STRUCTURE OF TWO-DIMENSIONAL SOLITONS IN THE CONTEXT OF A GENERALIZED KADOMTSEV—PETVIASHVILI EQUATION

L. A. Abramyan and Yu. A. Stepanyants

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The structure of steady-state two-dimensional solutions of the soliton type with quadratic and cubic nonlinearities and power-law dispersion is analyzed numerically. It is shown that steadily coupled two-dimensional multisolitons can exist for positive dispersion in a broad class of equations, which generalize the Kadomtsev—Petviashvili equation.

1. Kadomtsev and Petviashvili [1] have derived an equation that generalizes the well-known Korteweg—de Vries equation and describes quasiplanar disturbances in a quadratically nonlinear medium with weak dispersion. The basic approximation used in [1] is the assumption that the scale of the wave field in the direction of motion is much smaller than the scale in the transverse direction. Clearly, the same approximation can also be used to describe disturbances in other media having different types of nonlinearity and dispersion (see, e.g., [2, 3]).

A well-developed mathematical machinery is now available for obtaining exact solutions of the Kadomtsev—Petviashvili (KP) equation [4]. Of special interest among those solutions are two-dimensional solitons, for which it has been shown, first numerically [5] and later analytically [6], that they can exist only in media having positive dispersion. An intriguing problem is the existence and structure of two-dimensional solitons described by equations that are related to the KP equation, but for which analytical solutions have not yet been obtained.

In the present article we give soliton solutions obtained by numerical calculations for a generalized KP equation of the form

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+\frac{\alpha}{p}\frac{\partial u^p}{\partial x}+\beta\hat{H}_q[u]\right)=\gamma\frac{\partial^2 u}{\partial y^2}. \tag{1}$$

Here α, β, and γ are constant coefficients, $p > 0$ is an arbitrary number, $\hat{H}_q[u]$ is a linear operator, which in spectral form corresponds to the dispersion law $\omega = \beta k^q$; ω is the (angular) frequency, k is the wave number, and $q > 1$ is an arbitrary number. In particular, for $p = 2$ and $q = 3$ (in which case $H_q[u] \equiv \partial^3 u/\partial x^3$) Eq. (1) goes over to the classical KP equation.

2. We use the stabilized multiplier technique proposed by Petviashvili [5] to obtain soliton solutions of Eq. (1). We shall be concerned below with steady-state solutions of the form $u(x, y, t) = u(\zeta = x - Vt, y)$, and so we rewrite Eq. (1) in the variables ζ and y:

$$\frac{\partial}{\partial\zeta}\left(-V\frac{\partial u}{\partial\zeta}+\frac{\alpha}{p}\frac{\partial u^p}{\partial\zeta}+\beta\hat{H}_q[u]\right)=\gamma\frac{\partial^2 u}{\partial y^2}. \tag{2}$$

The change of variables

$$\xi=\zeta(V/\beta)^{1/(q-1)},\qquad \eta=y(V/\beta)^{1/(q-1)}(V/\gamma)^{1/2},\qquad v=u(V/\alpha)^{1/(1-p)} \tag{3}$$

permits us to reduce the basic equation to the dimensionless form

$$\frac{\partial}{\partial\xi}\left(-\frac{\partial v}{\partial\xi}+\frac{1}{p}\frac{\partial v^p}{\partial\xi}+H_q[v]\right)=\frac{\partial^2 v}{\partial\eta^2}. \tag{4}$$

We take the Fourier transform of Eq. (4) with respect to the variables ξ and η:

$$(k_\xi^2+k_\eta^2+k_\xi^{q+1})\tilde{v}(k_\xi, k_\eta)=(1/p)\,k_\xi^2\,\widetilde{v^p}(k_\xi, k_\eta), \tag{5}$$

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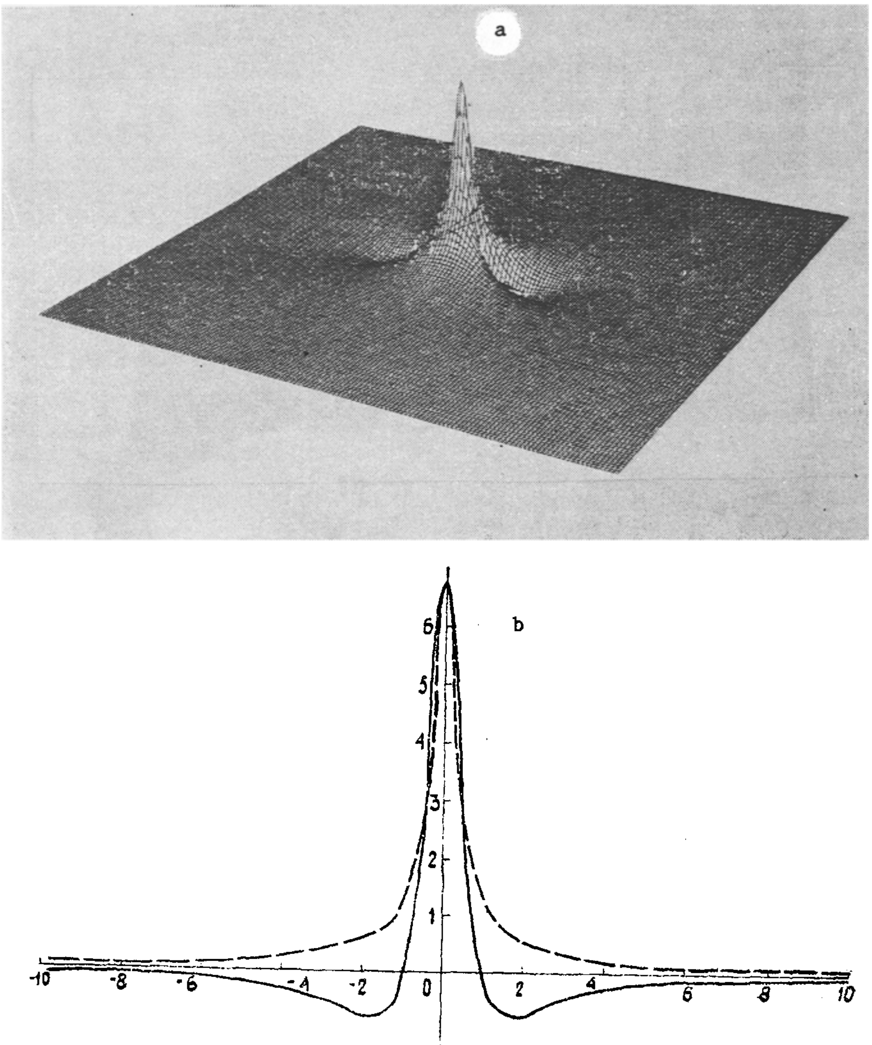

Fig. 1. a) Three-dimensional relief of an MKP two-dimensional soliton; b) cross sections of the two-dimensional soliton along (solid curve) and across (dashed curve) the direction of motion.

where $\tilde{v}(k_\xi, k_\eta)$ and $\tilde{v}^p(k_\xi, k_\eta)$ are the Fourier transforms of the functions $v(\xi, \eta)$ and $v^p(\xi, \eta)$, and k_ξ and k_η are the parameters of the Fourier transform with respect to the corresponding variables. Following [5], we multiply Eq. (5) by $\tilde{v}(k_\xi, k_\eta)$ and integrate the result with respect to k_ξ and k_η between infinite limits:

$$\iint_{-\infty}^{\infty} (k_\xi^2 + k_\eta^2 + k_\xi^{q+1}) (\tilde{v})^2 dk_\xi\, dk_\eta = (1/p) \iint_{-\infty}^{\infty} k_\xi^2 \tilde{v^p}\, \tilde{v}\, dk_\xi\, dk_\eta. \tag{6}$$

Hence it follows that if $\tilde{v}(k_\xi, k_\eta)$ is a solution of Eq. (5), then

$$M = p \frac{\iint_{-\infty}^{\infty} (k_\xi^2 + k_\eta^2 + k_\xi^{q+1}) (\tilde{v})^2 dk_\xi\, dk_\eta}{\iint_{-\infty}^{\infty} k_\xi^2 \tilde{v^p}\, \tilde{v}\, dk_\xi\, dk_\eta} \tag{7}$$

must be equal to unity. Accordingly, we form an iterative scheme for the numerical solution of Eq. (5) [or Eq. (4)] on the basis of Eq. (5):

$$\tilde{v}_n(k_\xi, k_\eta) = \frac{(1/p)\, k_\xi^2\, \widetilde{v_{n-1}^p}(k_\xi, k_\eta)}{k_\xi^2 + k_\eta^2 + k_\xi^{q+1}} M^r. \tag{8}$$

Here the index n indicates the iteration number, the multiplier M^r is introduced to suppress the computational instability that can occur without it, and the power exponent is chosen

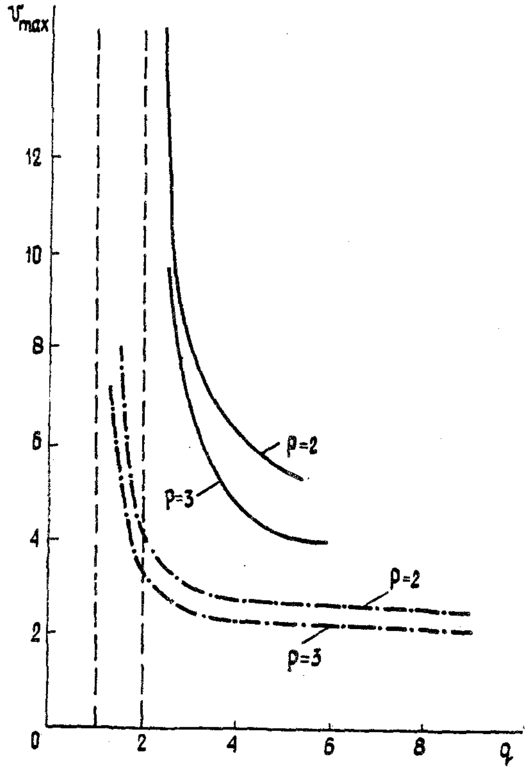

Fig. 2. Maximum value of the field in a two-dimensional soliton (solid curve) vs dispersion parameter for media with quadratic (p = 2) and cubic (p = 3) nonlinearities. The analogous graphs for one-dimensional solitons are represented by the dot-dash curves.

empirically so as to maximize the rate of convergence [this condition is usually satisfied if the degree of homogeneity of the right-hand side of Eq. (8) with respect to the unknown function $\tilde{v}$ is equal to zero]. The uncertainty associated with the condition $k_\xi = k_\eta = 0$ is easily removed by augmenting the definition of the function $\tilde{v}_n(0, 0)$ with an arbitrary number that does not affect the form of the solution. The convergence of the iterative process is tested by means of the parameter M. The process is said to be completed when the value of M differs from unity at most by 10^{-4}. In practice, this value is attained by the 30th iteration.

3. We have used the iterative scheme (8) to calculate the structure of two-dimensional solitons for various values of p and q. The starting function $v_0(\xi, \eta)$ is specified as an arbitrary smooth bell-shaped function. The calculations show that all two-dimensional solitons in media having quadratic (p = 2) and cubic (p = 3) nonlinearity with q > 2 are qualitatively similar in structure and closely resemble the structure of a KP two-dimensional soliton. As an example, Fig. 1 shows a two-dimensional soliton representing a solution of the modified KP (MKP) equation with cubic nonlinearity. The quantitative differences in the structure of the two-dimensional solitons for different values of p and q are attributable to the dependence of the maximum value v_{max} of the field at the vertex of the soliton and its characteristic space scale on the parameters p and q. The solid curves in Fig. 2 represent the dependence of v_{max} on q for two values of p = 2, 3. It is evident from the figure that the soliton amplitudes grow without limit in both cases as $q \to 2$. We have not been able to obtain a steady-state solution for the two-dimensional Benhamin–Ono equation [2] (p = 2, q = 2) or its modified analog (p = 3, q = 2). Apparently two-dimensional solitons do not exist in these cases. To a certain extent, the given situation is similar to what happens in dissipative media; according to the results of Pfirsch and Sudan [7], steady planar shock waves can exist only in cases when the dispersion law has the form $\omega \sim ik^{1+\varepsilon}$, where i is an imaginary unity and ε is any indefinitely small positive number.

Our calculations for the one-dimensional case [see Eq. (1), in which $\partial/\partial y = 0$] show that steady-state solitary waves are also possible here for power-law dispersion $\omega \sim k^{1+\varepsilon}$,

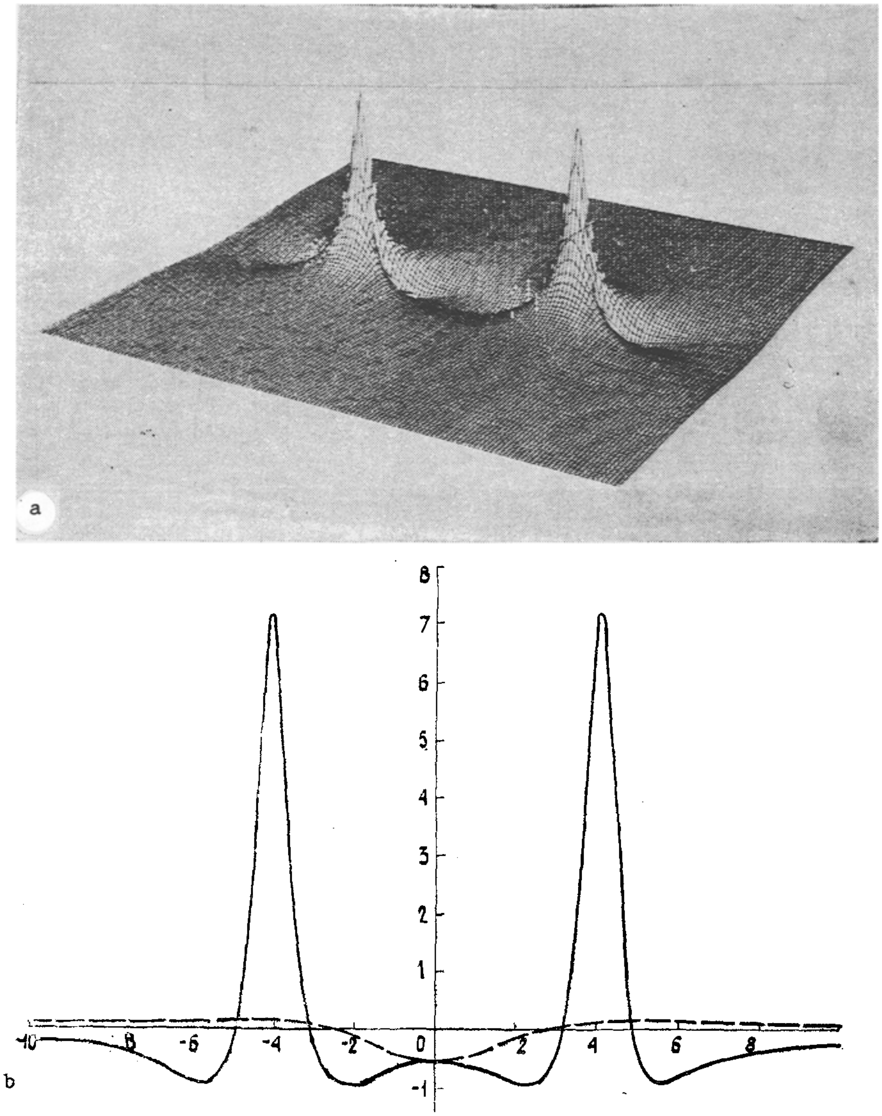

Fig. 3. a) Three-dimensional relief of an MKP two-dimensional bisoliton; b) cross sections of the bisoliton along (solid curve) and across (dashed curve) the direction of motion.

provided that $\varepsilon > 0$. The corresponding graphs of v_{max} as a function of q for p = 2, 3 are represented by the dot-dash curves in Fig. 2 [note that Eq. (1) is invariant under the substitutions $\beta \to -\beta$, $t \to -t$, $u \to -u$ for even-valued p in the one-dimensional case, so that solitons exist under such conditions both in media having positive dispersion and in media having negative dispersion]. The results therefore indicate that solitons exist in the one-dimensional case if the power exponent of the dispersion law $q > 1$, and in the two-dimensional case if $q > 2$. It seems natural to infer that solitons should exist in the three-dimensional case (see [11]) only for $q > 3$ (bearing in mind "true" three-dimensional solitons, since it is known that planar and axisymmetric solitons can indeed exist in three-dimensional space, but they are unstable under small perturbations [10, 11]).

We call attention to the nonmonotonic way in which the two-dimensional soliton field decays in the direction of motion (Fig. 1b). The existence of a local minimum in the structure of the soliton suggests the possible existence of coupled soliton pairs. Such pairs are sought by means of the same iterative scheme (8), but with a double-humped function of more or less arbitrary form specified as the starting function. If the distance between the humps is not too great, the iterative process converges to a solution of the previously known type shown in Fig. 1. Beginning with a certain distance, however, the iterative process converges to a double-humped bisoliton. As an example, Fig. 3 shows a bisoliton solution of the MKP equation. The calculations show that a whole family of bisolitons actually exists for specified values of p and q, and the height of their peaks depends on the distance between them. In this respect, the given solutions are perfectly analogous to those described [8] for the KP equation proper. The existence of local minima in the structure of the bisolitons suggests the possible existence of even more complicated steady-state formations, i.e., multisolitons, but their calculation is more formidable both in terms of computer resources and in terms of arriving at an accurate guess of the starting function. In addition to steadily coupled states, the given class of equations clearly also has transient (time-dependent) solutions in the form of oscillating coupled solitons (breathers), but the stability problem remains open for both steady-state and transient multisolitons.

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THEORY OF RELATIVISTIC CRM WITH SYNCHRONOUS ADIABATIC ELECTROMAGNETIC WAVE DECELERATION OF ELECTRON BEAM

N. S. Ginzburg

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An approximation of nonlinear theory of relativistic gyrotrons with variable magnetic fields is formulated. It is assumed that, for a single electron being decelerated by a high-frequency field, the condition of cyclotron resonance is satisfied identically over the entire interaction space. Other electrons captured by the wave, which undergo small oscillations, are decelerated with the resonant electron. Using the method of adiabatic invariants, a longitudinal amplitude distribution is determined for the high-frequency field that prevents escape of any electrons.

1. One useful method for increasing the efficiency of microwave devices is synchronous adiabatic electromagnetic wave deceleration of an electron beam. Under these conditions, a large fraction of electrons are captured by the electromagnetic wave as soon as they enter the interaction space, and the parameters of the electrodynamic or electrooptical system then vary smoothly so that the electrons are decelerated by the wave longer than they are accelerated, while after a few oscillation periods in the potential trap the electrons transfer much of their energy to the wave.

We will consider adiabatic deceleration in more detail for devices characterized by

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