CAUSTICS WITH ANOMALOUS PHASE SHIFTS

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We study the phase-shift correction to the eikonal of a field on a caustic. It is shown that, in addition to the usual phase loss on a caustic of $-\pi/2$, it is possible to have an anomalous phase shift $+\pi/2$. We give examples of spatial caustics with the anomalous phase shift which are found in anisotropic media, and of the analogous space–time caustics which arise in the propagation of a pulse in media with frequency dispersion. We study the uniform Airy asymptotic behavior, which is valid for the determination of a wave field in the vicinity of a nonsingular segment of a caustic with the anomalous phase shift.

1. Introduction

An additional phase shift on a caustic must be taken into account in the calculation of a wave field in many-ray conditions when

$$u(\mathbf{r}) = \sum_\nu A_\nu e^{ik\psi_\nu} = \sum_\nu \frac{A_\nu^0}{\sqrt{j_\nu}} e^{ik\psi_\nu}. \tag{1}$$

Here A_ν and ψ_ν are the amplitude and eikonal of the wave of ray ν: $j = n dS/n_0 dS_0$, divergence of the rays; $n = n(\mathbf{r})$, refractive index of the medium; dS, transverse cross section of the ray tube; and the index zero refers to the initial point of the ray $\mathbf{r}_0$, i.e., $n_0 \equiv n(\mathbf{r}_0)$, $dS_0 \equiv dS(\mathbf{r}_0)$, and $A_\nu^0 \equiv A_\nu(\mathbf{r}_0)$. The summation in (1) is carried out over all rays that arrive at the observation point. The phase-shift correction to the eikonal of a ray which has touched the caustic must therefore be included for a correct description of the interference picture of the waves.

Although the nature of the phase-shift correction is connected with the diffraction phenomena on the caustic, it can easily be interpreted in terms of geometrical optics [1]. Indeed, if the divergence j of the rays on the caustic has a zero of first order and is negative after the caustic ($j < 0$), we have for $j < 0$, independently of the caustic geometry,

$$j^{-1/2} = |j|^{-1/2} e^{\mp i(\pi/2)}. \tag{2}$$

Usually, further considerations [1-3] are used to select only the argument $-\pi/2$, and to consider caustics with the phase "loss" φ: $\varphi = k\psi - \pi/2$. However, it has not been clarified if caustics can occur with an extraordinary (anomalous) phase shift $+\pi/2$. It is shown below that these caustics are formed under certain specific conditions. These caustics will be called caustics with an anomalous phase shift.

It should be emphasized at the beginning that the anomalous phase shift on a caustic is not due to the geometry of the caustic, but to the specific physical properties of the medium where the wave propagates. It is important to note that in the majority of cases, one encounters caustics with the oridnary phase shift $-\pi/2$. We shall formulate more accurate conditions for the formation of caustics with the anomalous or ordinary phase shifts.

2. Formation of Caustics with Anomalous Phase Shifts in Anisotropic Media

We consider the caustics of a plane amplitude–phase screen in a one-dimensional anisotropic medium which correspond, e.g., to the propagation of a plane-wave beam. For each of the two independent normal waves, the wave field of the two-dimensional beam can be written in the form the plane-wave expansion

$$E(x, z) = \int_{-\infty}^{\infty} \tilde{E}_0(\varkappa) \exp\{i[k_z(\varkappa) z - \varkappa x]\} d\varkappa, \tag{3a}$$

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where

$$\tilde{E}_0(\varkappa) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_0(\xi)\, e^{i\varkappa\xi}\, d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_0(\xi)\exp[i\varphi_0(\xi) + i\varkappa\xi]\, d\xi, \tag{3b}$$

$A_0(x)$ and $\varphi_0(x)$ are the initial (for $z=0$) amplitude and phase of the field $E_0(x) = E(x, z=0)$, and $\tilde{E}_0(\varkappa)$ is the spatial spectrum of the initial beam (i.e., the directionality diagram of the beam). The dependence of $k_z(\varkappa)$ in (3a) is determined by the properties of the medium. In an isotropic medium, $k_z(\varkappa) = \sqrt{k^2 - \varkappa^2}$, and in anisotropic media, the function $k_z(\varkappa)$ has been studied in many works (e.g., in [4, 5] with the application to the magnetoactive plasma), and has, in general, a nonmonotonic character with bending points [4].

According to (3a) and (3b), we have

$$E(x, z) = \frac{1}{2\pi}\int\int_{-\infty}^{\infty} A_0(\xi)\exp[i\varphi(x, z, \xi, \varkappa)]\, d\xi\, d\varkappa, \tag{4}$$

where

$$\varphi(x, z, \xi, \varkappa) = \varphi_0(\xi) + k_z(\varkappa) z + \varkappa(\xi - x).$$

The points of stationary phase of this integral ξ_s and $\varkappa_s$ are given by the conditions

$$x = \xi + z\frac{dk_z}{d\varkappa}, \qquad \varkappa = -\varphi_0'(\xi) \equiv \varkappa_s, \tag{5}$$

which can be written in the form of the family of rays

$$x = \xi + z\frac{d}{d\varkappa}k_z(\varkappa_s) \equiv x(z, \xi), \tag{6}$$

where $\varkappa_s = -\varphi_0'(\xi)$, and ξ is the coordinate of the exit point of the ray from the initial plane $z=0$. The angle of tilt of the linear rays (6) is given by the functions $\varphi_0'(\xi)$ and $dk_z/d\varkappa$.

The caustic formed by rays (6) can be found from the condition $(\partial/\partial\xi)x(z, \xi) = 0$ and is described by the equations

$$\begin{gathered} z = \left\{\varphi_0''(\xi)\frac{d^2}{d\varkappa^2}k_z(\varkappa_s)\right\}^{-1} \equiv z_k(\xi), \\ x = \xi + \frac{d}{d\varkappa}k_z(\varkappa)\left\{\varphi_0''(\xi)\frac{d^2}{d\varkappa^2}k_z(\varkappa_s)\right\}^{-1} \equiv x_k(\xi). \end{gathered} \tag{7}$$

According to (7), the caustic occurs in the region $z > 0$ under the condition $\varphi_0''(\xi)(d^2/d\varkappa^2)k_z(\varkappa_s) > 0$. This condition imposes restrictions on the spatial modulation of the beam $\varphi_0(\xi)$, as well as on the properties of the medium $k_z(\varkappa)$. We note that in the singular direction where $d^2k_z/d\varkappa^2 = 0$, the caustic (7) asymptotically goes to infinity ($z_k \to \infty$, $x_k \to \infty$).*

The calculation of the integral in (4) by the two-dimensional method of stationary phase leads, as usual, to the equations of geometrical optics which describe the change of field along the ray (6):

$$E(x, z) = A_0(\xi_s)\left|1 - \frac{z}{z_k(\xi_s)}\right|^{-1/2}\exp[i\varphi(x, z, \xi_s, \varkappa_s) + i(\pi/4)\Delta], \tag{8}$$

where

$$\Delta = \left\{1 - \operatorname{sgn}\left[1 - \frac{z}{z_k(\xi_s)}\right]\right\}\operatorname{sgn}\varphi_0''(\xi_s), \tag{9}$$

and $\xi_s = \xi_s(x, z)$ is the root of Eq. (6), which defines the ray coordinate of the observation point (x, y). If there are several stationary points ξ_s we take, as in (1), a sum of expressions (8) which correspond to different $\xi_\nu = \xi_\nu(x, z)$.

The quantity Δ in (8) gives a correction to the phase shift of the field on caustic (7). For $\varphi_0''(\xi)(d^2k_z/d\varkappa^2) < 0$ when only imaginary caustic is realized ($z_k < 0$), the quantity Δ is equal to zero, which corresponds to the absence of caustic phase shift. Analogously, $\Delta = 0$ if $z_k > 0$ but $z < z_k$, i.e., on the segment of the ray prior to its contact with caustic (7). The correction to the phase shift ($\Delta \neq 0$) occurs only for $z > z_k > 0$. Then, according to (9),

*In this singular direction, one observes a slower spreading of the beam by diffraction than in the isotropic medium [5].

$$\Delta = 2\mathrm{sgn}\ \varphi_0''(\xi) = 2\ \mathrm{sgn}\frac{d^2}{d\varkappa^2}k_z(\varkappa_s),$$

and the corresponding phase factor in (8) is equal to

$$\exp\left(i\frac{\pi}{4}\Delta\right) = \exp\left(i\frac{\pi}{2}\mathrm{sgn}\frac{d^2k_z}{d\varkappa^2}\right) = \begin{cases}\exp\left(-i\frac{\pi}{2}\right), & (d^2k_z/d\varkappa^2) < 0,\\ \exp\left(+i\frac{\pi}{2}\right), & (d^2k_z/d\varkappa^2) > 0.\end{cases} \tag{10}$$

The usual caustic phase shift $e^{-i(\pi/2)}$ therefore arises when the caustic (7) is formed for $(d^2k_z/d\varkappa^2) <$), here, according to (7), $\varphi_0''(\xi) < 0$.* For example, in an isotropic medium $k_z(\varkappa) = \sqrt{k^2 - \varkappa^2}$, $d^2k_z/d\varkappa^2 = -k^2/k_z < 0$, and consequently, here only caustics with the usual phase shift are possible. It follows from (10) that caustics with the anomalous phase shift $e^{i(\pi/2)}$ are formed in an anisotropic medium under the condition $d^2k_z/d\varkappa^2 > 0$, i.e., in media with convex dependence $k_z(\varkappa)$ (Fig. 1b). This condition is realized, e.g., in a magnetoactive plasma under certain conditions [5] (see also [4]). We note that for the occurrence of the "anomalous" caustic, the beam must be modulated in a special way, viz., so that the inequality $\varphi_0''(\xi) > 0$ is satisfied (Fig. 1a).

3. Space – Time Caustics with an Anomalous Phase Shift

We shall consider now the space–time analogue of the previous problem, i.e., the propagation of a plane frequency-modulated pulse in a homogeneous medium with an arbitrary frequency dispersion $n = n(\omega)$. Suppose that the field of the pulse for $z = 0$ is equal to

$$E(0, t) = A_0(t)\exp[i\varphi_0(t)] \equiv E_0(t). \tag{11}$$

Then, analogously to (4), we have for $z > 0$ [6, 7]

$$E(z, t) = \frac{1}{2\pi}\int\int_{-\infty}^{\infty} A_0(\xi)\exp[i\varphi(z, t, \xi, \omega)]\,d\xi\,d\omega, \tag{12}$$

where

$$\varphi(z, t, \xi, \omega) = \varphi_0(\xi) + k(\omega)z + \omega(\xi - t),$$

and $k(\omega) = (\omega/c)n(\omega)$ is the wave number in the medium $n(\omega)$.

The method of stationary phase applied to the double integral (12) leads to the approximation of space–time geometrical optics [4, 7-9] supplemented by the equation of the caustic phase shift

$$E(z, t) = A_0(\xi_s)\left|1 - \frac{z}{z_k(\xi_s)}\right|^{-1/2}\exp[i\varphi(z, t, \xi_s, \omega_s) + i(\pi/4)\Delta], \tag{13}$$

where

$$\Delta = \left\{\mathrm{sgn}\left[1 - \frac{z}{z_k(\xi_s)}\right] - 1\right\}\mathrm{sgn}\left(\frac{dv_{gr}}{d\omega}\bigg|_{\omega=\omega_s}\right). \tag{14}$$

Here $\xi_s = \xi_s(z, t)$ is determined from the equation of the family of space–time rays

$$z = \left(\frac{dk}{d\omega}\right)^{-1}\bigg|_{\omega=\omega_s}(t - \xi) = v_{gr}(\omega_s)(t - \xi) \equiv z(t, \xi) \tag{15}$$

and corresponds to the initial moment of exit of the ray from the plane $z = 0$; $v_{gr}(\omega) = (dk/d\omega)^{-1}$ is the local group velocity of the wave in the medium; $\omega_s = -d\varphi_0(\xi)/d\xi = \omega_s(\xi)$ is a function which describes the initial frequency modulation of the pulse (11); and $z_k = z_k(\xi)$ is the coordinate of the space–time caustic formed by the family of rays (15).

The equations of the caustic of rays (15) can be found from (15) under the condition $(\partial/\partial\xi)z(t, \xi) = 0$ and have the form

$$\begin{aligned} z &= v_{gr}^2(\omega_s)\left\{\frac{d}{d\omega}v_{gr}(\omega_s)\frac{d\omega_s}{d\xi}\right\}^{-1} \equiv z_k(\xi),\\ t &= \xi + v_{gr}(\omega_s)\left\{\frac{d}{d\omega}v_{gr}(\omega_s)\frac{d\omega_s}{d\xi}\right\}^{-1} \equiv t_k(\xi). \end{aligned} \tag{16}$$

*The corresponding law of spatial modulation of the beam $\varphi_0(\xi)$ is the same as for the convergent initial (for $z = 0$) wave front.

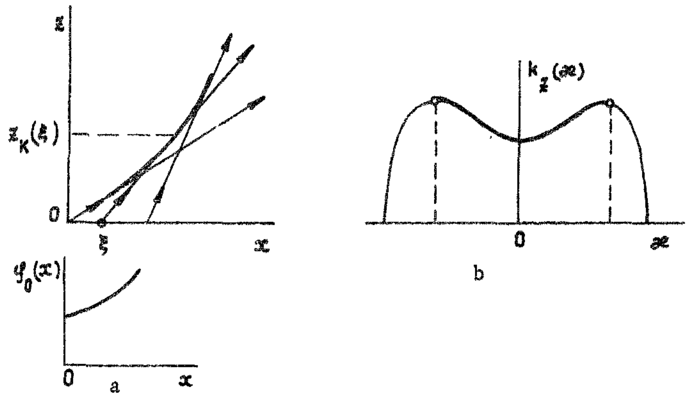

Fig. 1. Formation of an "anomalous" caustic in an anisotropic medium.

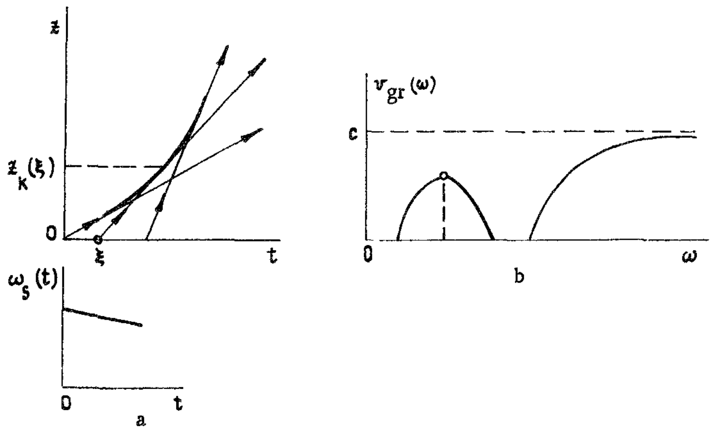

Fig. 2. Formation of the "anomalous" space–time caustic.

Hence it follows that a real caustic ($z_k > 0$) is formed only under the condition $\frac{d v_{gr}}{d\omega}\frac{d\omega_s}{d\xi} > 0$. In the opposite case, the caustic (16) is imaginary ($z_k < 0$).

The quantity Δ, which determines the phase shift correction on the caustic (16), is, according to (14), equal to zero for $z_k < 0$, and for $z < z_k$, if $z_k > 0$. For $z > z_k > 0$ we have

$$\Delta = -2\,\mathrm{sgn}\,\frac{d\omega_s}{d\xi} = -2\,\mathrm{sgn}\,\frac{dv_{gr}}{d\omega},$$

and hence we find

$$\exp\left(i\frac{\pi}{4}\Delta\right) = \exp\left(-i\frac{\pi}{2}\,\mathrm{sgn}\,\frac{dv_{gr}}{d\omega}\right) = \begin{cases}\exp\left(-i\frac{\pi}{2}\right), & \frac{dv_{gr}}{d\omega} > 0 \\ \exp\left(i\frac{\pi}{2}\right), & \frac{dv_{gr}}{d\omega} < 0\end{cases}. \tag{17}$$

It follows from (17) that the space–time caustic with an anomalous phase shift $\exp(i\pi/2)$ is formed in a dispersive medium with a decreasing dispersion characteristic $v_{gr}(\omega)$ (Fig. 2a).* This property is possessed by, e.g., the magnetoactive plasma in certain frequency intervals (Fig. 2b) [4, 9]. In a cold isotropic plasma $v_{gr}(\omega) = c\sqrt{1-\omega_p^2/\omega^2}$, where ω_p is the plasma frequency, we have $dv_{gr}/d\omega > 0$ and according to (17) one observes the usual caustic phase shift $\exp(-i\pi/2)$.

*The necessary condition for the formation of caustic (16) in the region $z > 0$ is, according to (16), that the initial pulse has a decreasing frequency modulation $\omega_s = \omega_s(\xi)$, $d\omega_s/d\xi < 0$ (for more detail, see [7]).

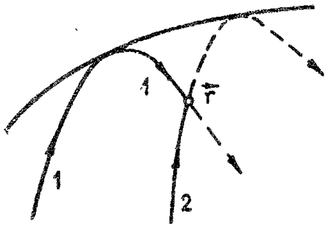

Fig. 3. Rays near a nonsingular caustic.

4. Wave Field in the Vicinity of a Nonsingular Caustic Segment with an Anomalous Phase Shift

In the presence of a nonsingular caustic, the geometrical optics formula (1) has the form

$$u(\mathbf{r}) = A_1 \exp(ik\psi_1) + A_2 \exp(ik\psi_2), \tag{18}$$

where the index 1 in the illuminated region refers to the ray which arrives at the observation point $\mathbf{r}$ after the contact with the caustic. The index 2, on the other hand, refers to the ray which arrives at $\mathbf{r}$ before the contact with the caustic (Fig. 3). The special feature of this caustic is that in the illuminated region, $A_1 = |A_1| \exp(i\pi/2)$ and $\psi_1 < \psi_2$ (for the usual caustic, $A_1 = |A_1| \exp(i\pi/2)$ and $\psi_1 > \psi_2$).

The uniform Airy asymptotic expression for the field in the vicinity of a nonsingular caustic has the form [10, 11]

$$u(r) = k^{1/6} \exp\left(ik\theta - i\frac{\pi}{4}\right) \{Av(k^{2/3}\zeta_0) + ik^{-1/3} Bv'(k^{2/3}\zeta_0)\}, \tag{19}$$

where the functions ζ_0, θ, A, and B can be algebraically expressed in terms of amplitudes $A_{1,2}$ and eikonals $\psi_{1,2}$ of the two rays (18). The expressions for the functions ζ_0, θ, A, and B take on a different form in the case of a caustic with the anomalous phase shift. This form can easily be found by "sowing together" asymptotically the caustic asymptotic expression (19) with the ray formula (18).*

Substituting into (19) the WKB asymptotic expression for the Airy function $v(\zeta)$ for $|\xi| \gg 1$, we obtain from (19), far from the caustic

$$u(\mathbf{r}) = A^- e^{ik\psi^-} + A^+ e^{ik\psi^+}, \tag{20}$$

where

$$\psi^{\mp} = \theta \mp \frac{2}{3}(-\zeta_0)^{3/2},$$
$$A^{\mp} = \frac{1}{2}\exp\left[-i\frac{\pi}{4}(1 \mp 1)\right][A(-\zeta_0)^{-1/4} \mp B(-\zeta_0)^{1/4}]. \tag{21}$$

Putting also $\psi^- = \psi_1$, $\psi^+ = \psi_2$, $A^- = A_1$, and $A^+ = A_2$, we find from (21)

$$A = (-\zeta_0)^{1/4}(iA_2 + A_1), \quad B = (-\zeta_0)^{-1/4}(iA_2 - A_1),$$
$$\theta = \frac{1}{2}(\psi_2 + \psi_1), \quad \zeta_0 = -\left[\frac{3}{4}(\psi_2 - \psi_1)\right]^{2/3}. \tag{22}$$

Taking into account (22), we write the principal term of the uniform asymptotic expression (19) in the following final form:

$$u(r) = \exp\left(ik\theta - i\frac{\pi}{4}\right)\{(-\zeta)^{1/4}(iA_2 + A_1)v(\zeta) + i(-\zeta)^{-1/4}(iA_2 - A_1)v'(\zeta)\}, \tag{23}$$

where

$$\zeta = -\left[\frac{3k}{4}(\psi_2 - \psi_1)\right]^{2/3}.$$

The obtained caustic asymptotic behavior (23) differs from the usual one [10, 11] only by the replacements A_2 by A_1, A_1 by A_2, ψ_1 by ψ_2, and ψ_2 follow

*The "sowing" procedure can be justified using the equations of the method of reference functions [10, 11] which follow from the substitution of (19) into the wave equation.

Expressions (23) are confirmed by the asymptotic behavior of the exact solutions (4) and (12), if the integrals there are calculated using a modification of the stationary-phase method valid in the case of two arbitrarily positioned stationary points [12].

In conclusion, we note that caustics with an anomalous phase shift of a more general type than $\exp(i\pi/2)$ can also occur. These caustics correspond to different degrees of degeneracy of the divergence of the ray tube j on the caustic, where $j = 0$.

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RADIATION OF A CHARGED PARTICLE ACCELERATED ALONG A FINITE PATH SEGMENT

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We examine the spectrum of radiation from a charged particle which moves with an initial speed v_1, a final speed v_2, and a smooth change in speed from v_1 to v_2 along a limited segment of the path.

The speed of a charged particle changes when it interacts with external fields or scattering centers. This change in speed often occurs in some limited region of space in which the particle is subject to external forces. Before entering this region, the particle speed has some initial value v_1, and after leaving the region in which the forces act it has the final v_2. The change in particle speed is accompanied by electromagnetic radiation. In this paper, we determine the radiation spectrum for a certain law of particle motion. Let a charged point particle move along the z axis, with the time dependence of its speed described by the expression

$$v(t) = \frac{v_1 + v_2}{2} + \frac{v_2 - v_1}{2}\,\mathrm{th}(\alpha t). \tag{1}$$

Obviously, for this law of motion, the particle speed at $t = -\infty$ is v_1, and at $t = +\infty$ it is v_2. The transition from v_1 to v_2 takes place near $t = 0$. When $t = 0$ the speed is the arithmetic mean of v_1 and v_2. The duration of the transition T from the initial speed v_1 to the final speed v_2 is of the order $1/\alpha$. We will henceforth assume for simplicity that

$$T = 1/\alpha. \tag{2}$$

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