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SCATTERING THEORY FOR A THREE-PARTICLE SYSTEM WITH TWO-BODY INTERACTIONS PERIODIC IN TIME

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For a three-particle system with two-body interaction potentials periodic in time, a scattering theory that extends Faddeev's three-particle scattering theory to the periodic case is constructed.

Scattering theory for a three-particle system with two-body potentials was constructed in Faddeev's well-known paper [1]. In the case when the interaction depends explicitly on the time, only two-particle problems have hitherto been considered. The corresponding questions for a three-particle system are much more complicated. In this case, one of the most important is the case of interactions that depend on the time periodically. For example, the problem of the behavior of a system of three particles in an external homogeneous electric field that is periodic in time, the field averaged over a period being zero, leads to this case.

As in the case of time-independent potentials [1], subsidiary scattering channels appear together with the main channel. Each subsidiary channel corresponds to a quasi-energy state (see [2]) of the two-body problem. A quasi-energy state is a function of the monodromy operator of the time-dependent Hamiltonian, and is the analog of a bound state in the case of a time-independent Hamiltonian. There thus arises the need for a preliminary investigation of the two-body problem and, in particular, the properties of quasi-energy states.

The fundamental problem of scattering theory is the construction of wave operators and the proof of their completeness. A meaningful definition of completeness in a two-body problem with interaction periodic in time was proposed for the first time in [3]. It takes the form that the image of the wave operator must be identical to the absolutely continuous subspace of the monodromy operator of the corresponding Hamiltonian $\hat{h}(t)$. Note that if the operator $\hat{h}(t)$ does not depend on the time, $\hat{h}(t) = \hat{h}$, completeness for $\hat{h}(t)$ is identical to completeness in the usual sense. Namely, the image of the wave operator is identical to the absolutely continuous subspace of the operator $\hat{h}$. The definition of completeness of the wave operator of [3] can be extended in a natural manner to the case of three particles with periodic interaction.

The device of an additional time [2, 4, 5] and the technique of scattering theory for time-independent Hamiltonians [6] made it possible to solve with comparative ease the scattering problem for two particles with interaction periodic in time [5]. The absence of a singular continuous spectrum of the monodromy operator of the "two-body" problem was noted in [7-9]. A condition for the number of quasi-energy states, with allowance for multiplicity, to be finite was obtained in [7].

The present paper is devoted to the scattering in a quantum three-particle system with two-body potentials periodic in time. We use basically Faddeev's scheme, taking into account the technical improvements of recent years [10, 11]. The explicit time dependence is eliminated by the device of introducing an "additional" time. For the Hamiltonians obtained, equations of the type of Faddeev equations are derived and used. Their analysis makes use of the results of the preliminary investigation of the two-body problem, and also the decrease of the resolvent of the kinetic-energy operator of the three particles when the spectral

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parameter tends to infinity (see Lemma 3.2 in Sec.3). The final result is also a new one for the time-independent three-particle problem.

In Sec.1 we formulate the main theorem. In Sec.2 we give necessary information of a general nature. In Sec.3 we derive the conditions for the wave operator to exist, be isometric, and complete. Section 4 is devoted to the verification of these conditions, i.e., the investigation of the Faddeev equations.

The results of the investigation were announced in [12].

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1. Main Theorem

We introduce the concepts and facts needed to formulate the theorem. Let $\hat{\Gamma}$ be a separable Hilbert space. For a linear operator A on $\hat{\Gamma}$ we denote by $\mathscr{D}(A)$, $R(A)$ the domain of definition and range of A. If $\mathscr{D}\subset\mathscr{D}(A)$, then by $A\restriction\mathscr{D}$ we shall denote the restriction of A to $\mathscr{D}$. A two-parameter family of unitary operators u(t, s), t, s $\in$ R, on $\hat{\Gamma}$ that satisfies the conditions 1) $u(t, s)u(s, \sigma)=u(t, \sigma)$, $t, s, \sigma\in R$, 2) $u(t, t)=I$, $t\in R$, 3) the family u(t, s) is strongly continuous with respect to the variables t and s is called a propagator. We also impose an additional condition of periodicity: 4) $u(t+2\pi, s+2\pi)=u(t, s)$, $t, s\in R$.

We introduce the space of functions 2π-periodic with respect to t with values in $\hat{\Gamma}$, $\Gamma = L^2(T, \hat{\Gamma})$, where $T = R/2\pi Z$, Z being the set of integers. The mapping $\mathscr{U}_\sigma: f(t)\to u(t, t-\sigma)f(t-\sigma)$, $t\in T$, $\sigma\in R$, $f\in\Gamma$, forms in Γ a strongly continuous group. By Stone's theorem, it defines on Γ a self-adjoint operator h:

$$(\exp(-i\sigma h)f)(t)=u(t, t-\sigma)f(t-\sigma),\quad \sigma\in R,\quad f\in\Gamma. \tag{1.1}$$

We shall call a self-adjoint operator h for which (1.1) holds an evolution operator. Suppose there is a family of operators A(t), t $\in$ T, in $\hat{\Gamma}$. Then by $\langle A(t)\rangle$ we denote the operator of multiplication on Γ by A(t). We give the lemma on perturbation of an evolution operator from [9].

LEMMA 1.1. Let h_0 be an evolution operator on Γ, $\mathscr{V}(t)$, $t\in T$, be a weakly measurable bounded operator function on $\hat{\Gamma}$, $\mathscr{V}(t)=\mathscr{V}(t)^*$, $t\in T$, and let $V=\langle\mathscr{V}(t)\rangle$. Then $h = h_0 + V$ is an evolution operator.

There is a simple but, for us, important example of this operator. Let $\hat{h}_0$ be a self-adjoint operator on $\hat{\Gamma}$, $\partial=-i\partial/\partial t$ a self-adjoint operator on Γ with the natural domain of definition, and $\mathscr{D}_0=\mathscr{D}(\partial)\cap\mathscr{D}(\langle\hat{h}_0\rangle)$. Then the operator $(\partial+\langle\hat{h}_0\rangle)\restriction\mathscr{D}_0$ is essentially self-adjoint; we denote its closure by h_0. Since ∂ and $\langle\hat{h}_0\rangle$ commute, h_0 is an evolution operator, and the propagator for it has the form $u_0(t, s) = \exp(-i(t-s)\hat{h}_0)$. Let the operator function $\mathscr{V}(t)$ satisfy the conditions of Lemma 1.1. We introduce the family of self-adjoint operators $\hat{h}(t)=\hat{h}_0+\mathscr{V}(t)$, $t\in T$, on $\hat{\Gamma}$. Then the operator $h\restriction\mathscr{D}_0=(\partial+\langle h(t)\rangle\restriction\mathscr{D}_0$, and the operator $h\restriction\mathscr{D}_0$ is essentially self-adjoint. We shall call the propagator u(t, s) for the evolution operator h in this or analogous situations the propagator for the family of self-adjoint operators $\hat{h}(t)$, t $\in$ T, or the propagator for the Hamiltonian $\hat{h}(t)$. By m(t), t $\in$ T, we denote the monodromy operator $m(t) = u(t + 2\pi, t)$.

Remark. In what follows, the conditions of Lemma 1.1 will always be satisfied. Therefore, without specifying it particularly, we shall always assume that an operator of the type $\partial + \langle\hat{h}(t)\rangle$ is an evolution operator.

We now turn to a quantum system of three particles of dimension m $\geq$ 3 with finite* masses. Let M be the linear manifold in R^{3m} determined by the center-of-mass equation $\sum_{k=1}^{3}\mu_k z_k=0$, where z_k and μ_k are the coordinate and mass of the particle with number k. The manifold M is isomorphic to R^{2m}. We obtain the simplest form of the correspondence between M and R^{2m} in terms of the Jacobi coordinates x_α, y_α, $\alpha=1, 2, 3$: $x_\alpha=z_\beta-z_\gamma$, $y_\alpha=z_\alpha-(\mu_\beta+\mu_\gamma)^{-1}(\mu_\beta z_\beta+\mu_\gamma z_\gamma)$; α, β, γ range over the set of even permutations of the numbers 1, 2, 3. The three-particle energy operator $\hat{H}(t)$ in $\mathscr{H}=L^2(M)$ has the form $\hat{H}(t)=\hat{H}_0+\sum_{\alpha=1}^{3}\mathscr{V}_\alpha(t, x_\alpha)$, $H_0=-(2m_\alpha)^{-1}\Delta_{x_\alpha}-(2n_\alpha)^{-1}\Delta_{y_\alpha}$, $t\in T$, where m_α and n_α are the reduced masses, $m_\alpha^{-1}=\mu_\beta^{-1}+\mu_\gamma^{-1}$, $n_\alpha^{-1}=\mu_\alpha^{-1}+(\mu_\beta+\mu_\gamma)^{-1}$. The real-measurable 2π-periodic (in time) potential $\mathscr{V}_\alpha(t, x_\alpha)$, the potential energy between the

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* If one of the masses is infinite, some modifications are required. However, the final results are essentially the same.

particles with numbers β and γ, $\beta \neq \gamma \neq \alpha$, satisfies condition A.

CONDITION A. $|\mathscr{V}_\alpha(t, x_\alpha)| \leqslant C(1+|x_\alpha|)^{-\varepsilon}$, $\varepsilon>2$.

In accordance with [7], this ensures that the number of quasi-energy states of the corresponding two-body problem is finite, i.e., the number of channels is finite. Let $\hat{K} = L^2(R^m)$; ρ_α and δ_α are the operators of multiplication by $(1+x_\alpha^2)^{-\varepsilon/4}$, $(1+y_\alpha^2)^{-\varepsilon/4}$, respectively. Let $V_\alpha = \langle \mathscr{V}_\alpha(t, x_\alpha)\rangle$, $q_\alpha = \rho_\alpha^{-1}V_\alpha$, and $h_{\alpha 0} = -i\partial/\partial t - \langle (2m_\alpha)^{-1}\Delta_{x_\alpha}\rangle$ be a self-adjoint operator on K; $r_{\alpha 0}(z)$ is the resolvent of $h_{\alpha 0}$. We subject the function $\mathscr{V}_\alpha$ to one further condition.

CONDITION B. If the equation $f_\alpha = -q_\alpha r_{\alpha 0}(+i0)\rho_\alpha f_\alpha$ has in K a solution, then $\rho_\alpha f_\alpha \in \mathscr{D}(r_{\alpha 0}(0))$ and $\rho_\alpha^{-2} r_{\alpha 0}(0)\rho_\alpha f_\alpha \in K$.

We explain Condition B. Faddeev's theory presupposed the absence at the spectral point zero of both a genuine eigenvalue and a virtual level. In reality, the case of a zero eigenvalue can be included in the treatment if it is assumed that the corresponding eigenfunctions decrease sufficiently rapidly at infinity. Our condition B generalizes this requirement to the case of an interaction periodic in the time.

By $\hat{h}_\alpha(t)$ we denote the Hamiltonian of the two-body problem $\hat{h}_\alpha(t) = -(2m_\alpha)^{-1}\Delta_{x_\alpha} + \mathscr{V}_\alpha(t, x_\alpha)$, $t \in T$. In addition, we set $\hat{H}_\alpha(t) = \hat{H}_0 + \mathscr{V}_\alpha(t, x_\alpha)$. Let $U(t, s)$, $u_\alpha(t, s)$ be the propagators for $\hat{H}(t)$, $\hat{h}_\alpha(t)$, respectively, and $m_\alpha(t) = u_\alpha(t+2\pi, t)$ be the monodromy operator. We define on $\hat{\mathscr{H}}$ the projection operator $\hat{P}_\alpha(t) = P_p(m_\alpha(t)) \otimes I^\alpha$, where I^α is the identity operator with respect to the variable y_α. Note that the rank of the operator $P_p(m_\alpha(t))$ is finite [7]. The role of the model space is played by $\hat{\mathscr{H}}^4$, and the model operator $\hat{H}^0(t)$ in $\hat{\mathscr{H}}^4$ is determined by the formula $\hat{H}^0(t) = \sum_0^3 \oplus H_\alpha(t)$, $t \in T$. Note that the model operator depends explicitly on the time. The "identification" operator $\hat{J}^0(t) : \hat{\mathscr{H}}^4 \to \hat{\mathscr{H}}$ is introduced as the row matrix $\hat{J}^0(t) = (I, \hat{P}_1(t), \hat{P}_2(t), \hat{P}_3(t))$, $t \in T$. Let $\hat{P}(t) = I \oplus \sum_{\alpha=1}^{3} \oplus \hat{P}_\alpha(t)$, $t \in T$, be a projection operator on $\hat{\mathscr{H}}^4$. We define for $\hat{H}(t)$ the wave operator

$$\hat{W}_\pm{}^0(t) = \operatorname*{s\text{-}lim}_{\sigma \to \pm\infty} U(t, \sigma)\hat{J}^0(\sigma)U^0(\sigma, t)\hat{P}(t),$$

where $U^0(t, s)$ is the propagator for $\hat{H}^0(t)$. The main result of the paper is the following theorem.

THEOREM 1.1. Suppose the potentials $\mathscr{V}_\alpha$ satisfy Conditions A and B. Then the wave operators $\hat{W}^0_\pm(t)$ exist, are isometric on $\hat{P}(t)\hat{\mathscr{H}}^4$, and are complete.

2. Preliminaries

Suppose $C_\pm = \{z : \pm \operatorname{Im} z > 0\}$. Let A be a self-adjoint or unitary operator on $\hat{\Gamma}$. Then by $P_{ac}(A)$ and $P_p(A)$ we denote the projection operator onto the absolutely continuous, $\hat{\Gamma}_{ac}(A)$, and point, $\hat{\Gamma}_p(A)$, subspaces of A.

The space $\Gamma = L^2(T, \hat{\Gamma})$ can be realized as $l_2(\hat{\Gamma})$ by means of the discrete Fourier transformation $\Phi : \Gamma \to l_2(\hat{\Gamma})$, $(\Phi f)_n = f_n = (2\pi)^{-1/2} \int_0^{2\pi} \exp(-int) f(t)\,dt$, $n \in Z$. We shall denote functions of the operator ∂, $\varphi(\partial)$, in the Fourier representation by $\{\varphi(n)\}$. For the pair of Hilbert spaces Γ_1, Γ_2, we denote by $B_{12} = B(\Gamma_1, \Gamma_2)$, $S_\infty(\Gamma_1, \Gamma_2)$ respectively, the classes of bounded and compact operators from Γ_1 into Γ_2. In the case $\Gamma_1 = \Gamma_2 = \Gamma$, we shall write $B(\Gamma)$, $S_\infty(\Gamma)$. Let $A(t)$, $t \in T$, be a family of operators that map from $\hat{\Gamma}_1$ to $\hat{\Gamma}_2$. Then by $\langle A(t)\rangle$ we denote the operator of multiplication by $A(t)$, mapping from Γ_1 to Γ_2.

Let $a(z)$, $z \in \Xi \subset C$, be a family of bounded operators from Γ_1 to Γ_2. Then by a or $a(\cdot)$ we shall denote the corresponding operator function $a : \Xi \to B_{12}$. Let $\omega^\pm = \{z : 0 \leqslant \pm \operatorname{Im} z \leqslant 1\}$. We define $X^\pm(B_{12})$ as the class of operator functions $a : \omega^\pm \to B_{12}$ that are Hölder with some exponent τ, $0 < \tau < 1$. In what follows, when it is clear what B_{12} is meant it will be omitted in the symbol of the class $X^\pm(B_{12})$. We define the class $X^0 = X^+ \times X^-$.

Let A be an operator on $\hat{\Gamma}$. By A we shall sometimes denote the operators $\langle A\rangle$, $\Sigma_1^N \oplus A$, $\Sigma_1^N \oplus \langle A\rangle$ on the spaces Γ, $\hat{\Gamma}^N$, Γ^N, respectively. For the self-adjoint operators h^0, h_α, ..., H, we shall denote the resolvents by $r^0(z)$ $r_\alpha(z)$, ..., $R(z)$, respectively. We shall always denote a coordinate variable by x_α, y_α, a momentum variable by ξ.

In what follows, we give necessary information about monodromy and evolution operators. Let h be some evolution operator and $u(t, s)$ the propagator generating it. Let $e_n = \langle \exp(-int) \rangle$, $n \in Z$, be an operator of multiplication on Γ. Then from (1.1) we find for any $\sigma \in R$, $n \in Z$, that $e_n^* \exp(-i\sigma h) e_n = \exp(-i\sigma(h-n))$. Thus, h and $h - n$, $n \in Z$, are unitarily equivalent. From this the periodic structure of the spectrum (with allowance for multiplicity) of the self-adjoint operator h is clear. From the definition of the monodromy operator there follows its periodicity with respect to t, $m(t) = m(t + 2\pi)$, $t \in R$, and from (1.1) we obtain the equation $\exp(-i2\pi h) = \langle m(t) \rangle$. We give without proof some results (see, for example, [7, 9]).

1. If ψ^λ is an eigenfunction of the evolution operator h, $h\psi^\lambda = \lambda\psi^\lambda$, then ψ^λ is equivalent to a strongly continuous function with respect to t and $\psi^\lambda(t) = e^{i\lambda t} u(t, 0) \psi^\lambda(0)$, $t \in R$. In addition, $\| \psi^\lambda(t) \|_{\hat{\Gamma}} = \| \psi^\lambda(0) \|_{\hat{\Gamma}}$, $t \in T$.

2. Under the conditions of 1), $\psi^{\lambda,n} = e_n^* \psi^\lambda$ are eigenfunctions for the self-adjoint operator h, $h\psi^{\lambda,n} = (\lambda+n)\psi^{\lambda,n}$; the functions $\psi^{\lambda,n}$, $\lambda \in (-1, 0]$, $n \in Z$, exhaust $\Gamma_p(h)$. The function $\psi^\lambda(t)$ is an eigenfunction for the monodromy operator $m(t)$, $m(t)\psi^\lambda(t) = \exp(-i2\pi\lambda)\psi^\lambda(t)$, $t \in T$; the functions $\psi^\lambda(t)$, $\lambda \in (-1, 0]$, exhaust $\hat{\Gamma}_p(m(t))$, $t \in T$.

3. The following equations hold:

$$P_{ac}(h) = P_{ac}(\langle m(t) \rangle), \quad P_p(h) = P_p(\langle m(t) \rangle). \tag{2.1}$$

We now consider the connection between the wave operator for the pair of evolution operators h^0 and h and the wave operator for the pair of corresponding propagators. Let h and h^0 be evolution operators on Γ and Γ_0, respectively, and $u(t, s)$ and $u^0(t, s)$ be the propagators corresponding to them. Let $\hat{p}(t)$, $t \in T$, be a family of orthogonal projectors on $\hat{\Gamma}_0$ that satisfy Condition C:

CONDITION C. The family of orthogonal projectors $\hat{p}(t)$, $t \in T$, is strongly continuous with respect to t. Let $p = \langle \hat{p}(t) \rangle$. The operators h^0 and p commute and $P_{ac}(h^0 p) = p$.

Let $J(t)$, $t \in T$, be a uniformly bounded weakly measurable family of operators that map from $\hat{\Gamma}_0$ to $\hat{\Gamma}$, $J = \langle \hat{J}(t) \rangle$. We define for $h_1 = h^0 p$, h, J the wave operator in the usual manner:

$$W_\pm = \underset{\sigma \to \pm\infty}{s\text{-lim}} \exp(i\sigma h) J \exp(-i\sigma h_1) P_{ac}(h_1). \tag{2.2}$$

We define for $u^0(t, s)$, $u(t, s)$, $\hat{p}(t)$, $\hat{J}(t)$ the wave operator

$$\hat{W}_\pm(t) = \underset{\sigma \to \pm\infty}{s\text{-lim}}\, u(t, \sigma) \hat{J}(\sigma) u^0(\sigma, t) \hat{p}(t). \tag{2.3}$$

The completeness of the wave operator is determined by the fulfillment of the equations $R(W_\pm) = \Gamma_{ac}(h)$, $R(\hat{W}_\pm(t)) = \hat{\Gamma}_{ac}(m(t))$, $t \in R$. From the periodicity of the propagator and from (2.3) there follows periodicity of the wave operator $\hat{W}_\pm(t) = \hat{W}_\pm(t + 2\pi)$, $t \in R$. The operators h^0 and p commuting, we obtain from (2.3) the relation $\hat{W}_\pm(t) = u(t, s)\hat{W}_\pm(s) u^0(s, t)$, $t, s \in R$. From this there follows strong continuity of the wave operator $\hat{W}_\pm(t)$ with respect to t. We prove a lemma on the connection of the wave operators $W_\pm$ and $\hat{W}_\pm(t)$.

LEMMA 2.1. Suppose Condition C is satisfied. Then the wave operators $W_\pm$ existing, being isometric, and complete is equivalent, respectively, to the wave operators $\hat{W}_\pm(t)$ existing, being isometric on $\hat{p}(t)\hat{\Gamma}_0$, and being complete.

Proof. From (1.1), (2.2), and (2.3)

$$(W_\pm f)(t) = (\underset{\sigma \to \pm\infty}{s\text{-lim}} \exp(i\sigma h) J \exp(-i\sigma h_1) P_{ac}(h_1) f)(t) = \underset{\sigma \to \pm\infty}{s\text{-lim}}\, u(t, t+\sigma) \hat{J}(t+\sigma) u^0(t+\sigma, t) \hat{p}(t) f(t) = \hat{W}_\pm(t) f(t).$$

From this follows the equivalence of the existence, and also $W_\pm = \langle \hat{W}_\pm(t) \rangle$. From this equation and

$$\| W_\pm P_{ac}(h_1) f \|_\Gamma^2 = \int_0^{2\pi} \| \hat{W}_\pm(t) \hat{p}(t) f(t) \|_{\hat{\Gamma}}^2 \, dt, \tag{2.4}$$

$$\| P_{ac}(h_1) f \|_\Gamma^2 = \int_0^{2\pi} \| \hat{p}(t) f(t) \|_{\hat{\Gamma}}^2 \, dt \tag{2.5}$$

the equivalence of being isometric follows. Going over in (2.4) to the adjoint operators and using instead of (2.5) the first equation of (2.1), we verify similarly that $W_\pm^*$ are isometric on $P_{ac}(h)$ and $\hat{W}_\pm^*(t)$ on $P_{ac}(m(t))$. This proves the equivalence of the completeness.

We now give some lemmas that will be helpful.

LEMMA 2.2: 1. Let ρ be the operator of multiplication on $L^2(R^m)$, $m\geqslant 3$, by $(1+x^2)^{-\varepsilon/4}$, $\varepsilon>2$. Then for every $z\in C_\pm$, $\sigma\in[0, 1]$

$$\||\Delta+z|^{-\delta}\rho\|\leqslant C_\varepsilon|\operatorname{Im} z|^{-\sigma/2}|z|^{-\sigma/4}, \quad z\in C_\pm. \tag{2.6}$$

2. Let l be the operator of multiplication on $L^2(R^m)$ by the bounded function $l(x)$, $l(x)\to 0$ as $|x|\to\infty$. Then $(-\Delta-z)^{-1}l\in S_\infty$, $\operatorname{Im} z\neq 0$.

Proof: 1. Suppose first $\sigma = 1$. We show that

$$\|(\Delta+z)^{-1}\rho\|\leqslant C_\varepsilon|\operatorname{Im} z|^{-1/2}|z|^{-1/4}, \quad z\in C_\pm. \tag{2.7}$$

Let e_λ be a spectral projector for $\sqrt{-\Delta}$. It is shown in [11] that $\|d(\rho e_\lambda\rho)/d\lambda\|\leqslant\hat{C}_\varepsilon^2$. Hence, for $z=a+ib$, $b\neq 0$ we obtain (2.7):

$$\|(\Delta+z)^{-1}\rho\|^2=\|\rho|\Delta+z|^{-2}\rho\|=\left\|\int_0^\infty((\lambda^2-a)^2+b^2)^{-1}(d(\rho e_\lambda\rho)/d\lambda)\,d\lambda\right\|\leqslant\hat{C}_\varepsilon^2\pi|b|^{-1}|z|^{-1/2}.$$

From (2.7) and the Heinz inequality (see, for example, [13]) (2.6) follows for $C_\varepsilon = \max(1, \pi\hat{C}_\varepsilon)$.

2. This proposition is well known.

We give results needed in what follows on the "two-body" problem from [7].

LEMMA 2.3. Let $\hat{h}_\alpha(t)=-(2m_\alpha)^{-1}\Delta_{x_\alpha}+\mathscr{V}_\alpha(t, x_\alpha)$, $t\in T$, in $\hat{K}=L^2(R^m)$, $m\geqslant 3$, and the potential $\mathscr{V}_\alpha$ satisfy Conditions A and B. Then $P_{ac}(h_\alpha)=I-P_p(h_\alpha)$, $\rho_\alpha^{-2}P_p(h_\alpha)\in B(K)$, $\rho_\alpha P_{ac}(h_\alpha)r_\alpha\rho_\alpha\in X^0$.

3. Conditions of Existence and Completeness of the Wave Operators

We first give a precise definition. We assume that the indices α, β, γ range over the values 1, 2, 3, and the index a over 0, 1, 2, 3. On $\mathscr{H}^4$ we define the evolution operator $H^0 = \partial + \langle H^0(t)\rangle$ and the projection operator $P = \langle\hat{P}(t)\rangle$. We introduce the operator of multiplication $J^0=\langle\hat{J}^0(t)\rangle : \mathscr{H}^4\to\mathscr{H}$. By direct calculation we verify that the operators P and H^0 satisfy Condition C, and we let $\mathscr{H}=H^0P=PH^0$. We define the wave operator

$$W_\pm^0=s\text{-}\lim_{\sigma\to\pm\infty}\exp(i\sigma H)J^0\exp(-i\sigma\mathscr{H})P_{ac}(\mathscr{H}). \tag{3.1}$$

In accordance with Lemma 2.1, $W_\pm^0=\langle\hat{W}_\pm^0(t)\rangle$ and Theorem 1 is equivalent to the following theorem.

THEOREM 3.1. Let the potentials $\mathscr{V}_\alpha$ satisfy Conditions A and B. Then $W_\pm^0$ exist, are isometric, and are complete.

We give auxiliary propositions needed in what follows.

LEMMA 3.2. For the operators ρ_α, δ_α, $\varepsilon>2$, $\hat{R}_0(z)$ on $\hat{\mathscr{H}}=L^2(M)$, $m\geqslant 3$,

$$\rho_\alpha\rho_\beta\hat{R}_0(z),\quad \rho_\alpha\hat{R}_0(z)\rho_\beta\in S_\infty,\quad \alpha\neq\beta,\quad \operatorname{Im} z\neq 0, \tag{3.2}$$

$$\rho_\alpha\rho_\gamma\delta_\beta^{-1}\in B,\quad \alpha\neq\gamma, \tag{3.3}$$

$$\|\rho_\alpha\hat{R}_0(n+i)\rho_\gamma\|\to 0,\quad n\geqslant 0,\quad n\to\infty,\quad \alpha\neq\gamma. \tag{3.4}$$

Remark. Proposition (3.4) is also obtained in [14].

Proof. Equations (3.2) and (3.3) are proved in [11]. We prove (3.4). We have $\rho_\alpha\hat{R}_0(z)\rho_\gamma=[\rho_\alpha\hat{R}_0(z)|(2n_\alpha)^{-1}\Delta_{y_\alpha}+z|^{1/4}][|(2n_\alpha)^{-1}\Delta_{y_\alpha}+z|^{-1/4}\rho_\gamma]$, $z=n+i$. It follows from (2.6) that the first term on the right-hand side is uniformly bounded. We consider the second term in the momentum variables ζ_α, ζ_γ, dual to y_α, y_γ. The kernel of this operator has the form (see [1]) $|(2n_\alpha)^{-1}\zeta_\alpha^2-z|^{-1/4}\tilde{\rho}_\gamma(\zeta_\alpha-\zeta_\alpha')\delta(\zeta_\gamma-\zeta_\gamma')$, where $\tilde{\rho}_\gamma$ is the Fourier transform of the function ρ_γ. In accordance with (2.6), the norm of the operator with kernel $|(2n_\alpha)^{-1}\zeta_\alpha^2-z|^{-1/4}\tilde{\rho}_\gamma(\zeta_\alpha-\zeta_\alpha')$ on $L^2(R^m)$ has for $\sigma = \frac{1}{4}$ the estimate $C|n+i|^{-1/16}$. It can be seen from this that $\||(2n_\alpha)^{-1}\Delta_{y_\alpha}+n+i|^{-1/4}\rho_\gamma\|_{\hat{\mathscr{H}}}\leqslant C|i+n|^{-1/16}$.

LEMMA 3.3. Suppose the potentials $\mathscr{V}_\alpha$ satisfy the conditions A and B. Then

$$\rho_\alpha^{-2}P_\alpha \in B, \tag{3.5}$$

$$P_\alpha \delta_\alpha^{-1}\delta_\gamma^{-1}\rho_\beta^2 \in B, \quad \alpha\neq\beta, \tag{3.6}$$

$$\rho_\alpha\delta_\alpha R(z), \quad \rho_\alpha R(z)\rho_\gamma \in S_\infty, \quad \alpha\neq\gamma, \quad \operatorname{Im} z\neq 0, \tag{3.7}$$

$$P_\alpha P_\gamma R(z)\in S_\infty, \quad \operatorname{Im} z\neq 0, \ \alpha\neq\gamma. \tag{3.8}$$

Proof. Equation (3.5) follows from Lemma 2.3; (3.6) follows from (3.5) and (3.3). To prove (3.7), it is sufficient to prove compactness of the operators $\rho_\alpha\delta_\alpha R_0(z)$, $\rho_\alpha R_0(z)\rho_\gamma$, $\alpha\neq\gamma$, $\operatorname{Im} z\neq 0$. In the Fourier representation, the operator $R_0(z)$ has the form

$$\Phi R_0(z)\Phi^* = \{\hat{R}_0(z-n)\}. \tag{3.9}$$

Hence and from (3.2), (3.4), and (2.6) we obtain (3.7).

In the case of (3.8), it is sufficient to prove compactness of the operator $P_\alpha P_\gamma R_\gamma(z)$, $\alpha\neq\gamma$, $\operatorname{Im} z\neq 0$. We have

$$P_\alpha P_\gamma R_\gamma(z) = (P_\alpha\rho_\alpha^{-1})\rho_\alpha R_\gamma(z)\rho_\gamma(\rho_\gamma^{-1}P_\gamma).$$

Hence and from (3.5) and (3.7) we obtain (3.8).

We introduce the auxiliary identification operator $J:\mathcal{K}^4\to\mathcal{K}$. It has the row form $J=J^0-\left(\sum_\alpha P_\alpha, 0,0,0\right)$.

We have the relation

$$\underset{\sigma\to\pm\infty}{s\text{-lim}}\,(J-J^0)\exp(-i\sigma\mathcal{H})P_{ac}(\mathcal{H})=0, \tag{3.10}$$

the proof of which is comparatively simple (see [10]) and is here omitted. Besides (3.1), we consider the wave operator

$$W_\pm = \underset{\sigma\to\pm\infty}{s\text{-lim}}\exp(i\sigma H)J\exp(-i\sigma\mathcal{H})P_{ac}(\mathcal{H}), \tag{3.11}$$

$$\tilde{W}_\pm = \underset{\sigma\to\pm\infty}{s\text{-lim}}\exp(i\sigma\mathcal{H})J^*\exp(-i\sigma H)P_{ac}(H). \tag{3.12}$$

By virtue of (3.10), $W_\pm^0 = W_\pm$. We assume that the wave operators $W_\pm$, $\tilde{W}_\pm$ exist. It is shown in [10] that isometry of the wave operators $W_\pm$ and $\tilde{W}_\pm$ follows from (3.8) and (3.10).

We now discuss the conditions of existence of the limits (3.11) and (3.12). Below, the continuity of the various operator functions with respect to the spectral parameter z is always understood in the uniform operator topology of the corresponding spaces. By Π we denote the complex plane with cut along the real axis. We shall say (cf. [15]) that the bounded operator G is smooth with respect to the self-adjoint operator A if for some closed set Ω of measure zero, $\Omega \subset \mathbb{R}$, the operator function $G(A-z)^{-1}G^*$ is continuous with respect to $z\in\Pi\backslash\Omega$.

We assume that $F=(HJ-J\mathcal{H})=\sum_{k=1}^{N}N_k^*G_k$, where the operators G_k and N_k are smooth with respect to the self-adjoint operators $\mathcal{H}$ and H, respectively. Then in accordance with the sufficient condition of [15], the limits (3.11) and (3.12) exist.

We introduce operators that act on $\mathcal{K}$: $V^\alpha=V-V_\alpha$, $P_\alpha^\perp=I-P_\alpha$, $N_{\alpha,0}=\rho_\alpha P_\alpha^\perp$, $N_{\alpha,1}=\delta_\alpha^{-1}P_\alpha V^\alpha$, the last being bounded by virtue of (3.3) and (3.5). We define operators that map from $\mathcal{K}^4$ to $\mathcal{K}$: $G_{\alpha,0}f=q_\alpha f_0$, $G_{\alpha,1}f=\delta_\alpha f_0$, $G_\alpha f=\delta_\alpha P_\alpha f_\alpha$, $f\in\mathcal{K}$. We calculate the operator F. Suppose $f\in\mathcal{D}(\mathcal{H})$; then

$$Ff=\sum_\alpha (HJ_\alpha - J_\alpha\mathcal{H})f_\alpha = \sum_\alpha (P_\alpha^\perp V_\alpha - V^\alpha P_\alpha)f_0 + V^\alpha P_\alpha f_\alpha.$$

Finally, we obtain

$$F=\sum_\alpha N_{\alpha,0}^* G_{\alpha,0} - N_{\alpha,1}^* G_{\alpha,1} + N_{\alpha,1}^* G_\alpha.$$

To prove Theorem 3.1, it remains to prove smoothness of the operators N(G) with respect to the self-adjoint operator $H(\mathcal{H})$. This is the subject of the following section.

4. Faddeev Equations

Let N denote any of the operators $N_{\gamma,k}$, $k=0,1$. We introduce the operators $\mathscr{L}_{\alpha,0}(z)=N_{\alpha,0}R_\alpha(z)N^*$, $\mathscr{L}^0_{\alpha,1}=\delta_\alpha^{-1}P_\alpha N^*$ and assume that

$$\mathscr{L}^0_{\alpha,1},\quad \mathscr{L}^0_{\alpha,0}\in X^0. \tag{4.1}$$

We introduce the operators $\mathscr{L}_\alpha(z)=\rho_\alpha R(z)N^*$, $\mathscr{L}_{\alpha,0}(z)=N_{\alpha,0}R(z)N^*$, $\mathscr{L}_{\alpha,1}(z)=P_\alpha\delta_\alpha^{-1}(I-V^\alpha R(z))N^*$, $F_\alpha(z)=\rho_\alpha P_\alpha R_\alpha(z)\delta_\alpha$, $\operatorname{Im} z\neq 0$. As in [10], we obtain an equation relating $\mathscr{L}_\alpha$, $\mathscr{L}_{\alpha,0}$, $\mathscr{L}_{\alpha,1}$: $\mathscr{L}_\alpha=\mathscr{L}_{\alpha,0}+F_\alpha\mathscr{L}_{\alpha,1}$. Hence and from the equation $R = R_\alpha - R_\alpha V^\alpha R$, as in [10], we deduce the Faddeev equations

$$\mathscr{L}_{\alpha,0}=N_{\alpha,0}R_\alpha\Big(N^*-\sum_{\gamma\neq\alpha}q_\gamma(\mathscr{L}_{\gamma,0}+F_\gamma\mathscr{L}_{\gamma,1}),\quad \mathscr{L}_{\alpha,1}=P_\alpha\delta_\alpha^{-1}\Big(N^*-\sum_{\gamma\neq\alpha}q_\gamma(\mathscr{L}_{\gamma,0}+F_\gamma\mathscr{L}_{\gamma,1}).$$

We write these equations in the matrix form

$$\mathscr{L}(z)=\mathscr{L}^0(z)-\mathscr{A}(z)\mathscr{L}(z), \tag{4.2}$$

where $\mathscr{L}=(\mathscr{L}_{\alpha,0},\mathscr{L}_{\alpha,1})$, $\mathscr{L}^0=(\mathscr{L}^0_{\alpha,0},\mathscr{L}^0_{\alpha,1})$ are column vectors, and the components of the operator matrix $\mathscr{A}(z):\mathscr{H}^6\to\mathscr{H}^6$ have the form

$$\mathscr{A}^{\gamma,0}_{\alpha,0}(z)=\rho_\alpha P_\alpha^\perp R_\alpha(z)q_\gamma,\quad \mathscr{A}^{\gamma,1}_{\alpha,0}(z)=\mathscr{A}^{\gamma,0}_{\alpha,0}(z)F_\gamma(z),\quad \alpha\neq\gamma,$$

$$\mathscr{A}^{\gamma,0}_{\alpha,1}=\delta_\alpha^{-1}P_\alpha q_\gamma,\quad \mathscr{A}^{\gamma,1}_{\alpha,1}(z)=\mathscr{A}^{\gamma,0}_{\alpha,1}F_\gamma(z),\quad \alpha\neq\gamma,\quad \mathscr{A}^{\alpha,k}_{\alpha,s}=0,\quad k,s=0,1.$$

Before we investigate the Faddeev equations, we give necessary lemmas and prove $\mathscr{H}$ smoothness of the operators G.

LEMMA 4.1. For the operators ρ_α, δ_α, $\hat{R}_0(z)$ for $\varepsilon > 2$, $m \geq 3$ the following relations hold: 1) $\rho_\alpha\hat{R}_0\rho_\gamma$, $\delta_\alpha R_0\delta_\alpha\in X^0$, 2) $\|\delta_\alpha\exp(i\sigma\Delta_{y_\alpha})\delta_\alpha\|\leqslant C(1+|\sigma|)^{-\tau}$, $\tau>1$.

The proof of 1) is in [11], of 2) in [16].

LEMMA 4.2. For the operators ρ_α, δ_α, $\varepsilon>2$, on $\mathscr{H}$

$$\rho_\alpha R_0\rho_\gamma,\quad \delta_\alpha R_0\delta_\alpha\in X^0, \tag{4.3}$$

$$\delta_\alpha R_\alpha\delta_\alpha\in X^0. \tag{4.4}$$

Suppose the potentials $\mathscr{V}_\alpha$ satisfy Conditions A and B. Then

$$\rho_\alpha P_\alpha^\perp R_\alpha\rho_\gamma\in X^0, \tag{4.5}$$

$$\rho_\gamma R_\alpha P_\alpha\delta_\alpha\in X^0,\quad \alpha\neq\gamma. \tag{4.6}$$

Proof. Equations (4.3) follows from (3.9) and from 1) of Lemma 4.1. Equation (4.4) is obtained from the commutativity of the operators δ_α and h_α, from 2) of Lemma 4.1, and from the representation $R_\alpha(z)=\pm i\int_0^\infty\exp(\mp i\sigma(H_\alpha-z))d\sigma$, $\pm\operatorname{Im} z>0$. From (4.3) and the formula $R_\alpha=R_0-R_\alpha V_\alpha R_0$ it follows that it is sufficient to prove (4.5) for $\alpha = \gamma$. In the mixed coordinates t, x_α, ζ_α, the operator $R_\alpha(z)$ has the form of the operator of multiplication by the operator-valued function $r_\alpha(z-(2n_\alpha)^{-1}\zeta_\alpha^2)$, which depends on ζ_α as on a parameter. Hence and from Lemma 2.3 the relation (4.5) follows for $\alpha = \gamma$. We prove (4.6). From (3.3), (3.5), and (4.4) we obtain $\rho_\gamma R_\alpha P_\alpha\delta_\alpha=(\rho_\gamma\rho_\alpha\delta_\alpha^{-1})(\rho_\alpha^{-1}P_\alpha)\delta_\alpha R_\alpha\delta_\alpha\in X^0$.

LEMMA 4.3. Suppose $\varepsilon > 2$. Then the operators $G_{\alpha,0}$, $G_{\alpha,1}$, G_α are smooth with respect to the self-adjoint operator $\mathscr{H}$.

The proof for $G_{\alpha,0}$, $G_{\alpha,1}$ follows from (4.3). For example, $G_{\alpha,0}\mathfrak{R}G^*_{\alpha,0}=q_\alpha R_0 q_\alpha\in X^0$. From (4.4), we obtain smoothness of the operator G_α with respect to the self-adjoint operator $\mathscr{H}$, $G_\alpha\mathfrak{R}G_\alpha^*=P_\alpha\delta_\alpha R_\alpha\delta_\alpha\in X^0$.

We investigate the properties of the matrix $\mathscr{A}$ component by component.

A. $\mathscr{A}^{\gamma,0}_{\alpha,0}$, $\alpha\neq\gamma$. It follows from (3.5) and (3.7) that the operator $\mathscr{A}^{\gamma,0}_{\alpha,0}(z)$, $\operatorname{Im} z\neq 0$, is compact, and from (4.5) we find that $\mathscr{A}^{\gamma,0}_{\alpha,0}\in X^0$.

B. $\mathscr{A}^{\gamma,1}_{\alpha,0}$, $\alpha\neq\gamma$. The compactness of this component for $\operatorname{Im} z \neq 0$ follows from the compactness of $\mathscr{A}^{\gamma,0}_{\alpha,0}(z)$. To investigate the smoothness of the operator function $\mathscr{A}^{\gamma,1}_{\alpha,0}$, we represent it in the more convenient form

$$\mathscr{A}^{\gamma,1}_{\alpha,0}=N_{\alpha,0}(R_\alpha-R_\gamma+R_\alpha V_\alpha R_\gamma)P_\gamma\delta_\gamma, \tag{4.7}$$

where we have used the relation $R_\alpha-R_\gamma=-R_\alpha(V_\alpha-V_\gamma)R_\gamma$. We prove the smoothness of all the terms on the right-hand side in (4.7). We represent the first term in the form $\rho_\alpha P_\alpha^\perp R_\alpha\rho_\gamma(\rho_\gamma^{-1}P_\gamma)\delta_\gamma$, and hence and from (4.5) and (3.5) it follows that it belongs to X^0. From (3.5) and (4.6) and the fact that the second term on the right can be represented in the form $(\rho_\alpha P_\alpha^\perp\rho_\alpha^{-1})\rho_\alpha R_\gamma P_\gamma\delta_\gamma$, we find that it belongs to X^0. The third term also lies in X^0, which follows from the fact that it can be represented in the form $\rho_\alpha P_\alpha^\perp R_\alpha q_\alpha\rho_\alpha R_\gamma P_\gamma\delta_\gamma$ and from (4.5) and (4.6).

C. $\mathscr{A}^{\gamma,0}_{\alpha,1}$, $\alpha\neq\gamma$. This operator does not depend on z, and the fact that it is bounded follows from (3.3) and (3.5).

D. $\mathscr{A}^{\gamma,1}_{\alpha,1}$, $\alpha\neq\gamma$. From (4.4) and (3.6) we find that $\mathscr{A}^{\gamma,1}_{\alpha,1}=(\delta_\alpha^{-1}P_\alpha V_\gamma\delta_\gamma^{-1})\delta_\gamma R_\gamma P_\gamma\delta_\gamma\in X^0$. Further, $\mathscr{A}^{\gamma,1}_{\alpha,1}(z)=(\delta_\alpha^{-1}P_\alpha V_\gamma)R_\gamma(z)\rho_\gamma\delta_\gamma(\rho_\gamma^{-1}P_\gamma)\in S_\infty$, $\operatorname{Im} z\neq 0$, by virtue of (3.5), (3.6), and (3.7).

Thus, $\mathscr{A}\in X^0$ and $\mathscr{A}(z)^2\in S_\infty$. The operator $I+\mathscr{A}(z)$ is invertible for $\operatorname{Im} z\neq 0$; this follows from the self-adjointness of the operator H as well (the proof completely repeats the corresponding argument in [1] and is here omitted).

We give the inversion lemma from [17].

<u>LEMMA 4.4.</u> Suppose the operator function $a\in X^0(B(\Gamma))$, $I-a(z)\in S_\infty$, $a(z)^{-1}\in B(\Gamma)$, $\operatorname{Im} z\neq 0$, and the operator function a is analytic with respect to $z\in\omega^\pm$. Then there exists a closed set Ω, $\Omega\subset R$, of Lebesgue measure zero such that $a(\lambda\pm i0)^{-1}\in B(\Gamma)$, $\lambda\in R\backslash\Omega$.

Applying this lemma for $a=I-\mathscr{A}^2$ and using the representation $(I+\mathscr{A})^{-1}=(I-\mathscr{A})(I-\mathscr{A}^2)^{-1}$, we find that for some closed set Ω, $\Omega\subset R$, of Lebesgue measure zero

$$(I+\mathscr{A}(\lambda\pm i0))^{-1}\in B,\quad \lambda\in R\backslash\Omega. \tag{4.8}$$

Note that the set Ω contains the singular spectrum of the self-adjoint operator H if it exists.

We now conclude the proof of Theorem 3.1. For this, we must verify the smoothness of the operators $N_{\alpha,k}$, $k=0, 1$, with respect to the self-adjoint operator H. From the smoothness of the operator function $\mathscr{A}$, we find from (4.8), (4.2), and (4.1) that the operator function $\mathscr{L}=(I+\mathscr{A})^{-1}\mathscr{L}^0$ is continuous with respect to $z\in\Pi\backslash\Omega$. This last result gives us somewhat more than we need; with allowance for the definition of the operators $\mathscr{L}$, this means that the operator function $N_{\alpha,k}R(z)N^*_{\gamma,k}$ is continuous with respect to $z\in\Pi\backslash\Omega$.

We now prove (4.1). We first prove that the operator $\mathscr{L}^0_{\alpha,1}$ is bounded. Suppose first $N=N_{\gamma,0}$. Then the boundedness of the operator $\mathscr{L}^0_{\alpha,1}$ follows from (3.3) and (3.5) for $\gamma\neq\alpha$ and $\mathscr{L}^0_{\alpha,1}=0$ for $\gamma=\alpha$. We obtain the boundedness of the operator $\mathscr{L}^0_{\alpha,1}$ for $N=N_{\gamma,1}$ from (3.6) and (3.5). Now consider the operator function $\mathscr{L}^0_{\alpha,0}$. Suppose $N=N_{\gamma,0}$. Then from (4.5) and (3.5) we find that $\mathscr{L}^0_{\alpha,0}=\rho_\alpha P_\alpha^\perp R_\alpha\rho_\gamma(\rho_\gamma^{-1}P_\gamma^\perp\rho_\gamma)\in X^0$. Suppose $N=N_{\gamma,1}$. Then $\mathscr{L}^0_{\alpha,0}=\sum_{\beta\neq\alpha}\rho_\alpha P_\alpha^\perp R_\alpha\rho_\beta(\mathscr{A}^{\gamma,0}_{\beta,1})^*\in X^0$, which follows from (4.5) and the boundedness of $\mathscr{A}^{\gamma,0}_{\beta,1}$. Theorem 3.1 is completely proved.

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CONTINUOUS MODELS OF PERCOLATION THEORY. II

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Percolation models in which the centers of defects are distributed randomly in space in accordance with Poisson's law and the shape of each defect is also random are considered. Methods of obtaining rigorous estimates of the critical densities are described. It is shown that the number of infinite clusters can take only three values: 0, 1, or ∞. Models in which the defects have an elongated shape and a random orientation are investigated in detail. In the two-dimensional case, it is shown that the critical volume concentration of the defects is proportional to a/l, where l and a are, respectively, the major and minor axes of the defect; the mean number of (direct) bonds per defect when percolation occurs is bounded.

Introduction

The present paper continues our [1], which gave a rigorous formulation of continuous percolation problems. For convenience of reference, we continue the numbering of the sections begun in [1].

4. Methods of Estimating Critical Quantities

4.1. The Method of Generations. In $\mathbb{R}^d$, we consider the continuous problem of percolation in which the shape of the defect is fixed and only the orientation can be random. Let $\mathbf{n}$ be the direction vector of a defect (see Sec.2 in [1]). For a fixed center, the orientation of a defect is determined by the probability measure μ on the sphere S^{d-1}. We restrict ourselves for the time being to measures such that for two independent random variables $\mathbf{n}_1$ and $\mathbf{n}_2$ subject to the distribution on S^{d-1} the difference $\mathbf{n}_2 - \mathbf{n}_1$ does not depend on $\mathbf{n}_1$. We shall say that such measures are symmetric. In particular, the uniform distribution is symmetric. It is readily seen that besides this the only symmetric distributions are those concentrated at k points that form the vertices of the regular (platonic) solids inscribed in S^{d-1}, so that the measure of each point is 1/k. For example, for d = 2 the class of symmetric distributions consists of the uniform distribution on the circle and the uniform distribution over the vertices of the regular k-gon (k = 1, 2, ...).

We shall say that defects are neighbors if they intersect. Let $K(\mathbf{r}, \mathbf{n}_1, \mathbf{n}_2)$ be the μ probability that a defect with center at 0 and direction $\mathbf{n}_1$ intersects a defect with center at $\mathbf{r}$ and direction $\mathbf{n}_2$. We denote $B(\mathbf{n}_1) = \int_{S^{d-1}}\int_{\mathbb{R}^d} K(\mathbf{r}, \mathbf{n}_1, \mathbf{n}_2)\, d\mathbf{r}\, \mu(d\mathbf{n}_2)$. We use the symmetry of the measure μ. Let μ' be the probability measure corresponding to the distribution of the difference $\mathbf{n}_2 - \mathbf{n}_1$. We distinguish in space a certain fixed direction e and denote by $A_{\mathbf{n}_1}$ the rotation about the origin that carries the direction $\mathbf{n}_1$ to e. Then

$$B(\mathbf{n}_1) = \int_{S^{d-1}}\int_{\mathbb{R}^d} K(\mathbf{r}, \mathbf{n}_1, \mathbf{n}_2)\, d\mathbf{r}\mu(d\mathbf{n}_2) = \int_{S^{d-1}}\int_{\mathbb{R}^d} K(A_{\mathbf{n}_1}\mathbf{r}, \mathrm{e}, \mathbf{n}_2 - \mathbf{n}_1)\, d\mathbf{r}\mu(d\mathbf{n}_2) =$$

$$\int_{S^{d-1}}\int_{\mathbb{R}^d} K(\mathbf{r}', \mathrm{e}, \mathbf{n}')\, d\mathbf{r}\mu'(d\mathbf{n}') = B = \mathrm{const}.$$

Thus, if the distribution of the orientation is symmetric, then for any defect (irrespective of its orientation) the mathematical expectation of the number of its neighbors is the same and equal to λB, where λ is the intensity of the Poisson field of the defect centers.

We distinguish some defect and call it the defect of generation 0. If we have already determined the defects of the generations 0, 1, ..., k − 1, then the defects of generation k are those that are neighbors of the defects of generation k − 1 and are not defects of the generation k − 2. Consider the random variable

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