz–Bruhat functions  $S(\mathcal{A})$ .

According to (3.8) the adelic unitary operator  $W(z)$  acts in the following way:

$$W(z)\psi(x) = \chi(2kx + kq)\psi(x + q), \quad (4.5)$$

where  $\chi(2kx + kq)$  is the additive character on adeles (2.6) and  $\psi \in L_2(\mathcal{A})$ . Since  $x, q, k \in \mathcal{A}$ , there exists prime  $p_n$  such that  $|2k_p x_p + k_p q_p|_p \leq 1$  for all  $p > p_n$ , and an infinite product of real and  $p$ -adic characters reduces to

$$\chi(2kx + kq) = \chi_\infty(2k_\infty x_\infty + k_\infty q_\infty) \prod_{p=2}^{p_n} \chi_p(2k_p x_p + k_p q_p). \quad (4.6)$$

When  $x, q, k$  are principal adeles (rational points) one has  $\chi(2kx + kq) = 1$  and

$$W(z)\psi(x) = \psi(x + q). \quad (4.7)$$

The adelic evolution operator  $U(t)$  can be defined by

$$U(t)\psi(x) = \int_{\mathcal{A}} \mathcal{K}_t(x, y)\psi(y)dy, \quad (4.8)$$

where  $U(t) = U_\infty(t_\infty) \prod_p U_p(t_p)$ ,  $t \in G \subset \mathcal{A}$  and  $\psi(x) \in L_2(\mathcal{A})$ . The kernel  $\mathcal{K}_t(x, y)$  for the harmonic oscillator is

$$\mathcal{K}_t(x, y) = \mathcal{K}_{t_\infty}^{(\infty)}(x_\infty, y_\infty) \prod_p \mathcal{K}_{t_p}^{(p)}(x_p, y_p), \quad (4.9)$$

where  $\mathcal{K}_{t_p}^{(p)}$  and  $\mathcal{K}_{t_\infty}^{(\infty)}$  are given by (3.10) and (3.13).

By virtue of (3.15) and an analogous relation in the real case, it follows that

$$U(t)W(z)U^{-1}(t) = W(T_t z). \quad (4.10)$$

Note that  $\psi_{\alpha\beta}(x)$  in (4.1) are the adelic orthonormal eigenfunctions of the evolution operator  $U(t)$ . Since  $\mathcal{K}_t^{(p)}(x, y)$  depend on  $t$  through  $\sin t$  and  $\tan t$ , the adelic time in  $U(t)$  cannot be a principal idele.

Let  $\hat{D}$  be an operator which acts in the Hilbert space  $L_2(\mathcal{A})$ . It is natural to define an expectation (average) value of the corresponding observable  $D$  in a state  $\psi(x) \in L_2(\mathcal{A})$  as

$$\langle D \rangle = (\psi, \hat{D}\psi) = (\psi^{(\infty)}, \hat{D}_\infty \psi^{(\infty)}) \prod_p (\psi^{(p)}, \hat{D}_p \psi^{(p)}) = \langle D_\infty \rangle \prod_p \langle D_p \rangle. \quad (4.11)$$

When  $\hat{D}$  is not a unitary operator, one has to take care of the convergence of the infinite product in (4.11). Let us note that one can consider operators  $\hat{D}$  composed only of a finite number of  $p$ -adic components different from the identify operators, i.e.,

$$\hat{D} = \hat{D}_\infty \prod_p \hat{D}_p = \hat{D}_\infty \prod_{p=2}^{p_n} \hat{D}_p, \quad (4.12)$$

where after some prime  $p_n$  all  $\hat{D}_p = 1$ . For any Schwartz–Bruhat function one can find a large enough  $p_n$  for which the unitary operators  $U(t)$  and  $W(z)$  may be effectively presented in the form (4.12).

Now one can introduce an operator

$$|x|_{(p_n)}^s = |x_\infty|^s \prod_{p=2}^{p_n} |x_p|_p^s, \quad (4.13)$$

where  $s \in \mathbb{C}$  and  $p_n$  is an arbitrary prime. An expectation value which corresponds to the operator (4.13) in the simplest vacuum state

$$\psi_{00}(x) = 2^{\frac{1}{4}} e^{-\pi x_\infty^2} \prod_p \Omega(|x_p|_p) \quad (4.14)$$

is

$$\langle |x|_{(p_n)}^s \rangle = (\psi_{00}, |x|_{(p_n)}^s \psi_{00}) = \sqrt{2} \Gamma\left(\frac{s+1}{2}\right) (2\pi)^{-\frac{s+1}{2}} \prod_{p=2}^{p_n} \frac{1-p^{-1}}{1-p^{-s-1}}, \quad \text{Re } s > -1. \quad (4.15)$$

When  $p_n \rightarrow \infty$ , we have

$$\langle |x|^s \rangle = \lim_{p_n \rightarrow \infty} \langle |x|_{(p_n)}^s \rangle = \sqrt{2} \Gamma\left(\frac{s+1}{2}\right) (2\pi)^{-\frac{s+1}{2}} \frac{\zeta(s+1)}{\zeta(1)} = 0, \quad (4.16)$$

where  $\zeta(s)$  is the Riemann zeta function.

In particular, from (4.16) we get

$$\langle |x| \rangle = 0. \quad (4.17)$$

Of interest is also a knowledge of the mean square deviation  $\Delta D$ , which is a measure of the dispersion around  $\langle D \rangle$ ,

$$\Delta D = [(\langle (D - \langle D \rangle)^2 \rangle)]^{\frac{1}{2}} = (\langle D^2 \rangle - \langle D \rangle^2)^{\frac{1}{2}}. \quad (4.18)$$

Using (4.18) we obtain

$$\Delta |x|_{(p_n)} = \frac{1}{2} \left( \frac{1}{\pi} \prod_{p=2}^{p_n} \frac{1-p^{-1}}{1-p^{-3}} \right)^{\frac{1}{2}} \left[ 1 - \frac{2}{\pi} \prod_{p=2}^{p_n} \frac{(1-p^{-1})(1-p^{-3})}{(1-p^{-2})^2} \right]^{\frac{1}{2}}. \quad (4.19)$$

By virtue of (4.16) one has

$$\Delta |x\rangle = \lim_{p_n \rightarrow \infty} \Delta |x\rangle_{(p_n)} = 0. \quad (4.20)$$

The expectation value of the momentum in the simplest vacuum state can be found in the following way:

$$\langle |k\rangle_{(p_n)}^s \rangle = (\tilde{\psi}_{00}(k), |k\rangle_{(p_n)}^s \tilde{\psi}_{00}(k)), \quad (4.21)$$

where  $\tilde{\psi}_{00}(k)$  is the Fourier transform of  $\psi_{00}(x)$  (4.14). Using (2.9) we get

$$\tilde{\psi}_{00}(k) = 2^{\frac{1}{4}} e^{-\pi k_\infty^2} \prod_p \Omega(|k_p|_p), \quad (4.22)$$

i.e.,  $\tilde{\psi}_{00} = \psi_{00}$ . It is clear that the above obtained results for coordinate  $x$  are also valid for the momentum  $k$ . In particular, one obtains

$$\langle |k\rangle \rangle = \langle |x\rangle \rangle = 0, \quad \Delta |k\rangle = \Delta |x\rangle = 0. \quad (4.23)$$

An uncertainty relation between the adelic position and momentum coordinates reads

$$\Delta x_{(p_n)} \Delta k_{(p_n)} = \frac{1}{4\pi} \prod_{p=2}^{p_n} \frac{1-p^{-1}}{1-p^{-3}} \left[ 1 - \frac{2}{\pi} \prod_{p=2}^{p_n} \frac{(1-p^{-1})(1-p^{-3})}{(1-p^{-2})^2} \right], \quad (4.24)$$

where the factor  $1/(4\pi)$  corresponds to the ordinary case.

One gets also interesting features applying the Mellin transformation (2.10) to the vacuum state  $\psi_{00}(x)$  (4.14), which can be considered as the simplest elementary function defined on adèles. It gives

$$\Phi(s) = \sqrt{2} \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s). \quad (4.25)$$

Since the Fourier transform  $\tilde{\psi}_{00} = \psi_{00}$ , one obtains  $\tilde{\Phi}(s) = \Phi(s)$ . Replacing  $\Phi$  and  $\tilde{\Phi}$  in the Tate formula (2.11) by (4.25), we have the well-known functional relation for the Riemann zeta-function:

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{\frac{s-1}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s). \quad (4.26)$$

## 5. CONCLUDING REMARKS

The adelic harmonic oscillator exhibits some remarkable mathematical properties. It is a simple, exact, and instructive adelic model. The simplest vacuum state is also the simplest elementary function, and its form is invariant under the Fourier transformation. Consequently, the Mellin transform of this vacuum state in the  $x$ - and  $k$ -representation is the same function which satisfies the Tate formula.

Some physical aspects of the adelic harmonic oscillator are very interesting. According to (3.4) the dimensionless position coordinate  $\xi$  may be presented in the Planck scale in the form

$$\xi = \sqrt{2\pi} \frac{x}{l_0},$$

where  $l_0 = \left(\frac{m_0 \omega_0}{\hbar}\right)^{\frac{1}{2}}$  is the Planck length,  $m_0 = \left(\frac{\hbar c}{2\pi G}\right)^{\frac{1}{2}}$ , and  $\omega_0 = \frac{2\pi}{t_0} = 2\pi c^2 \left(\frac{2\pi c}{\hbar G}\right)^{\frac{1}{2}}$ . In fact, our calculations are performed for  $l_0 = 1$ , and it seems most natural to take  $l_0$  as the Planck length. Thus, the results obtained for the adelic harmonic oscillator may be relevant to Planck scale physics. According to (4.23) one can measure distances which are smaller than the length  $l_0$ . Formula (4.24) contains a softening of the uncertainty relation. This is a consequence of the  $p$ -adic effects.



On the basis of the above considerations, one can suppose that at distances close to  $l_0$  there exist not only standard virtual particles but also  $p$ -adic ones. The adelic particles can interact by means of any of these virtual objects. In the above case of the adelic harmonic oscillator, just virtual particles of the  $p$ -adic vacuums lead to the unusual results. So  $p$ -adic effects appear through an interaction of some real particles with a  $p$ -adic virtual matter.

Standard quantum mechanics can be considered as an approximation of the adelic one when experimentally available distances are very large with respect to  $l_0$  (the Planck length). Namely, at very large distances ( $|x_\infty|_\infty \gg 1, |x_p|_p \gg 1$ ), for some reasons  $p$ -adic states are not occupied and adelic operators  $|x|_{(p_n)}, |k|_{(p_n)}$  have to be taken with  $p_n = 0$ , i.e.,  $|x|_{(0)} = |x_\infty|_\infty$  and  $|k|_{(0)} = |k_\infty|_\infty$ . For these operators ( $|x_\infty|_\infty$  and  $|k_\infty|_\infty$ ) calculations in standard and adelic quantum mechanics give the same results.

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