whose use leads to the expression (3.3).

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CRITICAL INDICES FOR MODELS WITH LONG-RANGE INTERACTION

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The leading eigenvalue and leading eigenfunction of the renormalization-group differential at a non-Gaussian fixed point are found. Expressions are obtained for the critical indices of models with long-range interaction.

Introduction

In [1,2], the Wilson equations were solved for an effective scalar Hamiltonian with free part determined by the long-range potential $U(\mathbf{x})\sim\mathrm{const}/|\mathbf{x}|^a$, $\mathbf{x}\to\infty$. This new non-Gaussian branch of fixed points of Wilson's renormalization group is separated from the Gaussian branch of fixed points at the point $a={}^3/_2 d$, and if d is not a multiple of 4 it describes the critical behavior of models with long-range interaction. The Hamiltonian has a nice representation which uses the procedure of analytic renormalization:

$$H=\ln(\mathrm{A.R.}\ {:}\exp(u(\varepsilon)\varphi^4){:}_{-\Delta(1-\chi)}),$$

where by φ^4 we denote the Hamiltonian

$$\varphi^4=\int\sigma^4(\mathbf{x})\,d^d\mathbf{x}=\int\sigma(\mathbf{k}_1)\ldots\sigma(\mathbf{k}_4)\,\delta(\mathbf{k}_1+\ldots+\mathbf{k}_4)\,d^{4d}\mathbf{k},$$

${:}\ldots{:}_{-\Delta(1-\chi)}$ is the transition to Wick polynomials with respect to the Gaussian field with propagator $-\Delta(1-\chi)(\mathbf{k})=-|\mathbf{k}|^{d-a}(1-\chi_R(\mathbf{k}))$, $\chi_R(\mathbf{k})$ is the characteristic function of the ball $\{\mathbf{k}\colon |\mathbf{k}|<R\}$, $u(\varepsilon)=u_1\varepsilon+u_2\varepsilon^2+\ldots$ is a formal numerical series in ε, and A.R. denotes a variant of analytic renormalization. This Hamiltonian can be expressed in the form of a formal power series $H=H_0+\varepsilon H_1+\varepsilon^2 H_2+\ldots$, where $\varepsilon=a-{}^3/_2 d$, d is the dimension of space.

Moscow State University. Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol.46, No.2, pp.232-241, February, 1981. Original article submitted February 26, 1980.

0040-5779/81/4602-0153$07.50

In this paper, we calculate the first two eigenfunctions (and their eigenvalues) of the differential of the renormalization group. Knowledge of the first eigenvalue makes it possible to find a closed expression for the critical index ν (in the form of a formal power series in ε) in the dimensions d = 1, 2, 3. In our theory, the index η is obvious and equal to $2 - (d/2 + \varepsilon)$. The remaining indices can be found from the relations of scaling theory (see [3-5]).

In the first and second orders of perturbation theory, the critical indices for models with long-range interaction were obtained by Fisher, Ma, and Nickel [6], Suzuki [7], Suzuki, Yamasaki, and Igarashi [8], and Ma [9].

In this paper, we study the critical indices for the non-Gaussian branch of fixed points of Wilson's renormalization group obtained in [1,2]. For the many definitions, the notation, and formulations of the theorems we refer the reader to [1,2]. The most necessary information will be recalled during the exposition.

1. Differential of the Renormalization Group at a Fixed Point

Let $H^{(0)}=\varepsilon H_1^{(0)}+\varepsilon^2 H_2^{(0)}+\ldots$ be a formal smooth Hamiltonian: $H_0 \in \mathcal{F}\mathcal{H}^\infty$. Then the action of the operator of the smoothed renormalization group (see [1]) on $H^{(0)}$ can be represented in the form

$$\mathcal{R}_{\chi,\lambda}^{(a)}(H^{(0)})=\ln : \exp \mathcal{R}_\lambda^{(a)} H^{(0)} :_{-\Delta(\chi_\lambda-\chi)} = : \exp \mathcal{R}_\lambda^{(a)} H^{(0)} :^{c}_{-\Delta(\chi_\lambda-\chi)}. \tag{1.1}$$

Here, $:\ldots:_{-\Delta(\chi_\lambda-\chi)}$ denotes the Wick operation with respect to a Gaussian measure with correlation function $\delta(\mathbf{k}_1+\mathbf{k}_2)(-\Delta(\chi_\lambda-\chi)(\mathbf{k}))$,

$$\Delta(\chi_\lambda-\chi)(\mathbf{k})=|\mathbf{k}|^{d-a}(\chi(\mathbf{k}/\lambda)-\chi(\mathbf{k})), \tag{1.2}$$

where $\chi(\mathbf{k})\in C_0^\infty(\mathbf{R}^d)$, $\chi(\mathbf{k})=\chi(|\mathbf{k}|)$,

$$\chi(\mathbf{k})\begin{cases} =0, & |\mathbf{k}|\geqslant R_1, \\ >0,\ <1, & R_1>|\mathbf{k}|>R_0, \\ =1, & R_0\geqslant|\mathbf{k}|, \end{cases}$$

$R_1 > R_0 > 0$ are numbers, and $a={}^3/_2 d+\varepsilon$. The symbol c means that only connected diagrams are taken. Finally, the scaling operator $\mathcal{R}_\lambda^{(a)}$ acts on the m-particle Hamiltonian

$$H=\int\limits_{|\mathbf{k}_i|<R} h(\mathbf{k}_1,\ldots,\mathbf{k}_m)\delta(\mathbf{k}_1+\ldots+\mathbf{k}_m)\sigma(\mathbf{k}_1)\ldots\sigma(\mathbf{k}_m)d\mathbf{k}$$

in accordance with the formula

$$\mathcal{R}_\lambda^{(a)} H=\lambda^{am/2-md+d}\int\limits_{|\mathbf{k}_i|<\lambda R} h(\mathbf{k}_1/\lambda,\ldots,\mathbf{k}_m/\lambda)\delta(\mathbf{k}_1+\ldots+\mathbf{k}_m)\sigma(\mathbf{k}_1)\ldots\sigma(\mathbf{k}_m)d\mathbf{k}, \tag{1.3}$$

and is extended by linearity to the complete space of formal Hamiltonians.

We denote by $D_{H^{(0)}}\mathcal{R}_{\chi,\lambda}^{(a)}$ the differential of the nonlinear renormalization-group transformation $\mathcal{R}_{\chi,\lambda}^{(a)}$ at the point $H^{(0)}$:

$$\mathcal{D}_{H^{(0)}}\mathcal{R}_{\chi,\lambda}^{(a)}H=\sum_{n=1}^{\infty}\frac{n}{n!}:\underbrace{\mathcal{R}_\lambda^{(a)}H^{(0)},\ldots,}_{n}\mathcal{R}_\lambda^{(a)}H^{(0)},\mathcal{R}_\lambda^{(a)}H:^{c}_{-\Delta(\chi_\lambda-\chi)}. \tag{1.4}$$

The differential at the point $H^{(0)}=0$ is given by

$$D_0\mathcal{R}_{\chi,\lambda}^{(a)}H=:\mathcal{R}_\lambda^{(a)}H:_{-\Delta(\chi_\lambda-\chi)}$$

We shall denote $D_0\mathcal{R}_{\chi,\lambda}^{(a)}$ by D_λ. Note that (1.4) can be rewritten in the form

$$D_{H^{(0)}}\mathcal{R}_{\chi,\lambda}^{(a)}H=D_\lambda(e^{H^{(0)}}H)e^{-H^{(0)}}. \tag{1.5}$$

To conclude this section, we give some information on analytic renormalization, since it will be frequently used in what follows. Namely, we shall be interested in the φ_d^4 theory with propagator $|\mathbf{k}|^{d-a}(1-\chi(\mathbf{k}))$ in the neighborhood of the points $a=a_0={}^3/_2 d$. Suppose we are given an arbitrary graph G in the φ_d^4 theory and F_G is the corresponding Feynman amplitude. The latter is a meromorphic function of $\varepsilon=a-{}^3/_2 d$. The renormalized amplitude is determined in accordance with the formula

$$\text{A. R. } F_G = \sum_{\substack{\{H_1,\dots,H_r\}\subset G\\ V(H_i)\cap V(H_j)=\phi}} F_{G|\{H_1,\dots,H_r\}} \prod_{j=1}^{r} O(H_j) \tag{1.6}$$

and is an analytic function of ε in a neighborhood of the origin. Here, the summation is over all possible sets $\{H_1,\dots,H_r\}$ of pairwise nonintersecting (in the vertices) subgraphs such that $V(H_1)\cup\dots\cup V(H_r)=V(G)$, where V(H) is the set of vertices of the graph H. $O(H)=\sum_{n=1}^{|V(H)|-1} a_n^{(H)}\left(\frac{1}{\varepsilon}\right)^n$ are certain polynomials (without free terms) in $1/\varepsilon$ of degree $|V(H)|-1$ associated with each graph, and O(H) depends only on H and not on G. An exception is the trivial subgraph of H with one vertex, for which $O(H)\equiv 1$. We recall that d is not a multiple of 4.

THEOREM 1. (See [10, 11, 12]) (additivity formula for Feynman amplitudes). With each connected graph H of the φ_d^4 theory one can associate a polynomial

$$O(H)=\sum_{n=1}^{|V(H)|-1} a_n^{(H)}\left(\frac{1}{\varepsilon}\right)^n$$

such that the renormalized amplitude $\text{A.R.}F_G$ is an analytic function of ε in some neighborhood of the origin.

We point out some additional properties of the polynomials O(H). $O(H)\equiv 0$ if: 1) H is one-particle-reducible or 2) the number of external lines of H is not equal to 4.

Suppose

$$O_1=1,\quad O_m=\sum_{G\in\mathscr{G}_m} O(\mathscr{G})=\sum_{n=1}^{m-1}\left(\frac{1}{\varepsilon}\right)^n \sum_{G\in\mathscr{G}_m} a_n^{(G)},\quad m\geqslant 2, \tag{1.7}$$

where $\mathscr{G}_m$ is the set of all the 1-particle-irreducible graphs of the φ_d^4 theory that can be constructed from the given m vertices (with four external lines).

We introduce the formal series

$$\rho(u)=\varepsilon\left\{\sum_{n=1}^{\infty}\frac{O_n}{n!}u^n\right\}\Big/\left\{\sum_{n=1}^{\infty}\frac{O_n}{(n-1)!}u^{n-1}\right\}=\sum_{n=1}^{\infty}c_n u^n. \tag{1.8}$$

It was shown in [1-2] that if the dimension d is not a multiple of 4 then $a_0={}^3/_2 d$ is a bifurcation value (see [2]), and the effective Hamiltonian is a renormalized projection Hamiltonian:

$$H=\text{A. R.}:\exp(u(\varepsilon)\varphi^4(\sigma)):^{c}_{-\Delta(1-\chi)}, \tag{1.9}$$

where $u(\varepsilon)=\sum_{j=1}^{\infty}u_j\varepsilon^j$ can be found from the equation

$$\rho(u)=0,\quad \varphi^4(\sigma)=\int\delta(\mathbf{k}_1+\dots+\mathbf{k}_4)\sigma(\mathbf{k}_1)\dots\sigma(\mathbf{k}_4)\,d\mathbf{k},\quad \Delta(1-\chi)(\mathbf{k})=|\mathbf{k}|^{d-a}(1-\chi(\mathbf{k})). \tag{1.10}$$

In other words, the Hamiltonian (1.7) with coupling constant determined by (1.10) is a fixed point of the renormalization transformation $\mathscr{R}^{\left(\frac{3}{2}d+\varepsilon\right)}_{\chi,\lambda}$ for any $\lambda\geq 1$. It is this fixed point that will interest us in what follows.

2. The Differential and Analytic Renormalization

A candidate for an even eigenfunction is the expression

$$\text{A. R.}:\varphi^2\exp(u(\varepsilon)\varphi^4):^{c}_{-\Delta(1-\chi)}, \tag{2.1}$$

Here, $\varphi^2=\int\delta(\mathbf{k}_1+\mathbf{k}_2)\sigma(\mathbf{k}_1)\sigma(\mathbf{k}_2)\,dk$. In the determination of expressions of the type (2.1), we encounter diagrams spanned by a certain number of vertices with four external lines and just one vertex with two external lines. In order to make direct use of the theorem on the additivity of analytic renormalization in the φ_d^4 theory, we shall assume that at the vertex with two external lines there is a loop. This loop changes $\varphi^2=\int\delta(\mathbf{k}_1+\mathbf{k}_2)\sigma(\mathbf{k}_1)\sigma(\mathbf{k}_2)\,dk$ by the factor $c(\varepsilon)$, which is an analytic function of ε: $c(\varepsilon)=\int\Delta(1-\chi)(\mathbf{k})\,dk$. This integral converges at large a, and at the point $a=^3/_2 d+\varepsilon$ we consider its analytic continuation, which, as is well known, is $c(\varepsilon)=\int\Delta\chi dk$. This factor is unimportant for determination of the eigenfunction.

LEMMA 1. Suppose

$$\mathscr{W}(u)=\sum_{n=1}^{\infty} O_n \frac{u_n}{n!}, \tag{2.2}$$

$$\mathscr{V}_2(u)=\sum_{n=0}^{\infty} O_n^{(2)} \frac{u^n}{n!}, \quad O_n^{(2)}=\sum_{\mathscr{G}\in\mathscr{G}_{m^2}} O(\mathscr{G}), \tag{2.3}$$

where $\mathscr{G}_m{}^2$ is the set of all graphs spanned by the m four-leg diagrams and one two-leg diagram. Then

$$\text{A. R.}: \varphi^2 \exp(u\varphi^4):^{c}_{-\Delta(1-\chi)}=\mathscr{V}_2(u):\varphi^2\exp(\mathscr{W}(u)\varphi^4):^{c}_{-\Delta(1-\chi)}. \tag{2.4}$$

Equation (2.4) is understood in the sense of equality of the formal series in u.

The proof of this lemma is similar to the proof of the theorem in [2] (see also [12]).

LEMMA 2. The Wick polynomials $:\varphi^2(\varphi^4)^n:_{-\Delta(1-\chi)}$ are eigenfunctions of the differential D_λ:

$$D_\lambda :\varphi^2(\varphi^4)^n:_{-\Delta(1-\chi)}=\lambda^{d/2+\varepsilon+2n\varepsilon}:\varphi^2(\varphi^4)^n:_{-\Delta(1-\chi)}. \tag{2.5}$$

This lemma can be proved by direct verification.

LEMMA 3. Suppose $H^{(0)}=\text{A.R.}:\exp(u\varphi^4):^{c}_{-\Delta(1-\chi)}$. Then

$$D_{H^{(0)}}\mathscr{R}^{(a)}_{\chi,\lambda}(\text{A. R.}:\varphi^2\exp(u\varphi^4):^{c}_{-\Delta(1-\chi)})=\lambda^{d/2+\varepsilon}\mathscr{V}_2(u)\frac{:\varphi^2\exp(\mathscr{W}(u)\lambda^{2\varepsilon}\varphi^4):_{-\Delta(1-\chi)}}{\text{A. R.}:\exp(u\varphi^4):_{-\Delta(1-\chi)}}. \tag{2.6}$$

The equality is understood in the sense of equality of the formal series in u.

Proof. Note that

$$\text{A. R.}:\varphi^2\exp(u\varphi^4):^{c}_{-\Delta(1-\chi)}\text{A. R.}:\exp(u\varphi^4):_{-\Delta(1-\chi)}=\text{A. R.}:\varphi^2\exp(u\varphi^4):_{-\Delta(1-\chi)}.$$

Indeed,

$$\text{A. R.}:\varphi^2\exp(u\varphi^4):^{c}_{-\Delta(1-\chi)}=\frac{d}{dv}\text{A. R.}:\exp(u\varphi^4+v\varphi^2):^{c}_{-\Delta(1-\chi)}|_{v=0}=$$

$$\frac{d}{dv}\ln\text{A. R.}:\exp(u\varphi^4+v\varphi^2):_{-\Delta(1-\chi)}|_{v=0}=\frac{\text{A. R.}:\varphi^2\exp(u\varphi^4):_{-\Delta(1-\chi)}}{\text{A. R.}:\exp(u\varphi^4):_{-\Delta(1-\chi)}}.$$

Therefore

$$D_{H^{(0)}}\mathscr{R}^{(a)}_{\chi,\lambda}(\text{A. R.}:\varphi^2\exp(u\varphi^4):^{c}_{-\Delta(1-\chi)})=\frac{D_\lambda(\text{A. R.}:\varphi^2\exp(u\varphi^4):^{c}_{-\Delta(1-\chi)}\text{A. R}:\exp(u\varphi^4):_{-\Delta(1-\chi)})}{\text{A. R.}:\exp(u\varphi^4):_{-\Delta(1-\chi)}}=$$

$$\frac{D_\lambda(\text{A. R.}:\varphi^2\exp(u\varphi^4):_{-\Delta(1-\chi)})}{\text{A. R.}:\exp(u\varphi^4):_{-\Delta(1-\chi)}}$$

By Lemmas 1 and 2,

$$D_\lambda(\text{A.R.}:\varphi^2\exp(u\varphi^4):_{-\Delta(1-\chi)})=D_\lambda(\mathscr{V}_2(u):\exp(\mathscr{W}(u)\varphi^4)\varphi^2:_{-\Delta(1-\chi)})=\lambda^{d/2+\varepsilon}\mathscr{V}_2(u):\varphi^2\exp(\lambda^{2\varepsilon}\mathscr{W}(u)\varphi^4):_{-\Delta(1-\chi)}.$$

This proves Lemma 3.

In what follows, we shall use as parameter of the renormalization group the variable τ: $\lambda=\exp(\tau/2)$.

THEOREM 2. Suppose $\rho(u)$ is given by (1.8). Let $H^{(0)}=\text{A.R.}:\exp(u\varphi^4):^{c}_{-\Delta(1-\chi)}$. Then

$$D_{H^{(0)}}\mathscr{R}^{(a)}_{\chi,\lambda}(\text{A. R.}:\varphi^2\exp(u\varphi^4):^{c}_{-\Delta(1-\chi)})=\left\{\exp\left[\frac{\tau}{2}\left(\frac{d}{2}+\varepsilon\right)\right]\mathscr{V}_2(u)\right\}\times$$

$$\left\{\exp\left(\tau\rho\frac{d}{du}\right)\left(\frac{1}{\mathscr{V}_2(u)}\text{A. R.}:\exp(u\varphi^4)\varphi^2:_{-\Delta(1-\chi)}\right)\right\}\Big/\ \{\text{A. R.}:\exp(u\varphi^4):_{-\Delta(1-\chi)}\}. \tag{2.7}$$

Proof. Indeed,

$$:\exp(\mathscr{W}(u)e^{\tau\varepsilon}\varphi^4)\varphi^2:_{-\Delta(1-\chi)}=\sum_{n=1}^{\infty}\frac{(e^{\varepsilon\tau}\mathscr{W})^n}{n!}:(\varphi^4)^n\varphi^2:_{-\Delta(1-\chi)}.$$

As in [2], we use the identity $(\exp(\varepsilon\tau)\mathscr{W}(u))^n=\exp(\tau\rho d/du)\mathscr{W}^n(u)$. Then

$$:\exp(\mathscr{W}(u)e^{\varepsilon\tau}\varphi^4)\varphi^2:=\exp\left(\tau\rho\frac{d}{du}\right)\sum_{n=1}^{\infty}\frac{\mathscr{W}^n}{n!}:(\varphi^4)^n\varphi^2:_{-\Delta(1-\chi)}=\exp\left(\tau\rho\frac{d}{du}\right)\left(\frac{1}{\mathscr{V}_2(u)}\text{A. R. }:\exp(u\varphi^4)\varphi^2:_{-\Delta(1-\chi)}\right),$$

which proves the theorem.

Consider the expression

$$\frac{\mathscr{V}_2'(u)}{\mathscr{V}_2(u)}=\left\{\sum_{n=1}^{\infty}\frac{O_n^{(2)}}{(n-1)!}u^{n-1}\right\}\Big/\left\{\sum_{n=0}^{\infty}\frac{O_n^{(2)}}{n!}u^n\right\}=\sum_{n=0}^{\infty}c_n^{(2)}u^n.$$

Here, the coefficients $O_n^{(2)}$ are polynomials in $1/\varepsilon$, and therefore, in general, the coefficients $c_n^{(2)}$ are also polynomials in $1/\varepsilon$.

We consider the product of two formal series in u:

$$\eta_2(u)=-\rho(u)\frac{\mathscr{V}_2'(u)}{\mathscr{V}_2(u)}=-\left(\sum_{n=1}^{\infty}c_nu^n\right)\left(\sum_{n=1}^{\infty}c_n^{(2)}u^n\right)=\sum_{n=1}^{\infty}r_nu^n. \tag{2.8}$$

In general, the coefficients r_n are also polynomials in $1/\varepsilon$. A remarkable circumstance is that the coefficients r_n do not depend on ε and are constants.

THEOREM 3. The quantities r_n, $n = 1, 2, \ldots$, are constants that do not depend on ε.

Proof. We shall prove this by induction on n in the same way as Theorem 2.5 is proved in [2]. It is readily seen that $r_1=-c_1c_0^{(2)}=-{}^1/_3$, so that the basis of the induction is established.

Suppose the coefficients $r_1,\ldots,r_{n-1}$ are analytic in ε (i.e., are constants). $h_2=\text{A.R. }:\exp(u\varphi^4)\varphi^2:^c_{-\Delta(1-\chi)}$ is analytic in ε in the neighborhood of the origin (see Theorems 2.2-2.4 in [1,2]). The differential of the renormalization group is also analytic in ε, and therefore the expression

$$D_{H(0)}\mathscr{R}^{\left(\frac{3}{2}d+\varepsilon\right)}_{\chi,\exp(\tau/2)}\text{A. R. }:\varphi^2\exp(u\varphi^4):^c_{-\Delta(1-\chi)}$$

is analytic in ε in the neighborhood of the origin. From this fact we derive the analyticity of c_n. By Theorem 1,

$$D_{H(0)}\mathscr{R}^{(a)}_{\chi,e^{\tau/2}}(\text{A. R.}:\varphi^2\exp(u\varphi^4):^c_{-\Delta(1-\chi)})=\left\{\exp\left[\frac{\tau}{2}\left(\frac{d}{2}+\varepsilon\right)\right]\mathscr{V}_2(u)\exp\left(\tau\rho\frac{d}{du}\right)\times\right.$$

$$\left.\left(\frac{1}{\mathscr{V}_2(u)}\text{A. R.}:\exp(u\varphi^4)\varphi^2:_{-\Delta(1-\chi)}\right)\right\}\Big/\{\text{A. R}:\exp(u\varphi^4):_{-\Delta(1-\chi)}\}.$$

We shall be interested in the series

$$\mathscr{V}_2(u)\exp\left(\tau\rho\frac{d}{du}\right)\left(\frac{1}{\mathscr{V}_2(u)}\text{A. R. }:\varphi^2\exp(u\varphi^4):_{-\Delta(1-\chi)}\right),$$

which is obviously analytic in ε in the neighborhood of the origin.

We consider the following operators, which act on the space of formal power series $a(u)=\sum_{j=0}^{\infty}a_ju^j$: $D=\rho(u)d/du$, $T=\mathscr{V}_2(u)$. We introduce a new notation for the coefficients of the series $\mathscr{V}_2(u)$:

$$\mathscr{V}_2(u)=\sum_{n=0}^{\infty}b_nu^n.$$

The matrix of the operator D has triangular form: $D=(d_{ij})_{i,j=0}^{\infty}$, $d_{ij}=jc_{i-j+1}$, where we assume that c_k are equal to 0 for $k \le 0$. The matrix of the operator T also has triangular form: $T=(t_{ij})_{i,j=0}^{\infty}$, $t_{ij}=b_{i-j}$, where $b_k = 0$ for $k < 1$. Clearly, T^{-1} is an operator of multiplication by $(\mathscr{V}_2(u))^{-1}$.

We denote by D_n, $(\exp(\tau D))_n$, T_n, $(T^{-1})_n$ the minors of the matrices D, $\exp(\tau D)$, T, T^{-1}, situated at the intersection of the first n + 1 rows and n + 1 columns. Because all the above matrices are triangular, we have the relations $(\exp\tau D)_n=\exp(\tau D_n)$, $(T_n)^{-1}=(T^{-1})_n$. The analyticity of all the elements of the matrix $\exp(\tau D)$ in τ also follows obviously from the triangular nature of the matrix D. From the triangular form there also follow the relations $Te^{\tau D}T^{-1}=e^{\tau TDT^{-1}}$, $T_ne^{\tau D_n}T_n^{-1}=(e^{\tau TDT^{-1}})_n$. The operator $S=TDT^{-1}$ is given by the formula

$$S = \mathscr{V}_2(u)\left(\rho(u)\frac{d}{du}\right)\frac{1}{\mathscr{V}_2(u)} = \eta_2(u)\rho(u) + \rho(u)\frac{d}{du}.$$

We calculate the dependence of the matrix of $\exp(\tau S_n)$ on r_n, $n \geq 2$ (note that $\exp(\tau S_n)$ does not depend on r_k for $k > n$). The matrix of the operator S_n has the form $S_n = (s_{ij}^{(n)})_{i,j=0}^n$, $s_{ij}^{(n)} = js_{i-j+1} + r_{i-j}$, where we assume $r_k = 0$ for $k \leq 0$. S_n depends on r_n only in one element – at the intersection of the first column and the n-th row. Suppose $F_n = (f_{ij})_{i,j=0}^n$, where $f_{n0} = 1$, $f_{ij} = 0$ for all the remaining indices. Then $S_n = (r_n + 0)F_n + S_n' = r_nF_n + S_n''$, where the matrix S_n'' does not depend on r_n. We show by induction that a similar analytic expression is also valid for the matrix S_n^k: $S_n^k = \alpha_k r_n F_n + E^{(k)}$, where the matrix $E^{(k)} = (e_{ij}^{(k)})_{i,j=0}^n$ is triangular and does not depend on r_n. It is readily verified that $F_nS_n'' = r_0F_n$, $F_nF_n = 0$, $E^{(k)}F_n = e_{n,n}^{(k)}F_n$. In addition, if $E = (e_{kl})_{k,l=0}^n$, $F = (f_{kl})_{k,l=0}^n$, $EF = (g_{kl})_{k,l=0}^n$ are triangular matrices, then $g_{nn} = e_{nn}f_{nn}$. Hence, $S_n^{k+1} = (\alpha_k r_n F_n + E^{(k)})(r_nF_n + S_n'') = (\alpha_k r_0 + e_{n,n}^{(k)})r_nF_n + E^{(k)}S_n'' = \alpha_{k+1}r_nF_n + E^{(k+1)}$, where $E^{(k+1)} = E^{(k)}S_n''$,

$$e_{n,n}^{(k+1)} = (nc_1 + r_0)e_{n,n}^{(k)}, \tag{2.9}$$

$$\alpha_{k+1} = \alpha_k r_0 + e_{n,n}^{(k)}. \tag{2.10}$$

This proves the assertion.

The solution of the recursion relations (2.9) and (2.10) with the initial conditions $e_{n,n}^{(1)} = nc_1 + r_0$, $\alpha_1 = 1$ give the result $\alpha_k = [(nc_1 + r_0)^k - (0 + r_0)^k]/[(n-1)c_1]$, i.e., $S_n^k = \{[(nc_1 + r_0)^k - (0 + r_0)^k]/[(n-1)c_1]\}r_nF_n + E^{(k)}$. Thus,

$$\exp(\tau S_n) = \sum_{k=0}^{\infty}\frac{\tau^k[(nc_1 + r_0)^k - (0 + r_0)^k]}{k!(n-1)c_1}r_nF_n + E_0 = \frac{\exp(\tau(nc_1 + r_0)) - \exp(\tau(0 + r_0))}{(n-1)c_1}r_nF_n + E_0, \tag{2.11}$$

where the matrix E_0 does not depend on r_n $c_1 = \varepsilon$ and, therefore,

$$\exp(\tau S_n) = \exp(\tau r_0)\frac{\exp(\tau n\varepsilon) - 1}{(n-1)\varepsilon}r_nF_n + E_0.$$

We now turn to the expression $D_{H^{(0)}}\mathscr{R}_{\chi, e^{\tau/2}}^{(a)}(\text{A.R.} :\varphi^2 \exp(u\varphi^4):_{-\Delta(1-\chi)})$, which we denote by h_2':

$$h_2' = \sum_{n=0}^{\infty}u^n\sum_{m=0}^{n}\frac{q_{nm}(\tau)}{m!}\text{A.R.} :\varphi^2(\varphi^4)^m :_{-\Delta(1-\chi)},$$

where $(q_{nm}(\tau))_{n,m=0}^{\infty} = Q(\tau) = \exp(\tau S)$.

As we have already noted, $\text{A.R.} :(\varphi^4)^n\varphi^2:_{-\Delta(1-\chi)}$ is analytic in ε, and so is h_2'. This last means that all the coefficients h_2' for u^j, $j = 0, 1, \ldots$, are analytic in ε. The coefficient of u^n is

$$\sum_{m=0}^{n}\frac{q_{nm}(\tau)}{m!}A.\text{R.} :\varphi^2(\varphi^4)^m :_{-\Delta(1-\chi)}. \tag{2.12}$$

All the elements $q_{nm}(\tau)$, except $q_{n0}(\tau)$, can be expressed in terms of $r_1, \ldots, r_{n-1}$, and, therefore, they are analytic in ε by the induction hypothesis. Further, in accordance with (2.11),

$$q_{n0}(\tau) = \exp(\tau r_0)\frac{\exp(\tau n\varepsilon) - 1}{(n-1)\varepsilon}r_n + q_{n1}'(\tau),$$

where the element $q_{n1}'(\tau)$ can also be expressed solely in terms of $r_1, \ldots, r_{n-1}$, and, therefore, is analytic in ε. Thus, from the analyticity of the coefficient (2.12) there follows analyticity of the term

$$\exp(\tau r_0)\frac{\exp(\tau n\varepsilon) - 1}{(n-1)\varepsilon}r_n\text{A.R.} :\varphi^2:_{-\Delta(1-\chi)}.$$

Therefore, the coefficient r_n is analytic in ε, which is what we wanted to prove. This proves Theorem 2.

3. Spectrum of the Differential of the Renormalization Group

We can now prove the following theorem.

THEOREM 3. $\text{A.R.} :\varphi^2 \exp(u(\varepsilon)\varphi^4):_{-\Delta(1-\chi)}^{c}$ is an eigenfunction of the differential of the renormalization group at the fixed point $H^{(0)} = \text{A.R.} :\exp(u(\varepsilon)\varphi^4):_{-\Delta(1-\chi)}^{c}$ with eigenvalue

$$\exp\left[\frac{\tau}{2}\left(\frac{d}{2} + \varepsilon + 2\eta_2(u(\varepsilon))\right)\right] = \lambda^{d/2+\varepsilon+2\eta_2(\varepsilon))}. \tag{3.1}$$

Here, $u(\varepsilon)$ is the coupling constant of the effective Hamiltonian.

Proof. Indeed,

$$D_{H(0)}\mathcal{R}^{(a)}_{\chi, e^{\tau/2}}\text{A. R.}: \exp(u(\varepsilon)\varphi^4)\varphi^2:^c_{-\Delta(1-\chi)} = \exp\left\{\frac{\tau}{2}\left(\frac{d}{2}+\varepsilon\right)\mathscr{V}_2(u)\right\}\times$$

$$\frac{\exp\left(\tau\rho\frac{d}{du}\right)\left(\frac{1}{\mathscr{V}_2(u)}\text{A. R.}: \varphi^2\exp(u(\varepsilon)\varphi^4):_{-\Delta(1-\chi)}\right)}{\text{A. R.}:\exp(u(\varepsilon)\varphi^4):_{-\Delta(1-\chi)}} = \exp\left\{\frac{\tau}{2}\left(\frac{d}{2}+\varepsilon\right)\right\}e^{\tau\eta_2}\frac{\text{A. R.}:\varphi^2\exp(u(\varepsilon)\varphi^4):_{-\Delta(1-\chi)}}{\text{A. R.}:\exp(u(\varepsilon)\varphi^4):_{-\Delta(1-\chi)}} =$$

$$\exp\left[\frac{\tau}{2}\left(\frac{d}{2}+\varepsilon+2\eta_2(u(\varepsilon))\right)\right]\text{A. R.}:\exp(u(\varepsilon)\varphi^4)\varphi^2:^c_{-\Delta(1-\chi)}.$$

Here, we have used the circumstance that $\rho(u)$ and $\eta_2(u)$ are formal power series in u with coefficients analytic in ε (actually, they do not even depend on ε), and $u(\varepsilon)$ is a formal power series in ε, the relation $\rho(u(\varepsilon)) = 0$ holding. The theorem is proved.

Remark. One can show that the second even eigenfunction of the differential of the renormalization group (which "bifurcates" from the eigenfunction $:\varphi^4:_{-\Delta(1-\chi)}$ of the differential of the renormalization group at the Gaussian fixed point) is given by the formula $\text{A.R.}:\varphi^4\exp(u(\varepsilon)\varphi^4):^c_{-\Delta(1-\chi)}$ and the corresponding eigenvalue is $\exp[\tau(\rho'(u(\varepsilon))-\varepsilon)]=\lambda^{2(\rho'(u(\varepsilon))-\varepsilon)}$. Note that the first order in ε of $(\rho'(u(\varepsilon)) - \varepsilon)$ is equal to -2ε, i.e., the second eigenvalue is already smaller than 1, which agrees with the general dynamical picture.

The question of the leading eigenfunctions will be considered in a separate paper.

4. Critical Indices

We now return to the old parameter λ of the renormalization group. Then the leading eigenvalue can be rewritten in the form $\lambda_2=\lambda^{d/2+\varepsilon+2\eta_2}$. The eigenfunction $\text{A.R.}:\varphi^2\exp(u(\varepsilon)\varphi^4):^c_{-\Delta(1-\chi)}$ "bifurcates" from the quadratic eigenfunction $:\varphi^2:_{-\Delta(1-\chi)}$ for the differential of the renormalization group at the Gaussian fixed point, and, as is well known, it is the leading eigenvalue λ_2 that determines the values of the critical indices (see [3, 5]). We recall also that d = 1, 2, 3. For d > 4, new quadratic eigenfunctions appear, and in this case the critical indices have a more complicated construction.

The index ν is given by the formula

$$\nu=1/(d/2+\varepsilon+2\eta_2). \tag{4.1}$$

The index η, which determines the order of decrease of the correlation function, is given by definition in our model and is equal to

$$\eta=2-(d/2+\varepsilon). \tag{4.2}$$

The remaining indices α, β, and γ can be found from ν and η in accordance with scaling theory (see [3, 5, 13]). In particular,

$$\gamma=\left(\frac{d}{2}+\varepsilon\right)\Big/\left(\frac{d}{2}+\varepsilon+2\eta_2\right)=1+\frac{4}{d}\frac{\varepsilon}{3}+\left(\frac{2}{d}\right)^2\frac{2}{9}\left(2\mathfrak{A}\left(\frac{d}{2}\right)-1\right)\varepsilon^2+\ldots, \tag{4.3}$$

where $\mathfrak{A}(x)=x(\psi(1)-2\psi(x/2)+\psi(x))$, $\psi(x)=\Gamma'(x)/\Gamma(x)$.

In [6], arguments based on the Callan–Symanzik equations were used to obtain the first two orders in ε, and they agree with the first two orders in (4.3).

It is interesting to make a comparison with the results of Yukhnovskii [14]. If we set d = 3, $\varepsilon = \frac{1}{2}$, then the value of the index η, as in [14], will be 0. For $\lambda = 2$, calculation of the first eigenvalue to the second order in ε gives the result $\lambda_2=2^{2+2\eta_2}\simeq 2.82$. In [14], a numerical calculation for λ_2 gave 2.947. The values for the critical index ν are, respectively, 0.66 and 0.64.

I thank P. M. Blekher for helpful discussions.

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LOW-FREQUENCY ASYMPTOTIC BEHAVIOR OF THE GREEN'S FUNCTIONS OF DEGENERATE BOSE SYSTEMS (KINETIC APPROXIMATION)

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On the basis of Bogolyubov's idea of a reduction in the description of nonequilibrium states, a closed system of equations is obtained for investigating the behavior of a spatially inhomogeneous degenerate system of Bose particles with weak interaction in the presence of an external alternating field. The connection between the low-frequency asymptotic behavior of the Green's functions and the kinetic characteristics of the system is established.

1. Introduction

In the present paper, we obtain equations of motion for the statistical operator of a spatially inhomogeneous degenerate system of identical Bose particles with weak interaction in the presence of an external field. We use the idea of Bogolyubov concerning the reduced description of nonequilibrium states [1] and the ergodic relations of the general theory of relaxation processes formulated in [2, 3]. The obtained equations are convenient for constructing perturbation theory in the low frequency of an external field, and also in the small spatial gradients and the small parameter of the effective interaction between the bosons. The entire treatment is in a model with condensate [4, 5]. Further, using the connection between the Green's functions and the variational derivatives [6], we find a closed system of integrodifferential equations for the quantities that determine the asymptotic behavior of the Green's functions in terms of the linearized collision integral of the quasiparticles.

2. Basic Parameters and Relations

In [5], Peletminskii and Sokolovskii considered the kinetics of a spatially inhomogeneous Bose system with weak interaction in a model with condensate. Here, we generalize the results of [5] to the case when the degenerate Bose system is in an inhomogeneous external field. The Hamiltonian of such a system has the form

$$H(t)=H+H_F(t). \tag{2.1}$$

Mechanical Engineering Institute, Voroshilovgrad. Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol.46, No.2, pp.242-250, February, 1981. Original article submitted July 23, 1979; revision submitted December 20, 1979.

 0040-5779/81/4602-0160$07.50