MANY-PARTICLE PROBLEM WITH LOGARITHMIC POTENTIALS AND ITS APPLICATION TO QUARK BOUND STATES

A.A. Khelashvili, V.Yu. Khmaladze,
and N.D. Chachava

The nonrelativistic many-body problem with logarithmic two-body potentials is solved in the hyperspherical formalism. In the diagonal approximation, an analytic expression is obtained for the eigenvalues of the hyperradial equation, and a mass formula is constructed. Meson–baryon mass relations are derived.

1. Introduction

In this paper, we investigate the spectrum of bound states in a relativistic system of many particles interacting through logarithmic two-body potentials. It is well known that the experimentally observed heavy quark–antiquark bound states can be fairly well described by a Schrödinger equation with potentials that increase at infinity [1-5]. The logarithmic potential introduced for the first time in [6] to explain the approximate equidistance of the spectra of the radial excitations of charmonium and bottomium is, though it does not have a fundamental theoretical derivation, a good approximation for a class of potentials that are singular at the origin and increase at infinity [7]. The characteristic scaling properties of the eigenvalues and eigenfunctions of the two-particle problem make it possible to obtain a number of exact rules for the ratios of the mass intervals and widths of leptonic decays of quarkonium, despite the fact that a Schrödinger equation with logarithmic potential cannot be solved exactly [8].

Recently, baryons have been successfully studied in a harmonic-oscillator model [9], though it is widely accepted that confinement potentials do not increase quadratically at large distances. Despite the reasonable agreement with the experimental data, the connection between the properties of the baryons and the underlying quark dynamics is here completely obscure. Therefore, the investigation of systems of three or more particles with the confinement potentials established in the meson spectrum is of undoubted interest. We should mention the promising results of [10,11], which used a nonsingular power-law potential with a small exponent [3]: $V=A+Br^{0.1}$. These studies also warrant attention from the technical point of view, since they demonstrate the fairly good convergence of hyperspherical expansions [12-15] for potentials that increase at infinity (see also [16]).

The study of power-law potentials in a hyperspherical basis is comparatively advantageous, since the hyperspherical images of such potentials are themselves similar power-law functions of the hyperradial length. For this reason, the many-particle problem can be related in reasonable approximations to the two-particle problem. The logarithmic potential belongs to the class of power-law potentials with zero exponent and, as will be shown below, leads to perspicuous relations without knowledge of certain technical details.

The paper is arranged as follows: First, for reference and completeness of the exposition we give the basic equations of the hyperspherical formalism; we then consider the logarithmic potential and in a reasonable approximation solve the hyperradial equations; we construct a mass formula for a system consisting of an arbitrary number of particles and, finally, derive meson–baryon mass relations on the basis of the obtained formula.

2. Hyperspherical Formalism in the Case of Unequal Masses

In this section, we shall follow [15] closely.* Suppose there is a system of A particles of masses m_i with radius vectors $\mathbf{r}_i$ (i = 1, 2, ..., A). By $\mathbf{R}_j$ we denote the radius vector of the center of mass of a

* Hyperspherical basis formulas for unequal masses were given earlier in [17].

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complex of j particles:

$$\mathbf{R}_j=\frac{1}{M_j}\sum_{k=1}^{j}m_k\mathbf{r}_k,\quad M_j=\sum_{k=1}^{j}m_k.$$

We introduce the radius vector of the center of mass of the complete system,

$$\mathbf{R}=\frac{1}{M}\sum_{k=1}^{A}m_k\mathbf{r}_k,\quad M=\sum_{k=1}^{A}m_k,$$

and also the remaining N = A − 1 relative radius vectors:

$$\xi_j=\sqrt{m_{j+1}M_j/MM_{j+1}}(\mathbf{r}_{j+1}-\mathbf{R}_j),\quad j=1,2,\ldots,N=A-1. \tag{2.1}$$

In the new coordinates, the kinetic energy operator of the system can be written in the form

$$\sum_{i=1}^{A}\frac{1}{2m_i}\nabla_{r_i}^2=\frac{1}{2M}\nabla_R^2+\frac{1}{2M}\sum_{j=1}^{N}\nabla_{\xi_j}^2.$$

From the definitions (2.1) we obtain the helpful relations

$$\mathbf{R}_j-\mathbf{R}_{j-1}=\sqrt{m_jM/M_jM_{j-1}}\xi_{j-1},\quad 2\leqslant j\leqslant N,\quad \mathbf{r}_{j+1}-\mathbf{r}_j=\sqrt{MM_{j+1}/M_jm_{j+1}}\xi_j-\sqrt{MM_{j-1}/m_jM_j}\xi_{j-1},\quad 2\leqslant j\leqslant N.$$

The last formula can also be generalized for j = 1 if it is assumed that $\xi_0 = 0$ and $M_0 = 0$. Applying it repeatedly, we arrive at the following general formula (i > j):

$$\mathbf{r}_i-\mathbf{r}_j=\sqrt{\frac{MM_i}{M_{i-1}m_i}}\xi_{i-1}+\sqrt{\frac{Mm_{i-1}}{M_{i-1}M_{i-2}}}\xi_{i-2}+\sqrt{\frac{Mm_{i-2}}{M_{i-2}M_{i-3}}}\xi_{i-3}+\ldots$$
$$+\sqrt{\frac{Mm_{i-(i-j-1)}}{M_{i-(i-j-1)}M_{i-(i-j)}}}\xi_{i-(i-j)}-\sqrt{\frac{MM_{i-(i-j+1)}}{M_{i-(i-j)}m_{i-(i-j)}}}\xi_{i-(i-j+1)}. \tag{2.2}$$

This means that the ξ vectors on the right-hand side of the equation are encountered with numbers from j − 1 to i − 1. For j = 1, the final (negative) term is absent.

A set of 3N − 1 angles (Ω) is introduced as usual [15]: 2N polar angles $\hat{\xi}_j$ for each vector ξ_j; N − 1 hyperspherical angles θ_i, which are determined by means of the lengths of the ξ_j: $\xi_j=\xi\sin\theta_N\ldots\sin\theta_{j+1}\cos\theta_j$ ($\theta_1=0$, $0<\theta_j<\pi/2$). Here, the hyperradial coordinate ξ is the length in the 3N-dimensional space: $\xi^2=\sum_{j=1}^{N}\xi_j^2$. In the hyperspherical coordinates introduced above, the kinetic energy operator of the relative motion takes the form [15]

$$T_\xi=-\frac{1}{2M}\sum_{j=1}^{N}\nabla_{\xi}^2=-\frac{1}{2M}\left\{\frac{\partial^2}{\partial\xi^2}+\frac{3N-1}{\xi}\frac{\partial}{\partial\xi}+\frac{L^2(\Omega)}{\xi^2}\right\}, \tag{2.3}$$

where $L^2(\Omega)$ is the operator of the angular momentum in 3N-dimensional space. A general prescription for constructing the eigenfunctions of this operator was given in [14]. The eigenfunctions of the operator $L^2(\Omega)$ are hyperspherical harmonics:

$$\mathcal{Y}_{[L]}(\Omega)=\mathcal{Y}_{l_1}^{m_1}(\hat{\xi}_1)\prod_{j=2}^{N}\mathcal{Y}_{l_j}^{m_j}(\hat{\xi}_j)\,{}^{(j)}P_{L_j}^{l_j,L_{j-1}}(\theta_j), \tag{2.4}$$

which satisfy the equation

$$\{L^2(\Omega)+L(L+3N-2)\}\mathcal{Y}_{[L]}(\Omega)=0. \tag{2.5}$$

In the expression (2.4), the functions ${}^{(j)}P_{L_j}^{l_j,L_{j-1}}(\theta_j)$ are related to Jacobi polynomials, $L_j=\sum_{i=1}^{j}(2n_i+l_i)$, $n_1=0$, and $\mathcal{Y}_{l_j}^{m_j}(\hat{\xi}_j)$ are ordinary spherical harmonics. For the indicated choice of the angular variables (Ω), we denote by [L] the set of the following 3N − 1 quantum numbers: the 2N orbital and magnetic quantum numbers l_j and m_j for each ξ_j ($j=1,\ldots,N$) and the N − 1 hyperspherical quantum numbers n_j (j = 2, ..., N) associated with the hyperspherical angles θ_j.

The number L is the principal orbital angular momentum, related to the numbers l_i and n_i by $L=\sum_{i=1}^{N}(2n_i+l_i)$, $n_1\equiv 0$.

After elimination of the center of mass and substitution of the expansion

$$\Psi(\mathbf{r}_{ij})\equiv\Psi(\xi,\Omega)=\sum_{[L]}\xi^{-(3N-1)/2}\mathcal{U}_{[L]}(\xi)\mathcal{Y}_{[L]}(\Omega)$$

using (2.3) and (2.5), we arrive at the system of hyperradial equations

$$-\frac{1}{2M}\left\{\frac{d^2}{d\xi^2}-\frac{\lambda(\lambda+1)}{\xi^2}\right\}\mathcal{U}_{[L]}(\xi)+\sum_{[L']}\langle\mathcal{Y}_{[L]}(\Omega)\,|\,V(\xi,\Omega)\,|\,\mathcal{Y}_{[L']}(\Omega)\rangle\mathcal{U}_{[L']}(\xi)=E\mathcal{U}_{[L]}(\xi),$$

where $V(\xi,\Omega)=\sum_{i>j}V_{ij}(\mathbf{r}_{ij})$ and $\lambda=L+{}^3/_2(N-1)$.

The main characteristic features in the case of unequal masses appear in the calculation of a matrix element of the potential. The two-body potentials depend on the relative radius vectors $\mathbf{r}_{ij}=\mathbf{r}_i-\mathbf{r}_j$, which in accordance with (2.2) can be expressed in terms of a linear combination of the vectors ξ_k. The coefficients of this combination depend in a complicated manner on the masses of the particles. In the hyperspherical approach, each of the two-body potentials can be expanded in a hyperspherical series. To this end, we introduce the kinematic rotation vector [15]

$$\mathcal{X}(\varphi)=\sum_{k=1}^{N}\sin\varphi_N\ldots\sin\varphi_{k+1}\cos\varphi_k\cdot\xi_k,\qquad\varphi_1\equiv 0.$$

By means of it, we can express any linear combination of the vectors $\mathbf{A}(\xi)=\sum_{k=1}^{N}\alpha_k\xi_k=c\mathcal{X}(\varphi)$, where $c^2=\sum_{k=1}^{N}\alpha_k^2$, and the angular parameters are determined in accordance with $\cos^2\varphi_j=\alpha_j^2\left(\sum_{k=1}^{N}\alpha_k^2\right)^{-1}$. Thus, for the relative radius vectors we have

$$\mathbf{r}_i-\mathbf{r}_j=a_{ij}\mathcal{X}(\varphi^{(i,j)})=a_{ij}\sum_{k=1}^{N}\sin\varphi_N^{(i,j)}\ldots\sin\varphi_{k+1}^{(i,j)}\cos\varphi_k^{(i,j)}\cdot\xi_k.\tag{2.6}$$

If we recall (2.2), it becomes obvious that the summation here extends in practice from j − 1 to i − 1. The angles are chosen in such a way as to make the remaining terms in this sum vanish. It is clear from Eq. (2.2) and our definitions that $a_{ij}=\sqrt{M(m_i+m_j)/m_im_j}$.

It is also obvious that the angles $\varphi^{(i,j)}$ in (2.6) depend on the masses of the particles. It is only in the case of equal masses that they take definite numerical values independent of the mass.

We can now make the necessary expansions of a plane wave and the potentials, as is done in [15]. The only difference arises because of the presence of the factor a_{ij} in the expansion (2.6). We therefore give the final result for central potentials:

$$V_{ij}(r_{ij})=2\pi^{3(N-1)/2}\sum_{K=0}^{\infty}(-1)^K\frac{\Gamma(K+{}^3/_2)}{\Gamma({}^3/_2)\Gamma(K+3N/2-{}^3/_2)}\sum_{[2K]}\mathcal{Y}_{[2K]}(\Omega)\int d\hat{q}\,\mathcal{Y}^*_{[2K]}(\varphi^{(i,j)}\hat{q})\times$$
$$\int_0^1 du\,u^2(1-u^2)^{(3N-5)/2}V_{ij}(a_{ij}\xi u)\,{}_2F_1(-K,K+3N/2-1,{}^3/_2;u^2),\tag{2.7}$$

where

$$\mathcal{Y}^*_{[L]}(\varphi,\hat{q})=\mathcal{Y}_{l_1}^{m_1*}(\hat{q})\prod_{j=2}^{N}\mathcal{Y}_{l_j}^{m_j*}(\hat{q})\,{}^{(j)}P_{L_j}^{l_jL_{j-1}}(\varphi)$$

and $\sum_{[2K]}$ denotes summation over all quantum numbers for which $\sum_{i=1}^{N}(2n_i+l_i)=2K$ $(n_1=0)$.

3. Logarithmic Potential

We consider the logarithmic potentials [6]

$$V_{ij}(r_{ij})=g_{ij}\ln\frac{r_{ij}}{r_0(ij)}.$$

In the expression (2.7), the integral contains the quantities $V_{ij}(a_{ij}\xi u)=g_{ij}\ln\frac{a_{ij}\xi}{r_0(ij)}+g_{ij}\ln u$. The integral of the first term is proportional to δ_{K0}, and therefore the matrix element of the potential has the structure

$$\langle\mathcal{Y}_{[L]}(\Omega)\,|\,V(\xi,\Omega)\,|\,\mathcal{Y}_{[L']}(\Omega)\rangle=\delta_{[L][L']}\sum_{i>j}g_{ij}\ln\frac{a_{ij}\xi}{r_0(ij)}+C_{[L][L']},$$

where

$$C_{[L][L']}=\langle\mathcal{Y}_{[L]}(\Omega)\,|\sum_{i>j}C_{ij}(\Omega)\,|\,\mathcal{Y}_{[L']}(\Omega)\rangle$$

and, in turn,

$$C_{ij}(\Omega)=2\pi^{3(N-1)/2}g_{ij}\sum_{K=0}^{\infty}(-1)^K\frac{\Gamma(K+^3/_2)}{\Gamma(^3/_2)\Gamma(K+3N/2-^3/_2)}\sum_{[2K]}\mathcal{Y}_{[2K]}(\Omega)\int d\hat{q}\,\mathcal{Y}^*_{[2K]}(\varphi^{(i,j)},\hat{q})\times$$

$$\int_0^1 du\,u^2(1-u^2)^{(3N-5)/2}\ln u\cdot{}_2F_1\left(-K,K+\frac{3N}{2}-1,\frac{3}{2};u^2\right). \tag{3.1}$$

Then the hyperradial equation takes the form

$$-\frac{1}{2M}\left\{\frac{d^2}{d\xi^2}-\frac{\lambda(\lambda+1)}{\xi^2}\right\}\mathcal{U}_{[L]}(\xi)-\left(E-\sum_{i>j}g_{ij}\ln\frac{a_{ij}}{r_0(ij)}\right)\mathcal{U}_{[L]}(\xi)+$$

$$\left(\sum_{i>j}g_{ij}\right)\ln\xi\,\mathcal{U}_{[L]}(\xi)=-\sum_{[L']}C_{[L][L']}\mathcal{U}_{[L']}(\xi). \tag{3.2}$$

It can be seen that except for the nondiagonal terms, which make the system of equations infinite dimensional, the functional form of (3.2) is identical to the two-particle radial equation. It is interesting that the coefficients of $\ln\xi$ are diagonal, and the entire nondiagonality is contained in the constant (ξ-independent) coefficients. In this property, the logarithmic potential is distinguished from all power-law potentials.

In the present paper, as an initial approximation, we limit ourselves to diagonal terms [16]. The validity of such an approximation depends on the convergence of the hyperspherical expansions, which in each concrete case must be investigated numerically. Here, we wish to draw attention to the unexpected appearance of a possibility of analytic solution. Indeed, introducing the dimensionless variable $\rho=\xi\sqrt{2M\left(\sum_{i>j}g_{ij}\right)}$, we can reduce the diagonal part of the system (3.2) to the simple equation

$$\left\{\frac{d^2}{d\rho^2}-\frac{\lambda(\lambda+1)}{\rho^2}+\varepsilon-\ln\rho\right\}\mathcal{U}_{[L]}(\rho)=0, \tag{3.3}$$

where $\varepsilon=\varepsilon_n^\lambda$, the dimensionless eigenvalues of this equation, are related to the energy E in a simple manner, and as a result we obtain for the mass of the system the formula

$$M(1,\ldots,A)=M_n{}^L=\sum_{i=1}^{A}m_i+C_{[L][L]}-\frac{1}{2}\sum_{i>j}g_{ij}\ln\frac{m_im_j}{m_i+m_j}-\frac{1}{2}\sum_{i>j}g_{ij}\ln\left[2r_0{}^2(ij)\left(\sum_{k>l}g_{kl}\right)\right]+\varepsilon_n{}^\lambda\left(\sum_{i>j}g_{ij}\right).$$

We discuss some general features of this mass formula. For fixed number of particles, the eigenvalues ε_n^λ of Eq. (3.3) depend only on the principal orbital angular momentum L and the radial quantum number n (n = 1, 2, ...). Therefore, the contribution of this number to the orbital and radial excitations does not depend on the masses of the components, as in the two-particle problem. There is however a difference from this last (as was shown above), namely, the angles $\varphi^{(i,j)}$ and, therefore, the coefficients $C_{[L][L]}$ (see (3.1)) depend on the masses in the case of unequal masses. As a result, the orbital (but not the radial) excitations will also depend on the masses in this case. In the case of equal masses, this dependence disappears, and the situation becomes completely similar to the quarkonium problem.

4. Application: Three-Quark Bound States and Meson–Baryon Mass Relations

As an application, we consider colorless three-quark bound systems in the ground (L = 0) state (baryons). In order not to lose the connection with mesons, we shall assume, as follows from the one-gluon exchange model [7], that the interaction potential between two quarks is half the potential between a quark and an antiquark: $V_{q_j q_j} = {}^1/_2 V_{q_i \bar{q}_j}$, i.e., $g_{ij} = \frac{1}{2} G_{ij}$, where G_{ij} is the quark–antiquark coupling constant. With allowance for this, we write the mass formula for the baryons in the form

$$M(1,2,3) = M_n^{L=0} = m_1 + m_2 + m_3 + \frac{1}{2}\left(\frac{1}{4} - \ln 2 + \varepsilon_n^{3/2}\right)\left(\sum_{i>j} G_{ij}\right) - \frac{1}{4}\sum_{i>j} G_{ij} \ln\left[\frac{m_i m_j}{m_i + m_j} r_0^2(ij)\left(\sum_{k>l} G_{kl}\right)\right], \quad (4.1)$$

where we have used the fact that $C_{[0][0]} = {}^1/_2({}^1/_4 - \ln 2)\left(\sum_{i>j} G_{ij}\right)$, i.e., for the states with L = 0 this coefficient does not depend on the masses.

We now compare this with the corresponding mass formula for mesons [18]:

$$M(1, \bar{2}) = M_n^{l=0} = m_1 + m_2 - \frac{1}{2} G_{12} \ln\left(\frac{2m_1 m_2}{m_1 + m_2} r_0^2(1,2) G_{12}\right) + \varepsilon_n{}^0 G_{12}. \quad (4.2)$$

In various potential models [3], including the logarithmic potential [18], it has been established that the vector particles in the systems of b, c, and s quarks can be described fairly well by parameters independent of the quark species. Being guided by this fact, we make below some assumptions about G_{ij} and obtain some mass relations. For example, if we assume that $G_{cc} = G_{ss}$ then it follows from (4.1) and (4.2) that

$$M(c\bar{c}) - M(s\bar{s}) = 2(M(ccs) - M(css)) = {}^2/_3[M(ccc) - M(sss)]. \quad (4.3)$$

If to this we add $G_{cc} = G_{cs}$, we then obtain the new relations

$$M(c\bar{s}) - M(s\bar{s}) = M(css) - M(sss), \quad M(c\bar{c}) - M(c\bar{s}) = M(ccc) - M(ccs). \quad (4.4)$$

To include the b-quark systems, it is necessary to assume $G_{bb} = G_{cc} = G_{bc}$. Then the corresponding expressions have the form (4.3) and (4.4), in which the substitution c → b or s → b is made.

For light (q ≡ u, d) quarks, the nonrelativistic models do not work well. However, one can consider systems of the type of atoms, in which light quarks are bound to heavy ones. We make the minimal assumption that the light quarks interact with the b, c, and s quarks with the same strength: $G_{qs} = G_{qc} = G_{qb}$. We then obtain relations in which the light quarks participate. For example, $M(c\bar{q}) - M(s\bar{q}) = M(cqq) - M(sqq)$, $M(ccq) - M(ssq) = M(c\bar{q}) - M(s\bar{q}) + {}^1/_2[M(c\bar{c}) - M(s\bar{s})]$, $2[M(csq) - M(ssq)] = M(c\bar{s}) + M(c\bar{q}) - M(s\bar{s}) - M(s\bar{q})$. Replacing here a c or s quark by a b quark, we obtain mass relations containing the latter.

We note that these mass relations are fairly general. They are obtained solely on the basis of the above assumptions about the coupling constants G_{ij}, with no regard paid to their numerical values nor any restrictions imposed on the parameters $r_0(ij)$. Mass relations of this kind can hold in the naive quark model when it is assumed that the hadron mass is the sum of the masses of the constituent quarks [19]. But here they have arisen for a fairly nontrivial potential.

Verification of our relations and also the prediction by means of them of states not yet discovered experimentally will become possible only after allowance has been made for the spins of the quarks. At the present time, there exist different models for including a spin–spin interaction [2, 5, 20]. This is an independent problem unrelated to the problem considered above, and it will therefore be treated separately.

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POLYNOMIAL CONSERVATION LAWS AND EXACT SOLUTIONS ASSOCIATED WITH ISOMETRIC AND HOMOTHETIC SYMMETRIES IN THE NONLINEAR SIGMA MODEL

G. G. Ivanov

In the nonlinear σ model, conserved tensor currents associated with the presence of isometric, homothetic, and affine motions in the space of values of the chiral field V^N are constructed. New classes of exact solutions in the SO(3)- and SO(5)-invariant σ models are obtained using the connection between the groups of isometric and homothetic motions of space–time and the isometric motions in V^N. Some methods for obtaining exact solutions in the four-dimensional σ model with nontrivial topological charge are considered.

1. Introduction

In [1], a study was made of chiral models of general form, in which the scalar multiplet $\varphi^A(x)$ takes its values in some real Riemannian manifold V^N. A connection was established between the isometric motions of the space–time V^n with the isometries in V^N, on the basis of which exact solutions in the self-gravitating σ model were obtained.

It is known [2] that nonlinear σ models considered on the background of flat space do not when $n > 2$ admit solutions with bounded action, i.e., solutions of instanton type. This prompted a number of authors to go over to the study of σ models that interact with the gravitational field [3-5]. As is shown in [3-5], solutions of instanton and meron types exist in such σ models. Nevertheless, the problem of investigating the solution space for four-dimensional σ models on the background of flat Minkowski space for Euclidean space remains open. Below, using the group-invariant approach, we investigate some classes of exact solutions in four-dimensional σ models on the background of flat space and their topological characteristics.

The material is arranged as follows. In Sec. 2, we construct conserved polynomial tensor currents associated with the isometric, homothetic, and affine motions in V^N. We show that the SO(N)-invariant

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