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Infinite Products of Relations, Set-Valued Series and Uniform Openness of Multifunctions

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Abstract. Using the notations of convergent series of sets and convergent products of relations, general open mapping theorems are presented which encompass classical results of Banach, Ptak, Khanh, and others.

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The Newton method is so important in mathematics in general and in nonlinear analysis and optimization in particular, that it seems worthwhile to revisit it from a new point of view. We observe that the openness theorem of Lyusternik-Graves for differentiable mappings, the openness theorem of Ursescu and Robinson [40, 43] for convex multifunctions and the openness results of Ptak, Khanh, Borwein and Zhuang, Penot [34–39, 21, 22, 6, 29] are instances of an assertion whose conclusion can be phrased in purely topological terms: the mapping (or the multifunction) is open in this sense that it transforms a neighborhood of a point into a neighborhood of some image point. Therefore, it is tempting to try to encompass these separate results into a general framework which would be entirely topological.

Here we tackle this aim by introducing the use of infinite products of relations (in terms of composition of multifunctions). This tool might appear to be useful elsewhere. An important special case consists in studying set-valued series. The use of series of sets is not new; it probably goes back to S. Banach. But in general it is used in a nonsystematic way (see, however, [14, 25, 42]).

For series whose general terms are bounded closed convex subsets of a Banach space (or of a locally convex topological vector space), the question can be reduced to ordinary series of vectors of a topological vector space through the use of the Hörmander or Rådström embeddings. In the general case, the question is more intricate, and some care must be given, for instance, in defining the Cauchy criterion (this is due to the absence of a simplification rule). We use this special case (treated in Section 1 with some preliminary material for the sake of self-containedness) to justify the definition we adopt for convergent products of relations.

1. Series of Sets and Convergent Sequences of Relations

It is natural to define a series of nonempty subsets $\sum X_n$ of a topological vector space (t.v.s.) X to be Kuratowski (resp. Mosco) convergent if the sequence (S_n) of partial sums, $S_n = X_0 + \ldots + X_n$, converges in the sense of Kuratowski-Painlevé (resp. Mosco). Let us note that it would suffice to dispose of a structure of topological group or semigroup on X.

However, the uniform structure of X inclines us to rather use hemiconvergence, also called Hausdorff convergence. Thus, $\sum X_n$ is said to be *hemiconvergent* with sum S if the sequence (S_n) hemi-converges to S:

$$\forall U \in \mathcal{N}_X(0) \ \exists n_U \in \mathbb{N} : \forall n \geq n_U , \qquad S_n \subset S + U , \ \ S \subset S_n + U .$$

Here $\mathcal{N}_X(0)$ denotes the family of neighborhoods of 0. Then obviously, the series $\sum \bar{X}_n$ with general term the closure $\bar{X}_n$ of X_n converges to $\bar{S}$ and it also converges in this sense that the partial sums S_n can be replaced by their closures S_N. Since the hyperspace 2^X of (nonempty) subsets of X can be given many topologies and convergences, the preceding definitions admit numerous variants. We rather adopt a different viewpoint.

Let us call $\sum X_n$ to be *selectionwise convergent* (in short *convergent*) if for any selection (x_n) of (X_n) (i.e. any sequence (x_n) with $x_n \in X_n$ for each n) the series $\sum x_n$ is convergent. We will see (Proposition 1.3 below) that any convergent series is hemiconvergent. The converse is not true.

EXAMPLE. The following series of subsets of $\mathbb{R}$ is hemiconvergent but not convergent: $X_n = [2^{-n-1}, \infty)$ (and its sum is $S = [1, \infty)$).

Let us say that a series $\sum X_n$ is a *Cauchy series* if

$$\forall V \in \mathcal{N}_X(0) \ \exists n_V \in \mathbb{N} : p \geq n \geq n_V \Rightarrow X_n + \ldots + X_p \subset V .$$

Then the sequence (S_n) of partial sums is a Cauchy sequence for the Hausdorff uniform structure on 2^X (but the converse is false, as shown by the preceding example). However, when each X_n is bounded and convex and X is locally convex the Radstrom simplification rule shows that the series $\sum X_n$ is a Cauchy series iff (S_n) is a Cauchy sequence.

1.1. PROPOSITION. *Any convergent series is a Cauchy series. If the space X is sequentially complete the converse holds.*

Proof. Suppose $\sum X_n$ is convergent but not Cauchy; there exists $U \in \mathcal{N}_X(0)$ such that for any $n \in \mathbb{N}$ one can find $q > p \geq n$ with $X_p + \ldots + X_q \not\subset U$. Thus, there exist sequences (p_i), (q_i), (x_i) with $p_{i+1} > q_i > p_i$, $x_i \in X_i$ for each $i \in \mathbb{N}$ such that for each $i \in \mathbb{N}$ one has $\sum_{k=p_i}^{q_i} x_k \notin U$: the series $\sum x_n$ is not Cauchy, hence is not convergent, a contradiction.

If X is sequentially complete and if $\sum X_n$ is Cauchy then any selection (x_n) of $\sum X_n$ is Cauchy, hence is convergent. □

1.2. *Remarks*. (a) If $\sum X_n$ is convergent (hence, is Cauchy) one has the following collective property for its selections: for each $U \in \mathcal{N}_X(0)$ there exists $n_U \in \mathbb{N}$ such that for any selection $\sum x_n$ of $\sum X_n$ one has

$$s - s_n \in U\ , \qquad \text{for } n \geq n_U\ ,$$

where

$$s_n = \sum_{0 \leq k \leq n} x_k\ , \qquad s = \lim s_n\ .$$

In fact, we can find $V \in \mathcal{N}_X(0)$ with $V + V \subset U$ and if n_V is as in the Cauchy condition for $\sum X_n$, for any selection $\sum x_n$ of $\sum X_n$ we have for $n \geq n_V, p \geq n$ with p so large that $s - s_p \in V$

$$s - s_n = s - s_p + s_p - s_n \in V + V \subset U\ .$$

(b) A series $\sum X_n$ of subsets of a normed vector space X is said to be *absolutely convergent* if the series $\sum \|X_n\|$ is convergent, where for a subset Z of X one sets $\|Z\| = \sup\{\|z\| : z \in Z\}$. Thus, in a Banach space any absolutely convergent series of subsets is convergent. Moreover, for any bijection φ of $\mathbb{N}$, the series $\sum X_{\varphi(n)}$ is convergent and has the same sum as $\sum X_n$. This concept can be extended to locally convex t.v.s. by using the family of semi-norms instead of the norms.

(c) Convex series of sets play a key role in openness results [18, 19, 26–31]. The connection relies on the following notion: a convex subset C of a t.v.s. is said to be CS-compact if for any series $\sum_{n\geq 0} t_n$ of nonnegative numbers with sum 1 the series $\sum_{n \geq 0} t_n C$ is convergent and its sum is contained in C.

1.3. PROPOSITION. *Any convergent series is hemiconvergent.*

Proof. Let S be the set of sums $s = \sum x_n$ of series which are selections of (X_n). Remark 1.2(a) shows that for any symmetric $U \in \mathcal{N}_X(0)$ there exists $n_U \in \mathbb{N}$ such that $S \subset S_n + U$ for $n \geq n_U$, where $S_n = X_0 + ... + X_n$. Now let us observe that the inclusion $S_n \subset S + U$ holds for $n \geq n_U$ since for any $s_n = x_0 + ... + x_n \in S_n$ we can complete the sequence (x_k) by taking $x_k \in X_k$ arbitrarily for $k > n$; by definition this yields a convergent series whose sum s satisfies $s_n - s \in U$. □

Now let us turn to infinite products of relations.

In the sequel we identify a multifunction $F\colon X \rightrightarrows Y$ with its graph $F \subset X \times Y$ and we denote by F^{-1} the multifunction given by $F^{-1} = \{(y, x) \in Y \times X : (x, y) \in F\}$. If $F\colon X \rightrightarrows Y$, $G\colon Y \rightrightarrows Z$ are multifunctions, their product $G \circ F$ is given by

$$G \circ F = \{(x, z) \in X \times Z : \exists y \in F(x) \cap G^{-1}(z)\}\ .$$

If (Z, d) is a metric space and $r \in \mathbb{P} := (0, \infty)$ we set

$$U_r = \{(x,y) \in Z^2 : d(x,y) < r\}, \qquad B_r = \{(x,y) \in Z^2 : d(x,y) \leq r\}.$$

For subsets C, D of Z and $w \in Z$, we set

$$d(w,D) = \inf\{d(w,z) : z \in D\}, \qquad e(C,D) = \sup\{d(w,D) : w \in C\}.$$

For $R_k \subset X^2$, $k = 1, ..., n$, we will write $\prod_{k=1}^{n} R_k$ for $R_n \circ R_{n-1} \circ ... \circ R_1$.

A number of results valid for metric spaces can be extended to the framework of uniform spaces which is adapted to uniform notions which also appear when one deals with topological vector spaces (t.v.s.) or, more generally, with topological groups. The following concept retains most of these notions and has a wider range of applications, as shown by Proposition 1.5 below.

1.4. DEFINITION [4]. A quasi-uniform space is a pair $(X,\mathcal{U})$ where $\mathcal{U}$ is a filter of reflexive relations on X such that for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ with $V \circ V \subset U$.

In other words, the family $\mathcal{U}$ is nonempty, hereditary ($U \in \mathcal{U}$, $U \subset V$ imply $V \in \mathcal{U}$), stable by finite intersections, such that for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ with $V \circ V \subset U$ and each $U \in \mathcal{U}$ contains the diagonal $\Delta_X = \{(x,x) : x \in X\}$. If for each $U \in \mathcal{U}$, one has that $U^{-1} = \{(y,x) : (x,y) \in U\}$ belongs to $\mathcal{U}$, then $(X,\mathcal{U})$ is a uniform space. Hemimetrics (i.e. metrics for which the symmetry property $d(x,y) = d(y,x)$ is lacking) give rise to quasi uniform structures, setting $U_r = \{(x,y) \in X^2 : d(x,y) < r\}$. Quasi uniform structures appear naturally on hyperspaces (see [2], for instance). More importantly, one disposes of the following result in which the topology induced by a quasi uniform structure $\mathcal{U}$ is the topology τ such that for each $x \in X$ the family $\mathcal{U}(x) = \{U(x) : U \in \mathcal{U}\}$ is a base of neighborhoods of x, where $U(x) = \{y \in X : (x,y) \in U\}$.

1.5. PROPOSITION (W.J. Pervin [32]). *Any topological space (X,τ) is quasi uniformizable in this sense that τ is the topology induced by some quasi uniformity $\mathcal{U}$ on X.*

1.6. DEFINITION. Given a sequence $(R_n)_{n\geq 0}$ of relations of a topological space X we say that the product $\prod_{n\geq 0} R_n$ is pointwise convergent if any sequence $(x_n)_{n\geq 0}$ of X such that $x_{n+1} \in R_n(x_n)$ for each $n \geq 0$ is convergent. Then, the product relation $R = \prod_{n\geq 0} R_n$ is given by: for $x \in X$ the image $R(x) = \prod_{n\geq 0} R_n(x)$ is the set of limits of such sequences with $x_0 = x$.

When X is a quasi uniform space, the product $\prod_{n\geq 0} R_n$ is said to be convergent if it is pointwise convergent and if for any entourage U of X there exists $n_U \in \mathbb{N}$ such that $\prod_{k\geq n} R_k \subset U$ for any $n \geq n_U$.

Our terminology stems from the following example.

1.7. PROPOSITION. *Suppose X is a topological vector space (or topological additive group) and (X_n) is a sequence of nonempty subsets of X. Let $R_n = \{(x, x+u) : x \in X,\ u \in X_n\}$. Then the product $\prod R_n$ is convergent iff the series $\sum X_n$ is convergent.*

Proof. Let us first observe that a sequence (y_n) of X is such that $y_{n+1} \in R_n(y_n)$ iff there exists a selection (x_n) of (X_n) such that $x_n = y_{n+1} - y_n$ for each $n \in \mathbb{N}$. In such a case we have $y_{n+1} = y_0 + \sum_{k=0}^{n} x_k$ and (y_n) converges iff $(s_n) := (\sum_{k=0}^{n} x_k)$ converges. Then Remark 1.2 shows that for each $V \in \mathcal{N}_X(0)$ there exists $n_V \in \mathbb{N}$ such that for any selection (x_n) of (X_n) one has $s - s_n \in V$ for $n \geq n_V$. If $y = \lim y_n$, where (y_n) is given inductively by $y_{n+1} = y_n + x_n$, we have $y = y_0 + s = y_{n+1} - s_n + s$, hence $(y, y_{n+1}) \in \hat{V}$, where $\hat{V} = \{(x,y) \in X^2 : x - y \in V\}$ is a basic entourage of X. □

Another fundamental example of a convergent product of relations is given by $R_n = \{(x,y) \in X^2 : d(x,y) \leq r_n\}$, where (X,d) is a complete metric space and $\sum r_n$ is a convergent series. Again the second requirement of Definition 1.6 is easy to check. Note that in this case R_n is reflexive, i.e. contains the diagonal $\Delta_X = \{(x,x) : x \in X\}$; but a product of relations can be convergent even if none of the R_n's is reflexive (take $R'_n = R_n \backslash \Delta_X$, where R_n is as above with $r_n > 0$).

2. **Uniform Openness Criteria**

The following concept will be convenient. It is an extension of a notion given in [30] for metric spaces (see also [10–12, 17, 23, 24]).

2.1. DEFINITION. Let $(X, \mathcal{U}_X)$, $(Y, \mathcal{U}_Y)$ be quasi uniform spaces. A multifunction $F\colon X \rightrightarrows Y$ is said to be uniformly open over $C \subset F$ if for each $U \in \mathcal{U}_X$ there exists $V \in \mathcal{U}_Y$ such that for any $c = (a,b) \in C$ one has $V(b) \subset F(U(a))$. For $C = F$, F is said to be uniformly open.

For $C = \{(a,b)\}$ this definition reduces to the openness of F at (a,b). As is well known, a surjective continuous linear map between two Banach spaces is uniformly open in the preceding sense. An example of a uniformly open nonlinear map is provided by the case $x \mapsto x^2$ from $X = [1, \infty)$ into itself. An example of an open map which is not uniformly open is the square root map from $X = [1, \infty)$ into itself.

The following result contains the essence of our treatment.

2.2. THEOREM. *Let $(X, \mathcal{U}_X)$, $(Y, \mathcal{U}_Y)$ be quasi uniform spaces and let $F\colon X \rightrightarrows Y$ be a sequentially closed multifunction. Let (R_n), (S_n) be sequences of relations on X and Y respectively such that*

(a) *for any $(x,y) \in F$, $n \in \mathbb{N}$,*

$$S_n(y) \subset S_{n+1}\Big(F\Big(R_n(x)\Big)\Big) ;$$

(b) *the product* $\prod_n R_n$ *is convergent and each* R_n *is reflexive;*
(c) *the family of relations* (S_n^{-1}) *converges pointwise to the identity mapping of* Y*: for each* $y \in Y$ *and each sequence* (y_n) *such that* $y_n \in S_n^{-1}(y)$ *for each* n *one has* $(y_n) \to y$*;*
(d) *the set of* $i \in \mathbb{N}$ *such that* $S_i \in \mathcal{U}_Y$ *is infinite.*

Then F *is uniformly open.*

Proof. Let $U \in \mathcal{U}_X$. Let $h \in \mathbb{N}$ be such that $\prod_{j=i}^{\infty} R_j \subset U$ for $i \geq h$. Let $i \in \mathbb{N}$, $i > h$ be such that $S_i \in \mathcal{U}_Y$. We will show that $S_i(b) \subset F(U(a))$ for each $(a, b) \in F$, what will prove the result.

Let $(a, b) \in F$ and let $y \in S_i(b)$. Let us set $(x_i, y_i) = (a, b)$ so that

$$(x_i, y_i) \in F , \qquad x_i \in \prod_{j=0}^{i-1} R_j(a) , \; y_i \in S_i^{-1}(y) .$$

Let us suppose that for $k = 1, ..., n$ we have constructed $(x_k, y_k) \in F$ such that

$$x_k \in R_{k-1}(x_{k-1}) \subset \prod_{j=0}^{k-1} R_j(a) , \qquad y_k \in S_k^{-1}(y)$$

and let us define (x_{n+1}, y_{n+1}) satisfying similar requirements. Taking $(x_n, y_n) \in F$ instead of (x, y) in assumption (a), we can find some

$$x_{n+1} \in R_n(x_n) , \qquad y_{n+1} \in F(x_{n+1})$$

such that $y \in S_{n+1}(y_{n+1})$. Thus, the sequence (x_n, y_n) is well-defined. Since (x_n) is a selection of $(R_{n-1}(x_{n-1}))$, it converges by assumption (b). Assumption (c) guarantees that (y_n) converges to y. Since F is sequentially closed, for $x = \lim x_n$ we have $(x, y) \in F$. Our choice of i ensures that $x \in U(a)$ so that $y \in F(U(a))$.□

Let us note the following consequences in which, for a metric space (Z, d) and $r \in \mathbb{R}_+$ we set

$$U_r = \{(w, z) \in Z^2 : d(w, z) < r\} , \qquad B_r = \{(w, z) \in Z^2 : d(w, z) \leq r\} .$$

The second case of the following corollary contains results of Ptak [28], Khanh [21], Penot [30].

2.3. COROLLARY. *Let* $F: X \rightrightarrows Y$ *be a multifunction between two metric spaces. Suppose either (the graph of)* F *is complete or (the graph of)* F *is closed and* X *is complete.*

Suppose that for two sequences $(r_n)_{n\geq 0}$, $(t_n)_{n\geq 0}$ *of positive numbers with* $s := \sum_{n\geq 0} r_n < \infty$, $(t_n) \to 0$ *one has:*

for each $n \in \mathbb{N}$ *and each* $(x, y) \in F$

$$U_{t_n}(y) \subset U_{t_{n+1}}\Big(F\big(U_{r_n}(x)\big)\Big)\,.$$

Then F *is uniformly open and for each* $x \in X$ *one has, with* $t = t_0$,

$$U_t\big(F(x)\big) \subset F\big(U_s(x)\big)\,.$$

Proof. Replacing X by its completion $\hat{X}$ and setting $\hat{F}(\hat{x}) = \emptyset$ for $\hat{x} \in \hat{X}\backslash X$ we reduce the first case to the second one, observing that $\hat{F} = F$ is complete in $\hat{X} \times Y$, hence is closed.

Setting $R_n = U_{r_n}$ in X^2, $S_n = U_{t_n}$ in Y^2, the assumption on F amounts to assumption (a) of Theorem 2.2. The other assumptions are immediately satisfied. Taking $U := \prod_{j=0}^{\infty} R_j = U_s$, $i = 0$, in the proof of Theorem 2.2 we get that $S_0(b) \subset F(U(a))$ for each $(a, b) \in F$, the new conclusion. □

We observe that this corollary has a quantitative content which is of interest (see [3, 16, 29] for instance for what concerns the notion of rate of openness). Moreover, for each $k \in \mathbb{N}$, replacing $(r_n)_{n\geq 0}$, $(t_n)_{n\geq 0}$ by $(r_{n+k})_{n\geq 0}$, $(t_{n+k})_{n\geq 0}$ we get that for $q_k = \sum_{n\geq k} r_n$

$$U_{t_k}\big(F(x)\big) \subset F\big(U_{q_k}(x)\big)\,,$$

what proves again uniform openness, since q_k is arbitrarily small.

The following refinement takes into account the important notion of almost open mapping [3, 6, 36]. Here $\mathrm{cl}\, Z$ denotes the closures of a subset Z of a topological space.

2.4. COROLLARY. *Let* X, Y *be metric spaces and let* $F\colon X \rightrightarrows Y$ *be a multifunction. Suppose either (the graph of)* F *is complete or (the graph of)* F *is closed and* X *is complete. Suppose that for a sequence* (α_n) *of* $\mathbb{R}_+$ *and for sequences* (r_n), (t_n) *of the set* $\mathbb{P}$ *of positive numbers satisfying* $s := \sum_{n\geq 0} r_n < \infty$, $(t_n) \to 0$, $\alpha_n < t_{n+1}$ *for each* n, *one has for each* $n \in \mathbb{N}$ *and each* $(x, y) \in F$

$$U_{t_n}(y) \subset \mathrm{cl}\Big[B_{\alpha_n}\Big(F\big(U_{r_n}(x)\big)\Big)\Big]\,.$$

Then F *is uniformly open and for each* $x \in X$ *and each* $k \in \mathbb{N}$ *one has, with* $q_k = \sum_{n\geq k} r_n$,

$$U_{t_k}\big(F(x)\big) \subset F\big(U_{q_k}(x)\big)\,.$$

Proof. It suffices to observe that for each $n \in \mathbb{N}$ since $\alpha_n < t_{n+1}$, we have with $D_n = F\big(U_{r_n}(x)\big)$,

$$\mathrm{cl}\, B_{\alpha_n}(D_n) \subset U_{t_{n+1}}(D_n)$$

so that the assumptions of the preceding corollary are satisfied. □

2.5. COROLLARY. *Suppose X, Y, F are as in the preceding two corollaries and that for each $r \in \mathbb{P}$ there exists some $t(r) \in \mathbb{P}$ such that for each $(x, y) \in F$,*

$$U_{t(r)}(y) \subset \operatorname{cl} F\Big(U_r(x)\Big) .$$

Then F is uniformly open and for any r, s in $\mathbb{P}$ with $s > r$ one has

$$U_{t(r)}(y) \subset F\Big(U_s(x)\Big) .$$

Proof. Given r, s in $\mathbb{P}$ with $r < s$ we set $\alpha_n = 0$ for each $n \in \mathbb{N}$, $r_0 = r$, and take r_n in $\mathbb{P}$, $n \geq 1$ (for instance, $r_n = 2^{-n}(s - r)$) in such a way that $\sum_{n\geq 1} r_n = s - r$. Then, given any sequence (α_n) with limit 0 in $\mathbb{P}$ we set $t_0 = t(r)$, $t_n = \min(\alpha_n, t(r_n))$ so that $(t_n) \to 0$ and for each $n \in \mathbb{N}$, and each $(x, y) \in F$

$$U_{t_n}(y) \subset U_{t(r_n)}(y) \subset \operatorname{cl} F\Big(U_{r_n}(x)\Big) = \operatorname{cl} B_{\alpha_n}\Big(F\Big(U_{r_n}(x)\Big)\Big) . \qquad \square$$

2.6. COROLLARY [3]. *Let X, Y, F be as in the preceding corollaries. Let $\theta > 0$ be such that for each $s \in (0, \theta)$ there exist $\alpha(s)$ and $\delta(s)$ in $\mathbb{P}$ with $\lim_{s\to 0} \delta(s) = 0$, and a sequence $(r_n(s))$ in $(0, \theta)$ with $s = \sum_{n\geq 0} r_n(s)$, $\alpha(r_n(s)) < \delta(r_{n+1}(s))$ for each $n \in \mathbb{N}$ and*

$$U_{\delta(s)}(y) \subset \operatorname{cl} B_{\alpha(s)}\Big(F\Big(U_s(x)\Big)\Big) \tag{$*$}$$

for any $(x, y) \in F$, $s \in (0, \theta)$. Then F is uniformly open and

$$U_{\delta(r_0(s))}(y) \subset F\Big(U_s(x)\Big)$$

for each $(x, y) \in F$ and each $s \in (0, \theta)$.

Proof. Given $s \in (0, \theta)$ we take $(r_n) = (r_n(s))$ and we set $t_n = \delta(r_n)$, $\alpha_n = \alpha(r_n)$ so that $\alpha_n < t_{n+1}$ and

$$U_{t_n}(y) \subset \operatorname{cl} B_{\alpha_n}\Big(F\Big(U_{r_n}(x)\Big)\Big)$$

for each $n \in \mathbb{N}$ and each $(x, y) \in F$. The result follows. □

2.7. *Remark.* If instead of our assumptions on (r_n) we suppose as in [3] that $r_0(t) = t$, $s_0(t) := \sum r_n(t) < \infty$, and again $(*)$ and $\alpha(r_n(t)) < \delta(r_{n+1}(t))$ for $t \in (0, \theta)$ we get

$$U_{\delta(t)}(y) \subset F\Big(U_{s(t)}(x)\Big)$$

for each $(x, y) \in F$ and each $t \in (0, \theta)$, making similar choices for q_n, r_n, t_n.

These conditions are satisfied when (r_n) is defined inductively by $r_n = p^{(n)} = r_{n-1} \circ p$, with $r_0 = I$, $r_i = p$, where p: $(0, \theta) \to \mathbb{R}$ is a Ptak-small function in this sense that $\sum r_n$ pointwise converges. Then the condition $\alpha \circ r_n < \delta \circ r_{n+1}$ amounts to $\alpha < \delta \circ r$. This particularly simple case occurs frequently. In particular, after obvious changes, it is present in the following proof, what shows the power of Ptak's approach.

2.8. COROLLARY [6]. *Let X, Y, F be as in the preceding corollaries. Suppose there exist $\theta > 0$ and mappings α, δ: $(0, \theta) \to \mathbb{P}$ with δ increasing,*

$$\lim_{t \to 0_+} \delta(t) = 0\,, \qquad \limsup_{t \to 0_+} t^{-1}\delta^{-1}\Big(\alpha(t)\Big) < 1$$

and

$$U_{\delta(s)}(y) \subset \operatorname{cl} B_{\alpha(s)}\Big(F\Big(U_s(x)\Big)\Big)$$

for any $(x, y) \in F$, $s \in (0, \theta)$. Then F is uniformly open and for some $a > 0$ and some $b > 0$, one has for any $t \in (0, a)$, $(x, y) \in F$

$$U_{\delta(t)}(y) \subset F\Big(U_{bt}(x)\Big)\,.$$

The following proof yields an estimate of b which is of interest for the computation of the modulus of openness: for any $b > \Big(1 - \limsup_{t \to 0_+} t^{-1}\delta^{-1}\Big(\alpha(t)\Big)^{-1}$ one can find $a > 0$ for which the last inclusion holds.

Proof. Let $c \in (0, 1)$ and let $\theta' > 0$ be such that $t^{-1}\delta^{-1}(\alpha(t)) < c$ for $t \in (0, \theta')$, so that, as δ is increasing, $\alpha(t) < \delta(ct)$ for $t \in (0, \theta')$. Taking $r_n(s) = c^n(1 - c)s$, $b = (1 - c)^{-1}$ we can apply the preceding results. □

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