CONVECTIVE INSTABILITY OF A ROTATING FLUID

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Izv. AN SSSR. Mekhanika Zhidkosti i Gaza, Vol. 4, No. 6, pp. 88–93, 1969

A physical system may be in thermodynamic equilibrium when participating as a whole in uniform rotational motion [1]. In particular, mechanical equilibrium of a liquid in a cavity rotating about a stationary axis with the constant angular velocity Ω ("solid-body" rotation of the liquid) is possible. If the liquid is uniform in composition and isothermal, then such equilibrium, as shown in [2], is stable for all Ω. However, in the case of a nonuniformly heated liquid, stability of the solid-state rotation is, generally speaking, impossible.

The appearance of two steady-state force fields is associated with uniform rotation: the centrifugal field and the Coriolis force field. The former field forces the liquid elements which are less heated and therefore more dense to move away from the axis of rotation, displacing the less dense liquid layers (centrifugation). If we maintain in the liquid a temperature gradient which prevents the establishment of equilibrium stratification of the liquid, then with a suitable value of this gradient (the magnitude obviously depending on Ω) undamped flows—convection—will develop in the liquid. Thus, while in conventional gravitational convection the gravity field is the reason for the appearance of the Archimedes buoyant forces, in the rotating cavity the mixing of the nonuniformly heated liquid is caused by the centrifugal field. As soon as the convective flows arise the Coriolis forces come into play. Account for the latter, as is shown below, prevents reducing in a trivial fashion the study of convective stability of rotating liquid to the well-studied problems of gravitational convection.

1. Assume that a liquid whose density is

$$\rho = \rho_0(1-\beta T) \qquad (1.1)$$

fills a cavity rotating with the constant angular velocity

$$\mathbf{\Omega} = \Omega\boldsymbol{\gamma}, \qquad \boldsymbol{\gamma}^2 = 1 \qquad (1.2)$$

If the temperature T, measured from an arbitrary zero, is not uniform throughout the liquid volume, then, generally speaking, motion develops in the cavity. In a reference system rotating with the angular velocity Ω the convection equations have the form

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho_0}\nabla\left(p - \frac{\rho_0\Omega^2 r^2}{2}\right) + \nu\nabla^2\mathbf{v} - r\Omega^2\beta T\mathbf{n} + 2\Omega(\mathbf{v}\times\boldsymbol{\gamma}).$$

$$\frac{\partial T}{\partial t} + \mathbf{v}\nabla T = \chi\nabla^2 T, \qquad \operatorname{div}\mathbf{v} = 0. \qquad (1.3)$$

Here **v** is the liquid velocity relative to the boundaries of the cavity, p is pressure, and **n** is the unit vector along the cylindrical coordinate r, measured from the axis of rotation. The last two terms in the first equation, when multiplied by ρ_0, yield the intensity of the buoyant (Archimedes) force in the centrifugal field and the intensity of the Coriolis force.

It is easy to see from (1.3) that in the steady equilibrium state (**v** = 0 and all quantities independent of time) the temperature T_0 and pressure p_0 satisfy the equations

$$\nabla^2 T_0 = 0, \qquad \nabla T_0 \times \boldsymbol{\gamma} = 0, \qquad \nabla p_0 = \rho\Omega^2 r\mathbf{n}. \qquad (1.4)$$

This implies that in equilibrium the temperature and density gradients are perpendicular to the axis of rotation and the centrifugal pressure is balanced by the hydrostatic pressure.

Let us study the stability of the equilibrium of a thin cylindrical layer of liquid bounded by solid surfaces of radii r_1 and r_2 ($r_2 - r_1 = \delta \ll r_1$). The equilibrium temperature differential between the boundaries of the layer is $\theta = T_2 - T_1 > 0$. Both cylindrical surfaces rotate with the same angular velocity (1.2) about the common axis of symmetry (the z-axis; the unit vector γ is directed along this axis). In the gravity field **g** the applicability of (1.3) to the problem in question is limited to the case of large centrifugal accelerations

$$r_1\Omega^2 \gg g. \qquad (1.5)$$

If the reverse inequality is satisfied, in the right-hand side of (1.3.1) we must include the term $-g\beta T$ and drop the terms which are quadratic in Ω. The Coriolis forces are linear in Ω and must be retained. The influence of the Coriolis forces on gravitational convection was examined in [3, 4].

The equilibrium temperature gradient in the thin liquid layer is determined from (1.4.1):

$$\nabla T_0 = \frac{\theta}{\delta}\mathbf{n} \qquad (\delta \ll r_1). \qquad (1.6)$$

We choose the units of distance δ, velocity ν/δ, temperature $\nu\theta/\chi$, and pressure $\rho_0\nu^2/\delta^2$. Linearizing (1.3), we write the equations for the critical perturbations in dimensionless form:

$$0 = -\nabla p + \nabla^2\mathbf{v} - RT\mathbf{n} + D(\mathbf{v}\times\boldsymbol{\gamma}),$$

$$0 = \nabla^2 T - \mathbf{nv}, \quad \operatorname{div}\mathbf{v} = 0$$

$$(R = r_1\Omega^2\beta\theta\delta^3/\nu\chi, \quad D = 2\Omega\delta^2/\nu). \qquad (1.7)$$

Two dimensionless parameters appear in these equations: the Rayleigh number R and the Taylor number D^2. We note that both parameters contain the angular velocity of rotation. This is associated with the fact that Ω appears in the expressions for the Archimedes and Coriolis forces and therefore plays a dual role. On the one hand the rotation is the cause of the centrifugal forces, without which convection would be impossible, and on the other hand the Coriolis forces which develop with rotation affect the structure of the critical perturbations, which leads in the final analysis to greater stability.

2. The solution of the convection equations (1.7), periodic in φ and z, is sought in the form

$$v_r = v(r)e^{im\varphi}e^{ikz},$$

$$v_\varphi = u(r)e^{im\varphi}e^{ikz}, \qquad v_z = w(r)e^{im\varphi}e^{ikz},$$

$$p = p(r)e^{im\varphi}e^{ikz}, \qquad T = \tau(r)e^{im\varphi}e^{ikz}. \qquad (2.1)$$

Substituting (2.1) into (1.7) and excluding p and w, we obtain

$$L^2 v + isDv' + a^2 R\tau - a^2 Du = 0,$$

$$a^2 Lu - k^2 Dv - isLv' = 0, \qquad L\tau - v = 0,$$

$$(L \equiv d^2 / dr^2 - a^2,\ a^2 \equiv s^2 + k^2,\ s \equiv m\delta / r_1). \qquad (2.2)$$

In writing these equations we have made significant use of the thinness of the liquid layer. In particular,

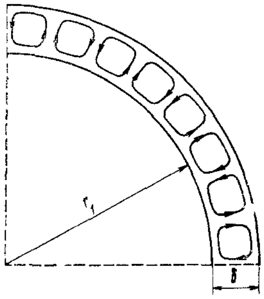

Fig. 1

terms of the type $1/r$, which appear in the exact equations along with $d(\cdot)/dr$, have been dropped everywhere, since their ratio is of order $\delta/r_1 \ll 1$.

The velocity perturbations must disappear at the solid boundaries of the layer. With account for (1.7.3) we have

$$v = v' = u = 0 \quad \text{for } r = r_1,\ r = r_2. \qquad (2.3)$$

We indicate the exact solution of (2.2) for the case in which the surfaces of the liquid layer are maintained at constant temperatures, i.e., $\tau = 0$ at the boundaries of the layer. In this case it is convenient to exclude u and τ from (2.2), which leads to a sixth-order equation for the radial component:

$$L^3 v + a^2 Fv = 0, \qquad F \equiv R - (k^2 / a^2) D^2. \qquad (2.4)$$

The solution of this equation must satisfy the conditions

$$v = v' = L^2 v = 0. \qquad (2.5)$$

at the boundaries of the liquid layer. The last condition follows from (2.2.1). The eigenvalues $F(a^2)$ of the boundary value problem (2.4), (2.5) define the connection between the parameters R and D for fixed values of a and k:

$$R = F(a^2) + (k^2 / a^2) D^2. \qquad (2.6)$$

Minimization of the function $R(a^2, k^2)$ makes it possible to determine the lower threshold value R_0 of the Rayleigh number and the form of the critical perturbation which disrupts the solid-body rotational stability. We see from (2.6) that the minimum $R = R_0$ is reached for $k = 0$ and is independent of D. Finding R_0 thus reduces to finding the minimum of the function $F(a^2)$. However, there is no need to do this, since for $k = 0$ the boundary value problem in question coincides with the problem of finding the equilibrium stability of a plane horizontal fluid layer which is heated from below in a gravity field [5]. Using the results of [5], we have $R_0 = 1708$, $a_0 = 3.13$. The basic critical motion is a system of two-dimensional rolls which fill the cylindrical layer and are oriented along the axis of rotation (Fig. 1). This figure shows a cross section of the cylindrical layer and indicates the streamlines of the basic critical motion.

The number of rolls is $2m = 2a_0 r_1/\delta$. Since a_0 is close to π, the width of each roll is close to the layer thickness δ.

We note that a similar sort of motion develops in a plane layer of conducting liquid heated from below in a longitudinal magnetic field [6]. Just like the Coriolis forces in the present problem, the longitudinal field does not affect the stability (the critical Rayleigh number is independent of the field intensity), but it does alter the form of the critical motion: in place of the usual Benard convective cells there appear rolls which are stretched out along the direction of the field.

The critical Rayleigh number is also independent of the Taylor number when the layer boundaries are thermally insulated ($\tau' = 0$). In this case the exact solution yields $R_0 = 720$, $a_0 = 0$.

These two exact solutions correspond to the two limiting cases in which the temperature perturbations either damp out completely at the layer boundaries ($\tau = 0$) or they do not decay at all ($\tau' = 0$). In the intermediate cases the boundary conditions for the temperature may be written in the form

$$\frac{\partial\tau}{\partial n} = -\lambda\tau \qquad (\infty \geqslant \lambda \geqslant 0). \qquad (2.7)$$

Here the differentiation is performed along the direction of the outward normal to the surfaces of the liquid layer. The parameter λ characterizes the rate of decay of the temperature perturbations at the layer boundaries. For $\lambda = \infty$ and $\lambda = 0$, condition (2.7) becomes the limiting cases $\tau = 0$ and $\tau' = 0$ of the boundary conditions considered above.

The boundary value problem (2.2), (2.3), (2.7) was solved by the Bubnov-Galerkin method. The approximating functions used were

$$v = c_1 x^2 (1 - x)^2, \qquad u = c_2 x (1 - x),$$

$$\tau = c_3 [1 + \lambda x (1 - x)] \qquad (0 \leqslant x \leqslant 1). \qquad (2.8)$$

The origin of the dimensionless coordinate x was located at the inner surface of the layer. The standard solution procedure leads to a system of three algebraic equations for the amplitudes c_i, for which the solvability condition yields

$$R = F(a^2, \lambda) + \frac{k^2}{a^2} G(a^2, \lambda) D^2,$$

$$F = \frac{28(504 + 24a^2 + a^4)B}{3a^2}, \qquad G = \frac{9B}{10 + a^2},$$

$$B = \frac{10\lambda(6 + \lambda) + a^2(30 + 10\lambda + \lambda^2)}{(14 + 3\lambda)^2}. \qquad (2.9)$$

Hence we see that the critical value R_0 again corresponds to $k = 0$ and is independent of D.

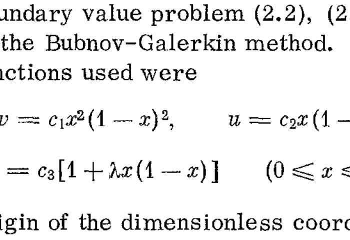

Fig. 2

The results of the calculation, made using the formula

$$R_0(\lambda) = \min F(a^2, \lambda), \qquad (2.10)$$

are shown in Fig. 2, where the ordinate is $a^* = a_0^2(\lambda)/a_0^2(\infty)$, $R^* = R_0(\lambda)/R_0(\infty)$.

For the limiting values of λ we obtain $a_0(\infty) = 3.12$, $R_0(\infty) = 1748$ and $R_0(0) = 720$. Comparing this with the exact solution, we conclude that the error in determining R_0, associated with the approximate nature of solution (2.8), does not exceed 2.5% over the entire range of values of λ.

The Taylor number appears in expressions (2.6) and (2.9) for the critical Rayleigh number in the combination (kD^2), where k is the wavenumber defining the perturbation wavelength along the cylinder axis. The basic critical motion with k = 0 can be realized only in a cylindrical layer of infinite length. In a layer of finite length h the perturbation wavelength $2l$ cannot exceed twice the layer length. If there are velocity nodes at the ends of the layer, then in the general case an integral number of half-waves must fit into the cylinder length. This requirement leads to a discrete spectrum of values of k.

However, with increase of k the critical R increase rapidly, and therefore we would expect experimental realization of only the initial portion of the k spectrum. Since $l \leqslant h$, then $k = \pi\delta/l \geqslant \pi\delta/h$. Assuming that $a \sim \pi$, for the thin layer $\delta \ll h$ we obtain $k \ll a$. For values of D which are not too large we can neglect terms with (kD^2) in minimizing the right-hand side of (2.9). The $R_0(\lambda)$ and $a_0(\lambda)$ which are then obtained (Fig. 2) are independent of D, so that the critical Rayleigh number R defined by (2.9) increases linearly with the Taylor number D^2. The rate of growth of R decreases for large values of D and as $D \to \infty$ minimization of the entire expression (2.9) yields $a_0^2 \sim (kD)^{2/3}$, $R \sim (kD)^{4/3}$.

3. The experiments were conducted with distilled water filling a cylindrical slot of length either 100 or 50 mm between two coaxial hollow plexiglass cylinders. The cylinders were rigidly interconnected and formed the rotating portion (rotor) of the setup. The liquid layer thickness was $\delta = 1.42$ mm and the inner radius was $r_1 = 27.5$ mm. The uniform rotational velocity was measured by a ST-MEI strobotach and could be varied in the 1300–5000-rpm range. For these velocities the centrifugal acceleration varied from 55 g to 800 g, so that condition (1.5) was well satisfied. The temperature difference across the boundaries of the liquid layer was measured by a thermopile consisting of 12 fine copper-constantan thermocouples whose junctions were located at the walls of the layer at equal distances from one another along the circumference. Thermocouples were also used to

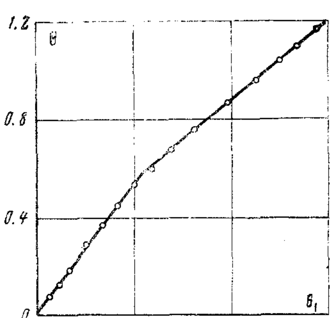

Fig. 3

measure the radial temperature difference in the rotor walls. The leads of all the thermocouples were bonded flush with the walls in grooves along the cylinder generators and were brought out to a slip-ring assembly. The radial heat flux through the rotor walls was created by a stationary heater and cooler. The latter consisted of water jackets in the form of hollow coaxial cylinders between which the rotor turned. The heater and cooler were fed from two jet-type ultra-

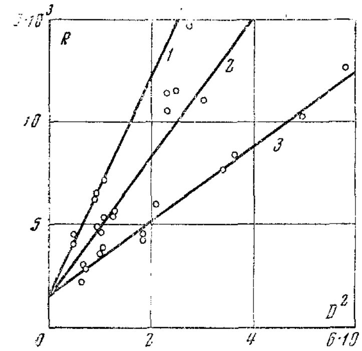

Fig. 4

thermostats. Each of the thermostats was equipped with an automatic programmer which made it possible to vary the temperature of the thermostatted liquid smoothly.

In the beginning of the experiment a steady-state radial temperature differential which obviously exceeded the critical value was created in the rotating rotor. Then this temperature difference was gradually reduced to zero in accordance with a set program which ensured quasi-steadiness of the process. A Kurnakov FPK-59 photoelectric pyrometer was used to record automatically the time variation of the temperature difference θ across the boundaries of the liquid layer and the radial temperature difference θ_1 in the rotor walls. A graph of $\theta = f(\theta_1)$ was plotted from measurements of the photorecording; Fig. 3 shows one such graph. The results were obtained with the rotor vertical, Ω = 1850 rpm, h = 50 mm. The critical temperature difference across the boundaries of the layer was determined from the break in the curve, which is caused by the change of the heat transfer regime after termination of convection.

The experimental points obtained for different values of Ω and h are grouped on the D^2R plane around three straight lines which diverge fanwise (Fig. 4). The lines were calculated with (3.2) and correspond to the following values of l: 1–2, 2–2.5, 3–3.33 cm. The continuations of all the straight lines intersect one another and the R-axis at the point $R_0 = 1460 \pm 50$. According to (2.10) this value of R_0 corresponds to λ = 5, $a_0 = 2.78$. Substituting these values into (2.9), we have

$$R = 1460 + 0.106k^2D^2. \qquad (3.1)$$

For comparison with experiment it is convenient to write this formula in the form

$$R = 1460 + \frac{0.0210}{l^2}D^2, \qquad (3.2)$$

where $2l$ is the perturbation wavelength in centimeters. As noted previously, the ratio h/l must be an integer. Comparison of the experimental curves with the theoretical curves (curves 1, 2, and 3), constructed with the use of (3.2), makes it possible to conclude that critical flows with l = 2, 2.5, and 3.33 mm for h = 10 cm, and with l = 2.5 cm for h = 5 cm were realized in the experiments. The largest-scale ($l \approx h$) possible flows, corresponding to the minimum R, were not realized in the experiments. A possible reason for this was the disturbance of the temperature field in the liquid layer by the thermocouple junctions. These junctions were located at a distance 2h/5 from the end of the layer.

Figure 5 shows on a logarithmic scale the theoretical straight line $A = D^2$, where $A = l^2(R - 1460/0.021$. This same figure shows the experimental points 1, 2, and 3 associated with the values l = 2, 2.5, and 3.33 cm. The results of measurements obtained with the rotor horizontal are indicated by the open points, those for the vertical position are indicated by the filled points. The satisfactory agreement of the theoretical and experimental data indicates the correctness of the comparison made. One of the reasons for the scatter of the experimental points is the noncoincidence of the hydrodynamic boundaries of the convective cells with the solid end surfaces: at the ends

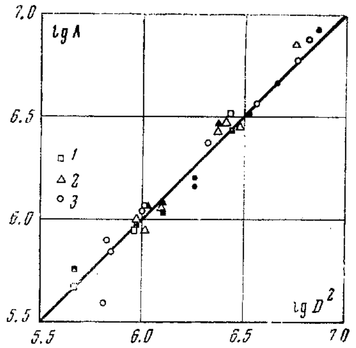

Fig. 5

of the layer the periodicity conditions used in the calculations are violated. Another possible reason for the scatter is the appearance of cells of different length because of the disturbing influence of the thermopiles.

It should be particularly emphasized that the experiments were conducted with a cylindrical layer whose length was many times greater than the thickness: $h/\delta \approx 70$. It would appear that the end effects would not be significant in such a thin layer. However, because of the strong dependence of the critical Rayleigh number on the wavenumber k the influence of the ends of the layer was dominant. As far as we know, the very important role of the longitudinal dimensions for geometrically thin liquid layers ($\delta \ll h$) has not been noted previously.

In conclusion we note that dimensional effects must also be significant in many analogous problems: for example, in studying the convective stability of a horizontal layer of a conducting liquid in a longitudinal magnetic field.

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