MANY-LOOP CALCULATIONS: THE UNIQUENESS METHOD AND FUNCTIONAL EQUATIONS

D.I. Kazakov

In the framework of the calculation of many-loop Feynman integrals – the uniqueness method – functional equations are obtained for the coefficient functions of the diagrams. Solution of a functional equation leads to calculation of an N-shaped diagram, the last of the 5-loop diagrams of the φ^4 theory. The obtained result makes it possible to extend by an order the tables constructed previously for the calculation of many-loop integrals.

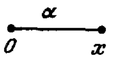
1. Introduction

In our earlier paper [1], we developed and applied the uniqueness method, which is directed to the calculation of many-loop Feynman integrals. It was shown that despite the great possibilities of the method there are limitations associated with the nonfulfillment of the uniqueness conditions simultaneously at all stages of the calculation in the case when the number of loops in the diagram is large (≥ 5).

In the present paper we show that functional equations can be obtained for the coefficient functions of the diagrams in which we are interested. Solving these equations, we can thus calculate integrals that cannot be found any other way. Augmenting the uniqueness method, the proposed method makes it possible to extend the class of exactly calculable diagrams. The use of the functional equations is illustrated by calculation of an N-shaped diagram in the φ^4 theory.

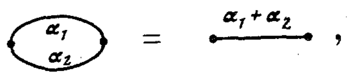
2. Derivation of Functional Equations

We recall first of all the notation and some necessary formulas of the uniqueness method. All calculations are made in a coordinate space of dimension $D = 4 - 2\varepsilon$. Integration is performed with respect to internal vertices. To the lines of the diagrams there correspond simple power factors of the form $1/(x^2)^\alpha$; α is called the line index and is written above the line:

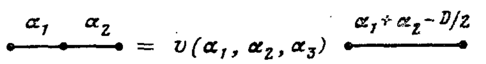


$$\Rightarrow \frac{1}{(x^2)^{\alpha}}.$$

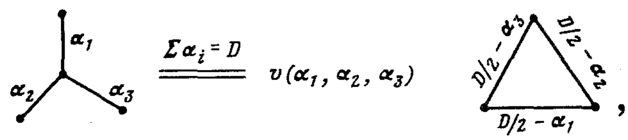
We shall need the following formulas [2,1]:



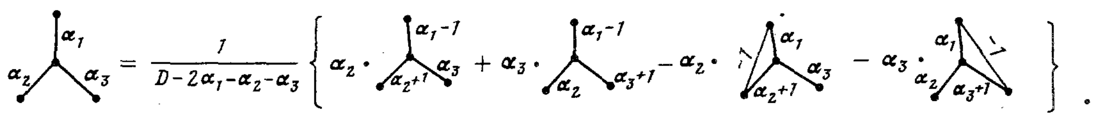
(1)



$$v(\alpha_1,\alpha_2,\alpha_3)=\prod_{i=1}^{3}\frac{\Gamma(D/2-\alpha_i)}{\Gamma(\alpha_i)},\quad \alpha_3=D-\alpha_1-\alpha_2, \tag{2}$$



(3)

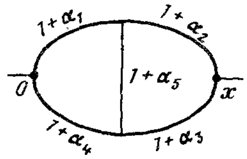


(4)

We now consider the characteristic two-loop diagram

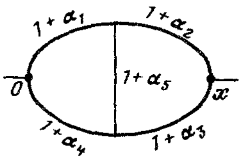
Joint Institute for Nuclear Research, Dubna. Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol.62, No.1, pp.127-135, January, 1985. Original article submitted January 16, 1984.

 0040-5779/85/6201-0084$09.50



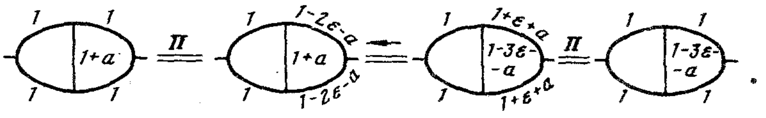
(A)

to which much attention has be paid in the literature [3, 2, 1]. The dependence of the integral on the unique dimensional argument can be separated explicitly. Suppose $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0$, $\alpha_5=a$. Then we have



$$= \frac{1}{(x^2)^{1+2\varepsilon+a}} \cdot F_\varepsilon(1+a),$$

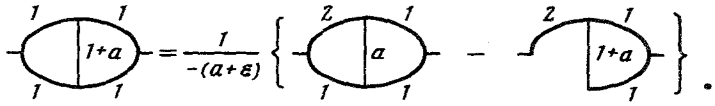
where $F_\varepsilon(1+a)$ is the coefficient function in which we are interested. We perform on the diagram the transformations (see [2])



Thus, we obtain a first equation for $F_\varepsilon(1+a)$:

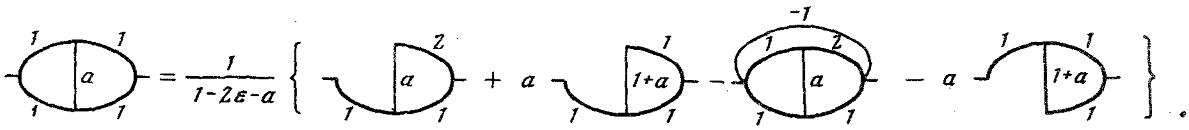
$$F_\varepsilon(1+a)=F_\varepsilon(1-3\varepsilon-a). \tag{5}$$

We now apply to the upper vertex the integration formula (3). We obtain



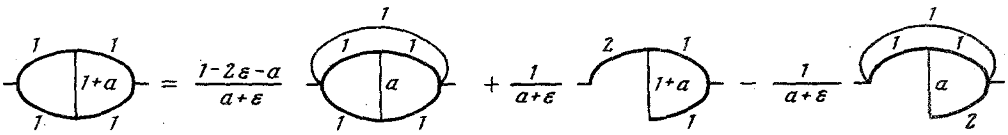
(6)

Applying the same formula but with a different distinguished line, we obtain



(7)

Combining (6) and (7), we obtain the required second equation:



or, analytically,

$$F_\varepsilon(1+a)=\frac{1-2\varepsilon-a}{a+\varepsilon}F_\varepsilon(a)+\frac{2(2a-1+3\varepsilon)\Gamma(-a-\varepsilon)\Gamma(a-1+2\varepsilon)}{(a+\varepsilon)\Gamma(a+1)\Gamma(2-3\varepsilon-a)}\frac{\Gamma^2(1-\varepsilon)}{\Gamma^2(1)}, \tag{8}$$

where we have used Eqs. (1) and (2).

Equations (5) and (8) are the required functional equations for the function $F_\varepsilon(1+a)$, and they must be solved simultaneously.

3. Solution of the Functional Equations

To simplify the inhomogeneous part of (8), we make the substitution

$$F_\varepsilon(1+a)=\frac{2\Gamma^2(1-\varepsilon)\Gamma(-a-\varepsilon)\Gamma(a+2\varepsilon)}{\Gamma^2(1)\Gamma(1+a)\Gamma(1-3\varepsilon-a)}G_\varepsilon(1+a). \tag{9}$$

Then the function G_ε satisfies the system of equations

$$G_\varepsilon(1+a)=G_\varepsilon(1-3\varepsilon-a), \tag{10}$$

$$G_\varepsilon(1+a)=-\frac{a}{a-1+3\varepsilon}G_\varepsilon(a)+\frac{1}{(a-1+3\varepsilon)}\left(\frac{1}{a+\varepsilon}+\frac{1}{a-1+2\varepsilon}\right). \tag{11}$$

To find the solution, we use the analytic properties of the unknown function. On the basis of the α representation it is known, for example, that [4]

$$F_\varepsilon(1+a)=\frac{\Gamma(1+a+2\varepsilon)}{\Gamma(1+a)}\int_0^1 d\alpha_1\dots d\alpha_5\delta\left(1-\sum\alpha_i\right)\alpha_5^a\left(\frac{D}{Q}\right)^{1+2\varepsilon+a}\frac{1}{D^{2-\varepsilon}}, \tag{12}$$

where $F_\varepsilon(1+a)$ is a meromorphic function regular at the point $a=0$ and having simple poles at the points $a=\pm n-2\varepsilon$ and $a=\pm n-\varepsilon$. The form of the inhomogeneous part of Eq. (11) suggests the same thing. Additional poles of the function $G_\varepsilon(1+a)$ arise because of the separation of the Γ functions in the denominator of Eq. (9). We shall therefore seek the solution of Eqs. (10) and (11) in the form of an infinite series of poles:

$$G_\varepsilon(1+a)=\sum_{n=1}^{\infty}f_n\left(\frac{1}{n+a+\varepsilon}+\frac{1}{n-a-2\varepsilon}\right)+\sum_{n=1}^{\infty}\varphi_n\left(\frac{1}{n+a}+\frac{1}{n-a-3\varepsilon}\right). \tag{13}$$

This automatically satisfies Eq. (10).

Substituting (13) in (11) and equating the residues at the poles, we obtain equations for f_n and φ_n:

$$f_n=-f_{n+1}\frac{n+\varepsilon}{n+1-2\varepsilon},\quad \varphi_n=\varphi_{n+1}\frac{n}{n+1-3\varepsilon}.$$

Their solution has the form

$$f_n=(-)^n\frac{\Gamma(n+1-2\varepsilon)}{\Gamma(n+\varepsilon)}c_1(\varepsilon),\quad \varphi_n=(-)^n\frac{\Gamma(n+1-3\varepsilon)}{\Gamma(n)}c_2(\varepsilon),$$

the inhomogeneous term in Eq. (11) determining the value of $c_1(\varepsilon)=\Gamma(\varepsilon)/\Gamma(2-2\varepsilon)$. We note that the first series in (13) is a particular solution of the inhomogeneous equation, whereas the second is a solution of the homogeneous equation. To find the coefficient $c_2(\varepsilon)$, we compare the obtained solution with a known solution for a particular value of a. For this, we consider the function $F_\varepsilon(1+a)$. By virtue of the uniqueness relations, this function is known exactly, i.e., in all orders in ε, for $a=0, -\varepsilon, -2\varepsilon, -3\varepsilon$. Comparing (9), (13), and the value of $F_\varepsilon(1)$, we obtain

$$c_2(\varepsilon)=-\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2-2\varepsilon)\Gamma(1-2\varepsilon)\Gamma(1+2\varepsilon)}.$$

As a result, we have

$$F_\varepsilon(1+a)=2\frac{\Gamma^2(1-\varepsilon)\Gamma(-a-\varepsilon)\Gamma(a+2\varepsilon)\Gamma(\varepsilon)}{\Gamma^2(1)\Gamma(1+a)\Gamma(1-a-3\varepsilon)\Gamma(2-2\varepsilon)}\times$$
$$\left\{\sum_{n=1}^{\infty}(-)^n\frac{\Gamma(n+1-2\varepsilon)}{\Gamma(n+\varepsilon)}\left(\frac{1}{n+a+\varepsilon}+\frac{1}{n-a-2\varepsilon}\right)-\frac{\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)\Gamma(1+2\varepsilon)}\times\right.$$
$$\left.\sum_{n=1}^{\infty}(-)^n\frac{\Gamma(n+1-3\varepsilon)}{\Gamma(n)}\left(\frac{1}{n+a}+\frac{1}{n-a-3\varepsilon}\right)\right\}. \tag{14}$$

To settle finally the question of the uniqueness of the solution (14), we must show that it is not possible to add to it an arbitrary solution of the homogeneous equation. Indeed, such a solution $\Delta(a)$ has the following properties:

a) $\Delta(0) = 0$ by virtue of the normalization on $F_\varepsilon(1)$;

b) $\Delta(\pm n) = 0$, $n = 1, 2, \dots$ by virtue of Eq. (8);

c) $|\Delta(x+iy)|<|\Delta(x)|$, where x lies in an interval between poles. This follows from the boundedness of the integral (12) and the particular solution (14).

d) $\Delta(z)$ does not have singularities, since they are all concentrated in the solution (14). It then follows from Carlson's theorem [5] that $\Delta(z) \equiv 0$. Thus, (14) gives us the necessary solution of Eqs. (5)

and (8).

The last sum in (14) is equal to $-\Gamma(1+a)\Gamma(1-a-3\varepsilon)$, by virtue of which the function $F_\varepsilon(1+a)$ can be represented in the form

$$\Gamma_\varepsilon(1+a)=2\frac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma^2(1)\Gamma(2-2\varepsilon)}\left\{\frac{\Gamma(-a-\varepsilon)\Gamma(a+2\varepsilon)}{\Gamma(1+a)\Gamma(1-a-3\varepsilon)}\sum_{n=1}^{\infty}(-)^n\frac{\Gamma(n+1-2\varepsilon)}{\Gamma(n+\varepsilon)}\left(\frac{1}{n+a+\varepsilon}+\frac{1}{n-a-2\varepsilon}\right)+\frac{\Gamma(-a-\varepsilon)\Gamma(a+2\varepsilon)\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)\Gamma(1+2\varepsilon)}\right\} \tag{15}$$

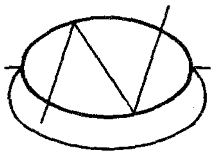
Unfortunately, we cannot obtain a closed expression for the first sum. For $\varepsilon = 0$, we obtain from (14)

$$F_0(1+a)=\frac{2}{a}\sum_{n=1}^{\infty}(-)^n\left[\frac{1}{(n+a)^2}-\frac{1}{(n-a)^2}\right]=-8\sum_{n=1}^{\infty}(-)^n\frac{n}{(n^2-a^2)^2}=\frac{2}{a}\left[\beta'(1+a)-\beta'(1-a)\right], \tag{16}$$

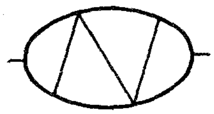
where $\beta(1+x)={}^1/_2[\Psi(1+x/2)-\Psi({}^1/_2+x/2)]$ (see [6]).

4. Calculation of an N-Shaped Diagram in the φ^4 Theory

In the five-loop approximation of the φ^4 theory, one diagram has not yet been calculated analytically:

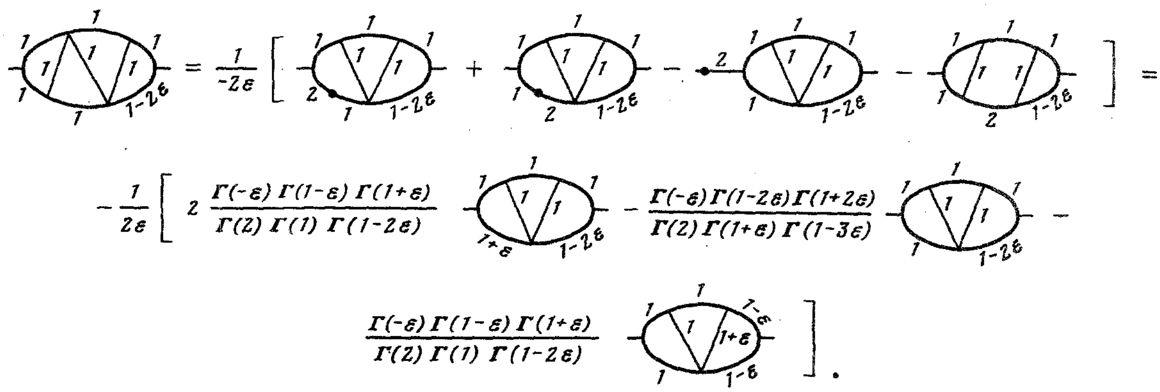


For its calculation, it is necessary to know the N-shaped diagram



to accuracy O(1). It was calculated numerically in [7], and in [1] the result $441/8\zeta$ (7) was guessed. Formula (16) makes possible an exact calculation.

We choose the indices of the lines in the N-shaped diagram in the following manner and apply formula (4) to the lower triple vertex:



Here, we have used (1)-(4). Thus, to calculate the N-shaped diagram to O(1) it is necessary to know the V-shaped diagram to accuracy $O(\varepsilon^2)$ or the two-loop diagram (A, $\alpha_i=\alpha_i\varepsilon$) to accuracy $O(\varepsilon^4)$. At the same time, the tables constructed in [1] contain expansions to $O(\varepsilon)$ and $O(\varepsilon^3)$, respectively.

To extend the tables, we use the solution (16). To this end, we expand in a series with respect to ε the function $F_\varepsilon(1+a\varepsilon)$, taking into account the symmetry property (5):

$$F_\varepsilon(1+a\varepsilon)=c_0+c_1\varepsilon+[c_2A+c_3B]\varepsilon^2+[c_4A+c_5B]\varepsilon^3+[c_6A+c_7B+c_8AB]\varepsilon^4+O(\varepsilon^5),$$

where we have introduced the notation $A\equiv(a+1)(a+2)$, $B\equiv a(a+3)$. Knowing the value of the function $F_\varepsilon(1+a\varepsilon)$ for $a=0, -1, -2, -3$, we find the coefficients $c_0, \dots, c_7$. We obtain

$$F_\varepsilon(1+a\varepsilon)=\frac{1}{1-2\varepsilon}\Big\{6\zeta(3)+9\zeta(4)+(21A-6B)\zeta(5)\varepsilon^2+\Big(45A-\frac{15}{2}B\Big)\zeta(6)\varepsilon^3-(23A-8B)\zeta^2(3)\varepsilon^3+$$
$$(147A-9B)\zeta(7)\varepsilon^4-\Big(\frac{135}{2}A-\frac{45}{2}B\Big)\zeta(3)\zeta(4)\varepsilon^4+c_8AB\varepsilon^4+O(\varepsilon^5)\Big\}. \tag{17}$$

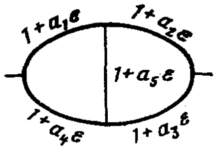
The coefficient c_8 cannot be determined from the known special values. It is readily seen that it is $c_8=\frac{1}{4!}\frac{d^4F_0(1+a)}{da^4}\Big|_{a=0}$. To find it, we expand the function $F_0(1+a)$ (16) with respect to a^2. We obtain

$$F_0(1+a)=8\sum_{n=0}^{\infty}a^{2n}(n+1)\Big(1-\frac{1}{2^{2n+2}}\Big)\zeta(2n+3). \tag{18}$$

Hence, for c_3 we find

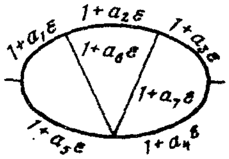
$$c_8=\frac{189}{8}\zeta(7). \tag{19}$$

This number makes it possible to complete formula (17), and also to construct a series expansion in ε for an arbitrary 2-loop diagram up to $O(\varepsilon^4)$, and for an arbitrary V-shaped diagram up to $O(\varepsilon^2)$, i.e., to extend by an order the tables constructed in [1]. They have the form



$$=\frac{\exp\Big\{-2\Big[\gamma\varepsilon+\frac{\zeta(2)}{2}\varepsilon^2\Big]\Big\}}{1-2\varepsilon}\{A_0\zeta(3)+A_1\zeta(4)\varepsilon+$$
$$A_2\zeta(5)\varepsilon^2+A_3\zeta(6)\varepsilon^3-A_4\zeta^2(3)\varepsilon^3+A_5\zeta(7)\varepsilon^4-A_6\zeta(3)\zeta(4)\varepsilon^4+O(\varepsilon^5)\},\quad A_0=6,\quad A_1=9,$$
$$A_2=42+30a+45a_5+10a^2+15a_5^2+15a_5a+10(a_1a_2+a_3a_4+a_1a_4+a_2a_3)+5(a_1a_3+a_2a_4),\quad A_3={}^5/_2(A_2-6),$$
$$A_4=46+42a+45a_5+14a^2+15a_5^2+33a_5a+50(a_1a_2+a_3a_4)+$$
$$31(a_1a_3+a_2a_4)+14(a_1a_4+a_2a_3)+6a_5a^2+6a_5^2a+24a_5(a_1a_2+a_3a_4)+12a_5(a_1a_3+a_2a_4)+12(a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4)+$$
$$12(a_1^2a_2+a_2^2a_1+a_3^2a_4+a_4^2a_3)+6(a_1^2a_3+a_3^2a_1+a_2^2a_4+a_4^2a_2),$$
$$A_5=294+402a+\frac{2223}{4}a_5+260a^2+\frac{3183}{8}a_5^2+516a_5a+386(a_1a_2+a_3a_4+a_1a_4+a_2a_3)+\frac{575}{2}(a_1a_3+a_2a_4)+$$
$$84a^3+\frac{567}{4}a_5^3+168(a_1^2a_2+a_2^2a_1+a_3^2a_4+a_4^2a_3+a_1^2a_4+a_4^2a_1+a_2^2a_3+a_3^2a_2)+$$
$$\frac{441}{4}(a_1^2a_3+a_3^2a_1+a_2^2a_4+a_4^2a_2)+\frac{945}{4}a_5a^2+252a_5^2a+\frac{693}{2}a_5(a_1a_2+a_3a_4+a_1a_4+a_2a_3)+\frac{945}{4}(a_1a_3+a_2a_4)a_5+$$
$$210(a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4)+14a^4+\frac{189}{8}a_5^4+42a_5a^3+\frac{189}{4}a_5^3a+\frac{525}{8}a_5^2a^2+\frac{357}{4}a_5^2(a_1a_2+a_3a_4+a_1a_4+a_2a_3)+$$
$$\frac{105}{2}a_5^2(a_1a_3+a_2a_4)+84a_5(a_1^2a_2+a_2^2a_1+a_3^2a_4+a_4^2a_3+a_1^2a_4+a_4^2a_1+a_2^2a_3+a_3^2a_2)+\frac{189}{4}a_5(a_1^2a_3+a_3^2a_1+a_2^2a_4+a_4^2a_2)+$$
$$\frac{357}{4}a_5(a_1a_2a_3+a_1a_3a_4+a_1a_2a_4+a_2a_3a_4)+28(a_1^3a_2+a_2^3a_1+$$
$$a_3^3a_4+a_4^3a_3+a_1^3a_4+a_4^3a_1+a_2^3a_3+a_3^3a_2)+14(a_1^3a_3+a_3^3a_1+a_2^3a_4+a_4^3a_2)+$$
$$42(a_1^2a_2^2+a_3^2a_4^2+a_1^2a_4^2+a_2^2a_3^2)+\frac{189}{8}(a_1^2a_3^2+a_2^2a_4^2)+42(a_1^2a_2a_3+a_1^2a_2a_4+a_1^2a_3a_4+a_2^2a_1a_4+a_2^2a_1a_3+a_2^2a_3a_4+a_3^2a_1a_4+$$
$$a_3^2a_2a_4+a_3^2a_1a_2+a_4^2a_2a_3+a_4^2a_1a_3+a_4^2a_1a_2)+\frac{315}{4}a_1a_2a_3a_4,\quad A_6=3(A_4-1), \tag{20}$$

where for brevity we have denoted $a^n\equiv a_1^n+a_2^n+a_3^n+a_4^n$;



$$= \frac{\exp\left\{-3\left[\gamma\varepsilon + \frac{\zeta(2)}{2}\varepsilon^2\right]\right\}}{1-2\varepsilon}\Big\{20\zeta(5)+\varepsilon[50\zeta(6)+(20+6(a_4+a_5+a_6+a_7))\zeta^2(3)]+$$

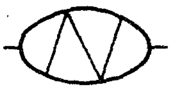
$$\varepsilon^2\Big[\zeta(7)\cdot 7\Big(\frac{380}{7}+20(a_1+a_3)+32a_2+17(a_4+a_5)+33(a_6+a_7)+6(a_1^2+a_3^2)+8a_2^2+4(a_4^2+a_5^2)+8(a_6^2+a_7^2)+$$

$$8(a_1+a_3)a_2+2(a_1a_4+a_3a_5)+6(a_1a_5+a_3a_4)+10(a_1a_6+a_3a_7)+6(a_1a_7+a_3a_6)+4a_1a_3+4(a_4+a_5)a_2+$$

$$12(a_6+a_7)a_2+2a_4a_5+4(a_4a_6+a_5a_7)+6(a_4a_7+a_5a_6)+10a_6a_7+{}^1/_4(a_4+a_5+a_6+a_7)+{}^1/_8(a_4+a_5+a_6+a_7)^2)+$$

$$\zeta(3)\zeta(4)\cdot 3(20+6(a_4+a_5+a_6+a_7))\Big)\Big]+O(\varepsilon^3)\Big\}. \quad (21)$$

Formula (21) makes it possible to complete readily the calculation of the N-shaped diagram. The result has the form



$$= \frac{1}{x^2}\frac{441}{8}\zeta(7),$$

in agreement with the prediction made earlier [1]. The constructed expansions (20) and (21) can be used subsequently as tables for finding the values of integrals.

5. Conclusions

We note finally that the functional equations obtained in this paper can also be obtained in the same manner for more complicated diagrams. It is possible that in this way a general form of expression for a diagram may be perceived. Hitherto, all exactly calculated integrals have been represented in the form of a product of Γ functions and their derivatives, and thus could be expanded in a series in ζ functions. It is not clear whether this is true in the general case or whether a general formula for the result can be obtained. But, solving functional equations for the coefficient functions of diagrams, we can at least represent the result in the form of a single series of the type (14).

From the practical point of view, the tables (20) and (21) are sufficient for calculating the singular (with respect to ε) parts of diagrams up to the five-loop approximation inclusively. The problem consists solely of reducing the considered diagram to a tabulated diagram, as was done in Sec. 4. It is evident that the achieved accuracy will for a long time be sufficient in actual calculations of renormalization-group anomalous dimensions of operators and other quantities determined by the singular contributions of Feynman diagrams.

I am grateful to D. V. Shirkov, P. P. Kulish, and A. V. Radyushkin for helpful discussions and advice.

LITERATURE CITED

1. D. I. Kazakov, Teor. Mat. Fiz., 58, 343 (1984).
2. A. N. Vasil'ev, Yu. M. Pis'mak, and Yu. R. Khonkonen, Teor. Mat. Fiz., 47, 291 (1981).
3. K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachov, Nucl. Phys. B, 174, 345 (1980).
4. E. Speer, Ann. Inst. H. Poincaré, 23, 1 (1975); V. A. Smirnov, Teor. Mat. Fiz., 44, 307 (1980).
5. Yu. V. Novozhilov, Introduction to the Theory of Elementary Particles [in Russian], Nauka, Moscow (1972).
6. I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, New York (1965).
7. K. G. Chetyrkin, S. G. Gorishny, S. A. Larin, and F. V. Tkachov, Phys. Lett. B, 132, 351 (1983).