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Statistical Moments of the Hypsometric Curve and Its Density Function[1]

John M. Harlin[2]

A method, applicable to multivariate designs, describing the form of the percentage hypsometric curve is developed in this research. Emphasis is placed on the quantitative aspects of curve form, rather than on average slopes, inflection points, or hypsometric integrals. A question of uniqueness arises when values, like the integral, are used as landform surrogates in process-response models involving drainage basins. It is demonstrated that the hypsometric curve has a much greater potential for quantitative landform analysis than can be realized through employment of the integral value alone. Unlike the integral, the functional form of the curve is unique to a particular area, depicting, among other things, evolutionary changes in the form of drainage basins. The technique involves treating the "decumulation" of the hypsometric curve in its mirror image as a cumulative distribution function. Statistical moments of the curve, and expectations of (x) *for the curve's density function are derived, projecting a vector of curve-form attributes.* KEY WORDS: geomorphology, hypsometric analysis, statistical moments.

INTRODUCTION

Hypsometric analysis was first used in fluvial geomorphology when Langbein *et al.* (1947) employed hypsometric (area-altitude) analysis to express the overall slope of selected drainage basins. The hypsometric curve found repeated usage throughout the 1950's (Strahler, 1952; Miller, 1953; Schumm, 1956; Coates, 1956), as the curve and its integral were incorporated into research dealing with erosional topography. After this original wave of "developmental" research, hypsometric analysis seemed to vanish from the literature with the exception of occasional textbook references. Part of the reason for this evanescence is that historically the technique demonstrated inability in adapting to multivariate designs. The integral value is readily available as a continuous parameter, but the form of the curve, not lending itself easily to quantification, has often been left to casual description.

Dependence on the integral for basin description presents immediate

[1] Manuscript received 9 March 1977; revised 26 April 1977.

[2] Department of Geography, University of Maryland (Baltimore County), Baltimore, Maryland 21228.

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problems. For example, Schumm (1956) found that a strong linear relationship exists between the integral and both relief ratio and stream gradient. However, once more than 25 percent of the mass is removed from the basin, relief ratio and gradient remain essentially constant throughout an area's development. Therefore, meaningful correlations between the integral and other basin parameters are not forthcoming. Carson and Kirkby (1972) indicate that when a basin attains "maturity" the integral settles down to a value of about 50 percent and thereafter changes little, adding that this may be a property of basins, or may show the hypsometric method to be "insensitive." Chorley and Kennedy (1971) state that at a hypsometric integral of 60 percent or less, the integral more or less stabilizes itself, irrespective of the absolute relief, as the basin is geometrically transformed, suggesting that there may exist a multitude of forms for any particular integral. The hypsometric method, however, has a much greater potential for quantitative landform analysis than can be realized through employment of the integral value alone.

Unlike the integral, the functional form of the curve is unique to a particular area, depicting, among other things, evolutionary changes in the form of the basin. Tanner (1957), Scheidegger (1961), and Evans (1972) have proposed that, since the hypsometric curve is not Gaussian, there is a significant skewness and kurtosis which can be used to characterize an area. The focus of this paper is to outline a technique to quantify the form of the curve so as to produce not only skewness and kurtosis for the curve, but also for its density function, and do it in a manner that is prohibitive in neither cost nor time. The model involves treating the "decumulation" of the hypsometric curve in its mirror image as a cumulative distribution function. Statistical moments of the curve, and expectations of (x) for the curve's density function are derived, projecting a dependent vector of curve-form attributes that can be utilized in subsequent models dealing with process-response within the drainage basin environment.

THE MODEL

Since the hypsometric curve (Fig. 1), drawn in mirror image from its usual form, has the characteristics of the cumulative distribution of a random variable (equals zero to the left of $x = 0$, equals 1 to the right of $x = 1$, and is monotonically increasing) moments and centers of gravity for the distribution can be established indicating very subtle changes in relative amounts of erosion occurring within various sections of the drainage basin. The first of two techniques to be described here treats the curve as a distribution function about which centroids for both the x- and y-axes are developed. The second, third, and fourth moments are then derived about the centroids that yield measures of skewness and kurtosis for the hypsometric curve.

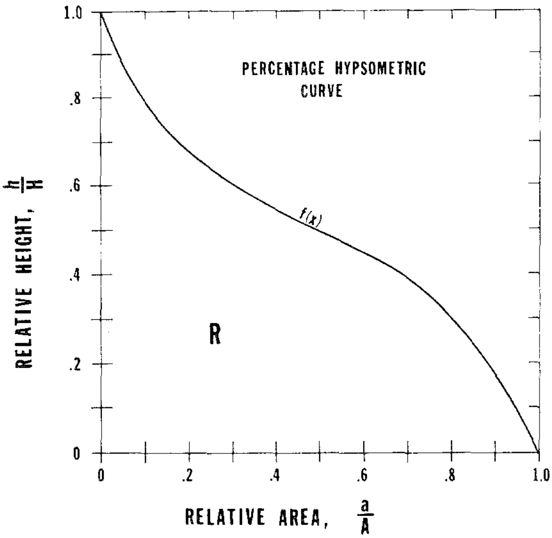

Figure 1. Hypsometric curve, after Strahler (1957).

The Hypsometric Curve as a Cumulative Distribution Function

The hypsometric curve is, by definition, monotonic and can be represented by a continuous polynomial function with the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \tag{1}$$

There are various ways in which the functional form of the curve may be generated. In this instance the polynomial is produced by polynomial regression. Allen (1975) used polynomial regression to obtain mathematical functions for beach profiles. These polynomials, termed "beach equations," were found to be extremely tractable in mathematical designs describing beach changes. The hypsometric curve may well have an advantage over other profiles when the regression format is employed, in that hypsometric curves are monotonic with few inflections. The relatively simple forms can be fitted to low-order (2nd and 3rd) polynomials which are highly significant. All hypsometric equations analyzed thus far have been significant at the 0.01 level when compared to actual plots.

Referring again to Figure 1, the hypsometric curve takes on the value $[f(x)]$ within a unit square defined by the area and altitude axes. The hypsometric integral can therefore be defined as equal to the region "R" under the function $f(x)$, i.e., define:

$$A\ (\text{area}) = \iint_R dx\, dy \tag{2}$$

Then define $f(x)$ to be the value of y on the upper boundary of R at the value x, so that R is seen to be bounded by the three curves:

$$x = 0, \qquad y = 0, \qquad \text{and } y = f(x)$$

Define:

$$\mu'_{10} = (I/A) \iint x\, dy\, dx \tag{3}$$

$$= (1/A) \int_0^1 xF(x)\, dx \tag{4}$$

where μ'_{10} equals the x-mean or centroid, and

$$\mu'_{01} = (1/A) \iint y\, dy\, dx \tag{5}$$

$$= (1/A) \int_0^1 \tfrac{1}{2}(f(x))^2\, dx \tag{6}$$

where μ'_{01} equals the y-mean, or centroid. Thereafter define:

$$\mu_{i0} = (1/A) \iint (x-\mu'_{10})^i\, dy\, dx \tag{7}$$

$$= (1/A) \int_0^1 (x-\mu^i_{10})^i \left[\int_0^{f(x)} dy\right] dx \tag{8}$$

$$= (1/A) \int_0^1 (x-\mu'_{10})^i f(x)\, dx \tag{9}$$

and

$$\mu'_{0j} = (1/A) \iint (y-\mu'_{01})^j\, dy\, dx \tag{10}$$

$$= (1/A) \int_0^1 \left[\int_0^{f(x)} (y-\mu_{01})^j\, dy\right] dx \tag{11}$$

The cumulative distribution approach, hereafter termed C.D.F., generates 9 indices of form elements—the first four moments about both the x- and y-axes, and the hypsometric integral. When the model is programmed for the final analysis the third and fourth moments about the ordinate are dropped. They add little additional information about the curve, and prove to be extremely unwieldy once expanded for programming. As previously stated, a second technique will be explored as an expression of form involving the hypsometric curve's density function.

Density Function for the Hypsometric Curve

A new function must now be developed having density equal to 1, in the form:

$$g(x) = [F(1)-F(0)]^{-1}F'(x), \qquad 0 \leqslant x \leqslant 1 \tag{12}$$

The least-square method is used to calculate the polynomials, and it has been found that no significant reduction in sum of squares occurs after the generation of third-degree equations. Therefore, $g(x)$ will be a polynomial of about order two:

$$g(x) = g_0+g_1x+g_2x^2 \tag{13}$$

The hypsometric curve is of the form

$$F(x) = f_0+f_1x+f_2x^2+f_3x^3 \tag{14}$$

Thus, given eq (4), and remembering that eq (12) will be in the form of eq (13), it must now be determined what g_0, g_1, and g_2 are as functions of f_0, f_1, f_2, and f_3.

Given eq (12), define:

$$E(x) = \int_0^1 xg(x)\, dx \tag{15}$$

which will denote the mean for a random variable X where the nth moment about that mean can now be expressed as $E[X-E(X)]^n$.

Now that the form attributes have been outlined for both the C.D.F. and the density function, there remains the problem of programming a system of rather complex functions in such a manner that renders the model an efficient and expedient tool for geomorphologists. Returns on investments of time devoted to research should be high, and initial trials of the simple technique that follows have shown the method to be a useful device involving minimal effort and expense.

THE PROGRAM

There are various approximation packages available for integration, however, with respect to the model now described, considerable computer time would be necessary for any study involving several drainage basins. Hence, a conversion follows, solving each equation as a summation expression involving only the coefficients (a_0, a_1, a_2, and a_3) obtained from the regression output allowing for exact integration. Given from eq (2):

$$\begin{aligned} A &= \iint_R dx\, dy \\ &= \int_0^1 f(x)\, dx \end{aligned} \tag{16}$$

where $f(x)$ equals a polynomial of the form:

$$a_0+a_1x+a_2x^2+a_3x^3$$

(a) define:

$$A = a_0x|_0^1+a_1x^2|_0^1+a_2x^3|_0^1+a_3x^4|_0^1 \tag{17}$$

$$= \sum_{k=0}^{3} a_k/(k+1), \qquad \text{(hypsometric integral)} \tag{18}$$

(b) define:

$$\mu'_{10} = (1/A) \int_1^0 x(a_0+a_1x+a_2x^2+a_3x^3)\, dx \tag{19}$$

$$= (1/A)[a_0x^2/2|_0^1+a_1x^3/3|_0^1+a_2x^4/4|_0^1+a_3x^5/5|_0^1] \tag{20}$$

$$= (1/A)\sum_{k=0}^{3} a_k/k+2, \qquad (x\text{-centroid}) \tag{21}$$

(c) likewise define:

$$\mu'_{20} = \left[(1/A)\sum_{k=0}^{3} a_k\Big/(k+3)\right]-\mu_1^{20}, \qquad \text{(second moment on } x\text{)} \tag{22}$$

(d) and:

$$\mu_{30} = \left[(1/A)\sum_{k=0}^{3} a_k \Big/ k+4\right] - \left[(3\mu'_{10}/A)\sum_{k=0}^{3} a_k \Big/ (k+3)\right] + 2(\mu'_{10})^3,$$

(third moment on x) (23)

(e) finally:

$$\mu_{40} = \left[(1/A)\sum_{k=0}^{3} a_k \Big/ (k+5)\right] - \left[(4\mu'_{10}/A)\sum_{k=0}^{3} a_k \Big/ (k+4)\right]$$
$$+ \left[(6/A)(\mu'_{10})^2 \sum_{k=0}^{3} a_k \Big/ (k+4)\right] - 3(\mu'_{10})^4,$$

(fourth moment on x) (24)

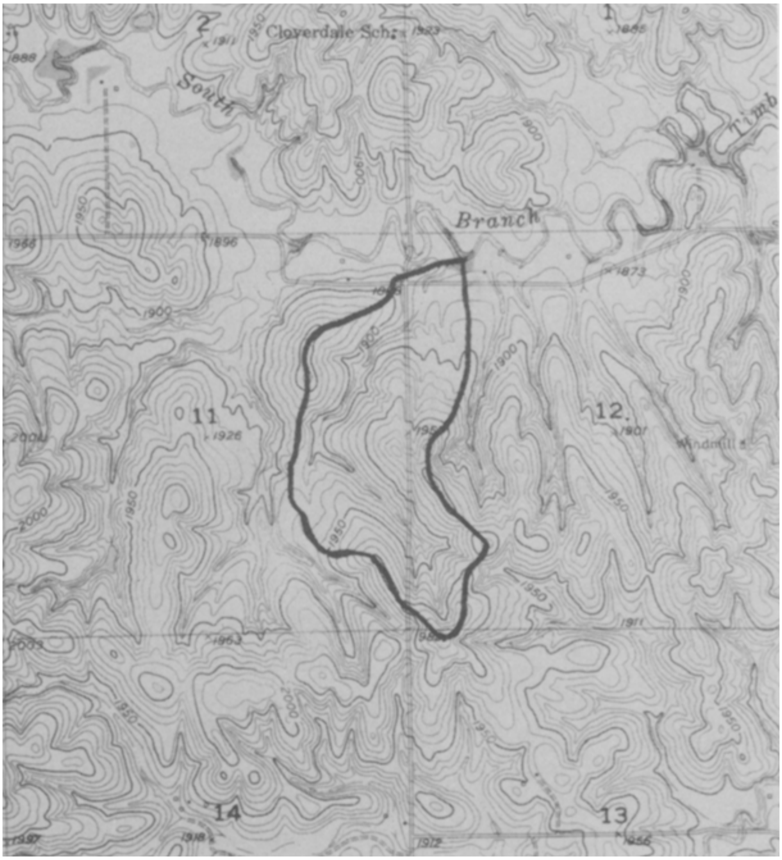

Figure 2. Wolbach, Nebraska.

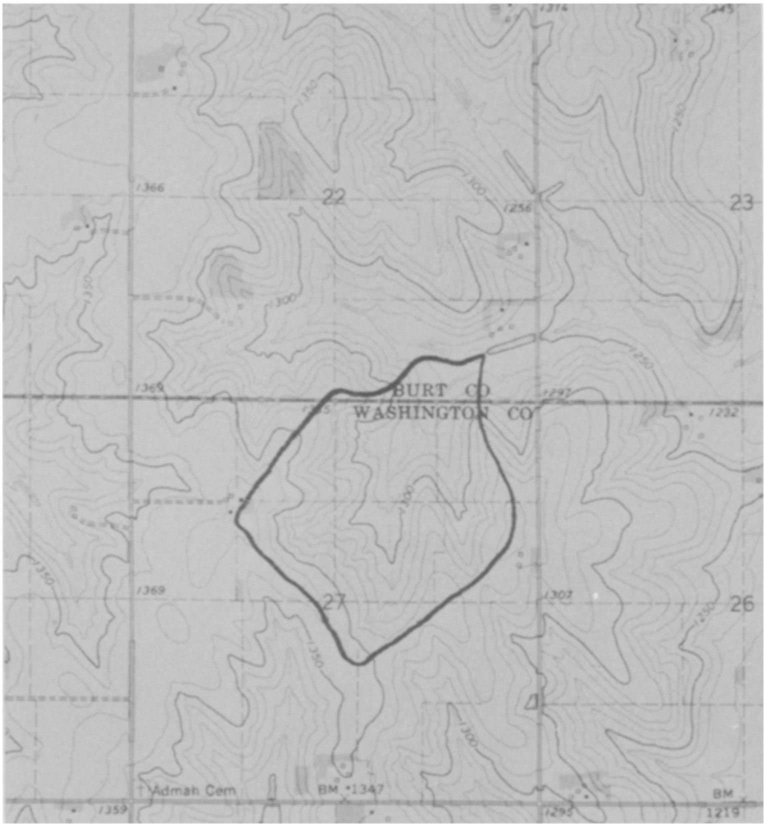

Figure 3. Nickerson, Nebraska.

In similar fashion summations were derived for the density approach. The conversion of the original equations is a rather tedious process. However, the summations now involve only the coefficients from regression, making the model easily adaptable to computer language. A program has been devised that reads any number of cubic polynomial coefficients and computes moments, skewness, and kurtosis for both the C.D.F. and density functions. The program reads each set of four coefficients using PL/1's stream input. There is no constraint upon the format of the numbers, other than they must be in the order a_0, a_1, a_2, a_3. After the moments, etc. have been computed each is printed and labeled along with the identification of the respective drainage basin. Copies of the program are available from the author.

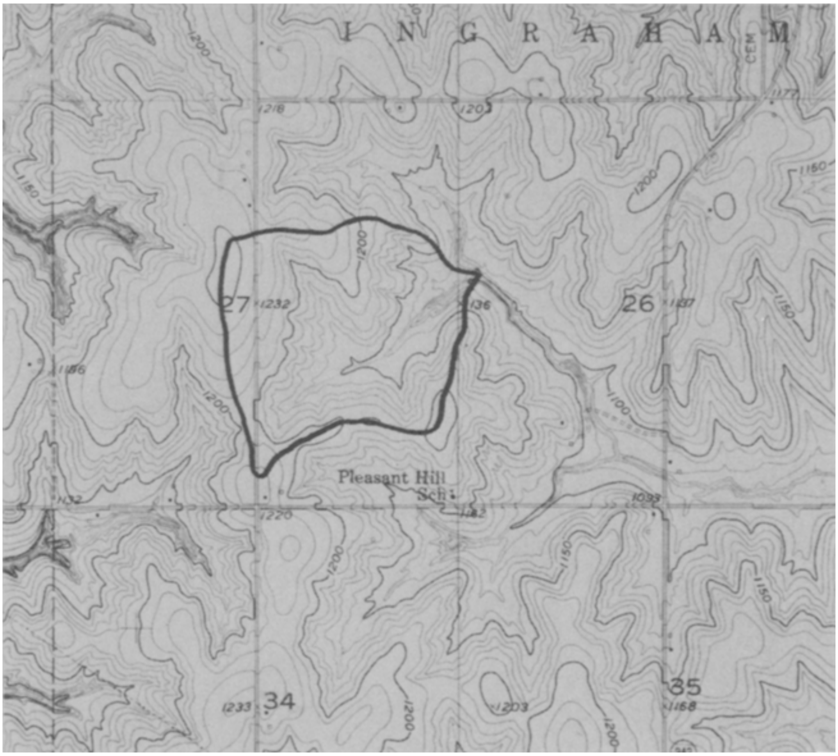

Figure 4. Glenwood, Iowa #1.

PRELIMINARY APPLICATION

Third- and fourth-order basins were chosen from upland areas in Iowa and Nebraska to test the model's ability to segregate basins that, in some cases, are very similar in form. In each case the drainage basins are carved in deep Peorian loess overlying drift topography. The maps depicted in Figures 2–6 are taken from 7.5 minute topographic sheets. References to specific drainage basins will be made as to the name of the topographic sheet from which the basins were drawn.

As a drainage basin develops, mass is removed, and the centroids for the forms shown in Figures 7–11 begin to converge on the origin, having upper limits of 0.5. Skewness will equal zero with the unit square where there is no mass removed, or where $\int_0^1 f(x)\,dx = 1$. In practical application, skewness may approach zero as the centroid approaches the median for the curve, or 0.5. The curve will become more and more positively skewed with headward development of the main stream and its tributaries as these streams encroach

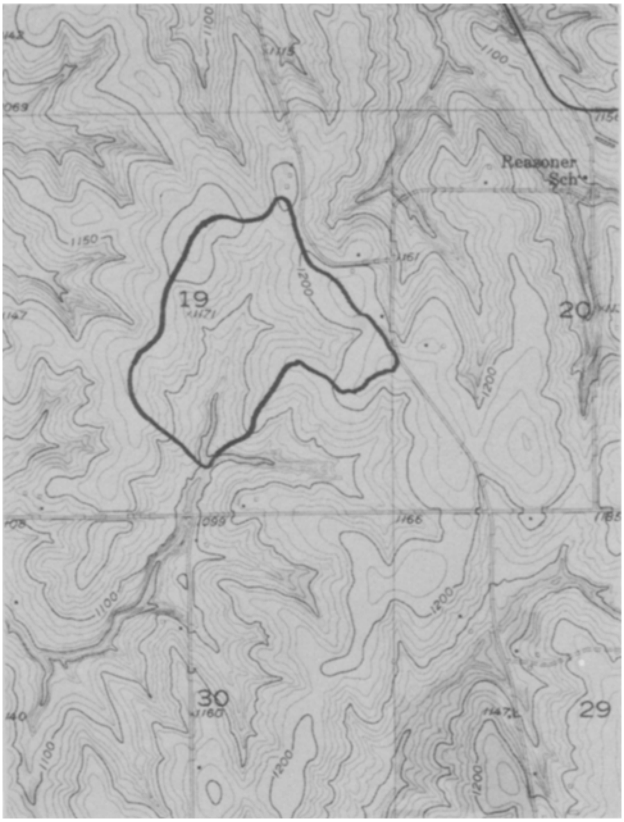

Figure 5. Glenwood, Iowa #2.

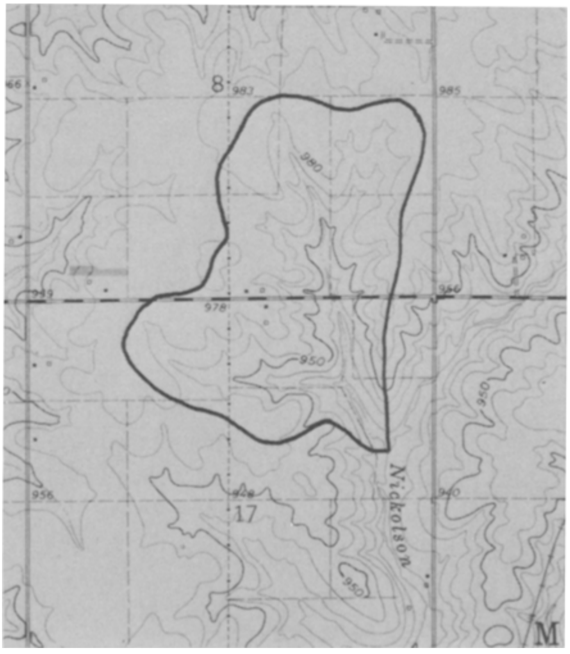

Figure 6. Legrand, Iowa.

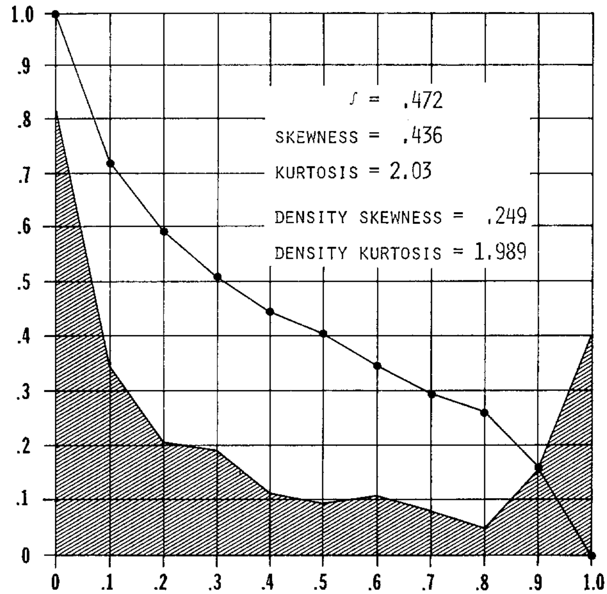

Figure 7. Hypsometric curve for Wolbach, Nebraska (shaded portion represents density of the curve).

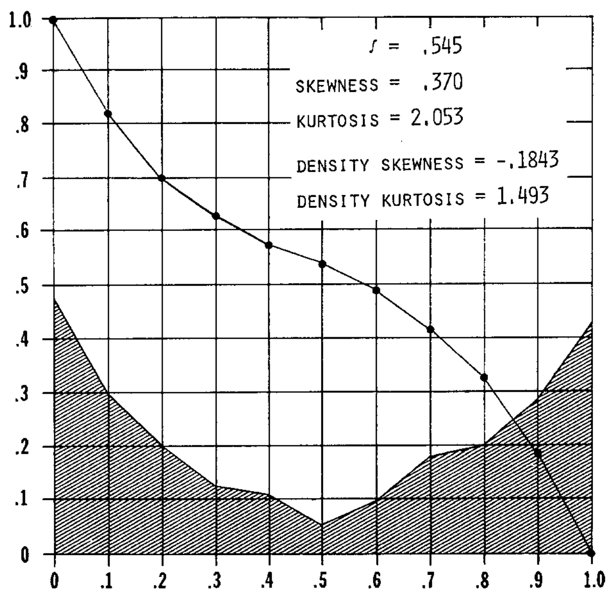

Figure 8. Hypsometric curve for Nickerson, Nabraska.

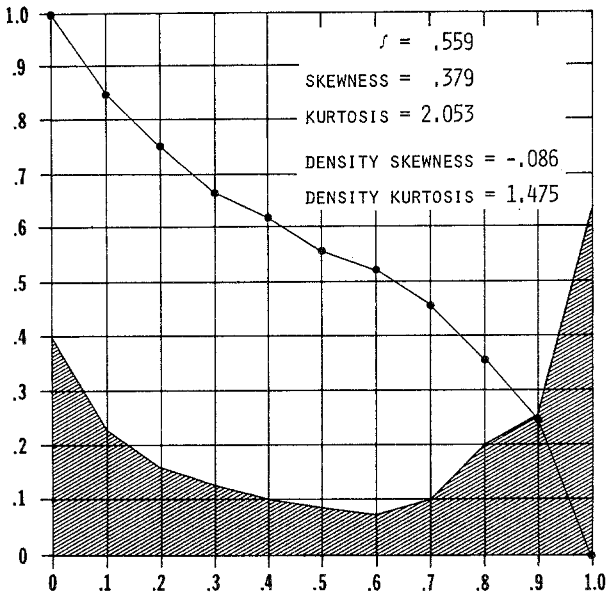

Figure 9. Hypsometric curve for Glenwood, Iowa #1.

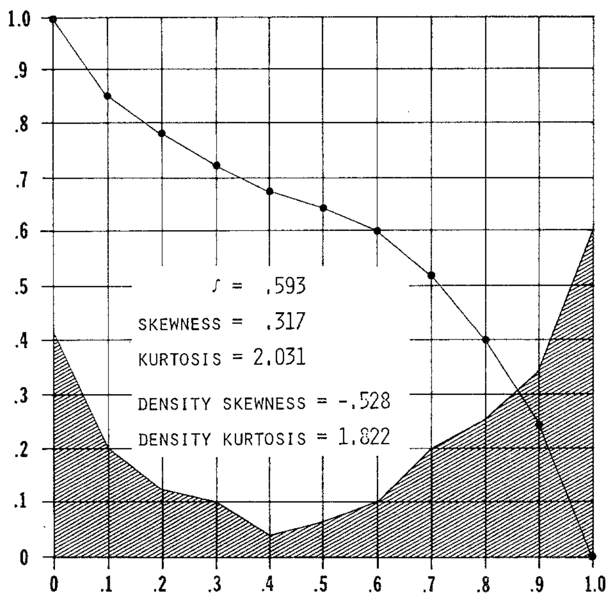

Figure 10. Hypsometric curve for Glenwood, Iowa #2.

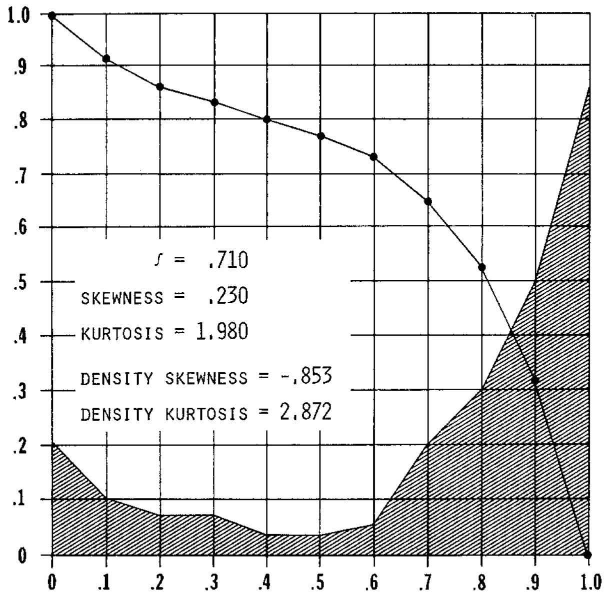

Figure 11. Hypsometric curve for Legrand, Iowa.

on flat-upland parcels in the upper reaches of the basin. When the curve demonstrates little development, as indicative of Strahler's (1952) inequilibrium phase, the curve is only slightly skewed, e.g., skewness equals 0.23 for Legrand (Fig. 11) with an integral of 0.710. As the integral decreases the skewness value becomes larger, e.g., skewness equals 0.44 for Wolbach (Fig. 7) with an integral of 0.47.

The values for skewness and kurtosis in particular seem to be more unique to specific curve forms than the integral. There is not a perfect linear relationship between skewness and the hypsometric integral. For example, Glenwood (Fig. 9) with an integral of 0.559, has skewness of 0.379. This is greater than the skewness value for Nickerson (Fig. 8) with an integral of 0.545. An explanation for this anomaly can be found in the amount of headward development that has occurred in the Glenwood basin (Fig. 4) relative to the total amount of material removed.

The ratio of a range of central values compared to a range that encompasses a large part of the distribution has been used in statistics to measure kurtosis. When the distribution is that of the hypsometric curve, and advanced erosion has occurred in both the upper and lower reaches of a basin, the value for kurtosis is relatively high, i.e., Nickerson and Glenwood, Figures 3 and 4, respectively.

The density function approach may be a more precise tool than the

C.D.F. This part of the model monitors changes (rates of change) all along the hypsometric function. Where change is rapid, density will be great. For areas demonstrating little development, hence little headward erosion and large tracts of flat-upland, change is concentrated in the lower reaches of the basin. A critical moment in basin development occurs when the density function changes from high negative skewness, as in the case of Legrand (Fig. 11), to positive skewness, as in the case of Wolbach (Fig. 7). This indicates that the inertia for erosion has shifted within the basin. Positive values would indicate when accelerated forms of erosion, like mass wasting, are more probable in the basin's upper reaches. When density skewness equals 0.0 an equal amount of change is occurring, or has occurred, in the upper and lower reaches. Further research may show that this is an indication of equilibrium.

CONCLUDING REMARKS

The true potential of the hypsometric method in quantitative landform analysis cannot be realized through application of the integral value alone. Indices pertaining to the form of the hypsometric curve, from both the theoretical and practical points of view, are more unique to an area than is the hypsometric integral.

The time and effort necessary for a statistical analysis of the hypsometric curve is no longer a prohibitive factor in research involving drainage basins. The method of curve-form description outlined here is readily accessible through a short, 67-step computer program. The program prints out moments, values of skewness and kurtosis for the hypsometric curve, and corresponding hypsometric density function, for less than 3 cents per basin.

The technique is especially applicable to small basins that demonstrate little variation in form and perhaps no variation in the integral. The original data is not difficult to produce for smaller drainage basins. The time-consuming process of collecting hypsometric data is one reason the technique may have originally gained little acceptance. However, as stated above small basins do not require a great deal of time, and the resulting vector of unique basin attributes provide much higher returns for time spent over topographic maps. With digitizers appearing in many departments even very large drainage basins can be processed with little effort.

In summary, preliminary analysis has shown the model to be a rigorous tool, sensitive to subtle differences in hypsometric form, providing the geomorphologist greater latitude when the hypsometric method is utilized in multivariate designs.

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