

The Generation of Multidimensional Autoregressive Series by the Herringbone Method¹

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The generation of isotropic artificial series in two or three dimensions by the autoregressive process is of considerable interest for the purpose of modeling environmental properties such as ore grade or reservoir porosity. The relations needed to produce bilateral symmetry using the one-sided autoregressive recursion equations have been attained on the square net and on the isometric lattice by an alternation procedure. In the case of the square net, the one-sided autoregressive (AR) process is alternated between the two diagonals of the net, while in three dimensions, the alternation takes place among the four body diagonals of the isometric cell.

KEY WORDS: autoregressive processes, multidimensional time series, spatial models, herringbone method.

INTRODUCTION

The simulation of such geological properties as elevation, ore grade, porosity, or sedimentary facies is becoming increasingly important for the prediction of urban development, ore reserve estimation, oil field recovery, or exploration potential. A critical factor in any simulation is a quantitative knowledge of the variability, and the incorporation of this variability into the simulation. The variability can be measured by the use of covariograms (Dijkstra, 1976) or by the use of semivariograms (Jowett, 1955, p. 161). For the geologists these will usually be constructed by making one or more traverses to form one-dimensional views of the observed space series. A comparison among commonly observed covariograms and semivariograms shows that a wide variety can be described in terms of a limited number of autoregressive (AR) and moving average (MA) models adapted from statistical time series analysis (Sharp, 1982). If simple

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multidimensional simulations could be developed that would incorporate the variance determined from linear traverses, then multidimensional simulations could become routine for the purposes of prediction and modeling.

Although procedures for the simulation of one-dimensional patterns have been well known for some time, increased interest has arisen in multidimensional simulations as the result of the advances being made in space series analysis (Cliff and Ord, 1981; Ripley, 1981). A great deal of this effort has been directed toward description and parameter estimation of the proposed models (Whittle, 1954; Besag, 1974; Tjøstheim, 1981; and Haggett et al., 1977). Relatively few descriptions of actual simulations are available. These may be termed: the nearest-neighbor, joint-probability model, the turning-band method, and the unilateral autoregressive.

The nearest-neighbor, joint-probability model (Martin, 1974) starts with a sequence of random normal deviates as in the form of a table or array. This is put into a two-sided, nearest-neighbor filter (Smith and Freeze, 1979, p. 1546) to form a set of simultaneous joint probabilities which will match the required autocorrelation. The determination of the simulated values requires the inversion of a $p \times p$ matrix. For small two-dimensional simulations this is not difficult but becomes increasingly more time consuming as the desired number of points to be simulated increases. The turning-band method (Matheron, 1973, p. 461) consists of constructing artificial series with known autocorrelation along several lines spaced at equal angles (Journel, 1974, p. 676-677). The required spatial values are determined as a linear combination of the values projected from these lines to the desired point. The quality of the simulation depends on the number of linear combinations used. The unilateral AR method (Schwarzacher, 1980) has been used to simulate stratigraphically bounded layers. If a stratigraphic section is specified, as on the left side of a diagram, then the individual laminae are simulated by a series of parallel one-dimensional series migrating from left to right (Schwarzacher, 1980, p. 224, 232).

THE SPATIAL AUTOREGRESSIVE MODELS

It has recently been shown that the first-order AR process in one dimension can be converted to use on a square net by taking linear combinations of the AR processes along the orthogonal axes (Aroian and Sharp, 1985) and the process would follow the relationship

$$z_{i,j} = \phi''(z_{i-1,j} + z_{i,j-1}) + a_{i,j}$$

where $z_{i,j}$ are the values of the geological parameter at the lattice coordinates i and j , ϕ'' is a weighting parameter, and $a_{i,j}$ is a random impulse drawn from a normal distribution with zero mean and variance $\sigma_a^2: N(0, \sigma_a^2)$. For series on a

square lattice $0 \leq \phi'' \leq \frac{1}{2}$ and

$$\sigma_a^2 / \sigma_z^2 = 1 - \phi''(\rho_{10} + \rho_{01})$$

where σ_z^2 is the variance of the observed series of z values, ρ_{10} and ρ_{01} are the autocorrelations at lag 1 along the I and J axes, respectively. The analogous first-order AR process in three dimensions for an isometric lattice is given by

$$z_{i,j,k} = \phi'''(z_{i-1,j,k} + z_{i,j-1,k} + z_{i,j,k-1}) + a_{i,j,k}$$

In this case for the series to be stationary $0 < \phi''' < \frac{1}{3}$, and the variance ratio $\sigma_a^2 / \sigma_z^2 = 1 - \phi'''(\rho_{100} + \rho_{010} + \rho_{001})$.

A practical computer simulation can be performed by letting the I axis represent rows of an array, the J axis representing columns, and (in three dimensions) the K axis representing successive layers. To start a simulation on a square net, initial values may be generated along the top row, $i = 1$, and along the left most column, $j = 1$, by use of the one-dimensional AR process. The remainder of the array is filled by starting with the second row and computing each line from left to right using the recursive relationship for a square net. This is repeated row for row until the square array is filled (Fig. 1). For three dimensions, it would first be necessary to create a one-dimensional series along each of the three orthogonal axes and then to form a basal sheet using the procedure of the

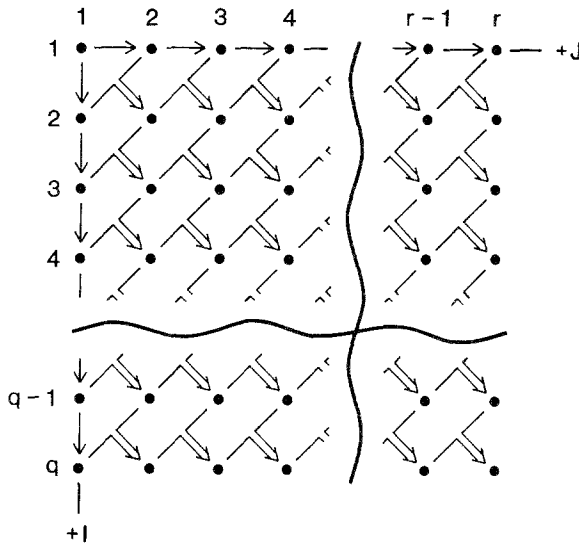


Fig. 1. A unilateral sequential procedure for generating a two-dimensional first-order autoregressive process along the NW-SE diagonal.

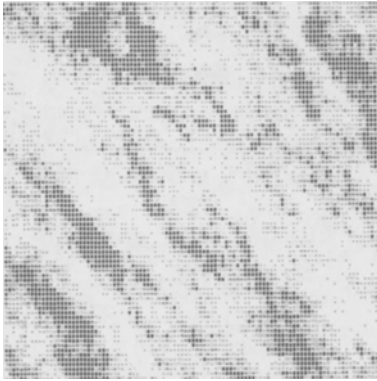


Fig. 2. A contour map of a two-dimensional artificial series obtained by a first-order autoregressive process generated unilaterally along the NW-SE diagonal.

simulation for a square net. Once this has been completed, the recursive relationships for an isometric lattice may be used to fill in progressively the three-dimensional array, one layer at a time, by starting at the bottom and progressing upwards.

When this procedure is in fact performed, it is found that the artificial series is stratified along the NW-SE diagonal of the net (Fig. 2). That is, the artificial series is nicely autocorrelated along the NW-SE diagonal but is uncorrelated along the NE-SW diagonal. This was not initially expected, but it is now clear from the equations that a correlated series can only be formed in the direction in which the equations are recursively applied.

HERRINGBONE METHOD

The difficulty with the procedure described above is the stratified diagonal nature of the artificial series generated. Yet ideally the correlations along both diagonals should be equal. After some trial and error attempts to resolve this difficulty, it was realized that the desired symmetry could be attained through a process analogous to lattice row twinning; that is, by alternating the series. For example, if any row of a square net is generated by proceeding from left to right (Fig. 1), then the next row is generated by reversing the pattern and proceeding from right to left; successive rows are generated by reversing direction at the end of each row in the configuration of a herringbone.

For the square net, the procedure would be as follows. At the starting point ($i = 1, j = 1$), a draw is made at random from the designated normal distribution, $N(0, \sigma_a^2)$. From there, the first row (alternately this could be done columnwise) is generated recursively by using the ordinary one-dimensional AR process

$$z_{1,j} = \phi z_{1,j-1} + a_{1,j}$$

for the values of $j = 2$ to r (Fig. 3a). After reaching the end of the row, the procedure returns along the second row moving from right to left (Fig. 3b). This is initiated by calculating the last point on the second row ($i = 2, j = r$) with the one-dimensional relation that is

$$z_{2,r} = \phi z_{1,r} + a_{2,r}$$

The remainder of the second row is completed by proceeding from right to left using the general equation for the square lattice which starts at the next to last point, by calculating the relationship

$$z_{2,r+1-j} = \phi''(z_{1,r+1-j} + z_{2,r+2-j}) + a_{2,r+1-j}$$

for values of j from 2 to r (Fig. 3b). The third row is initiated by again using the one-dimensional relationship such that

$$z_{3,1} = \phi z_{2,1} + a_{3,1}$$

and is completed by proceeding from left to right (Fig. 3c) using the general equation for the square lattice which starts at the second point by calculating

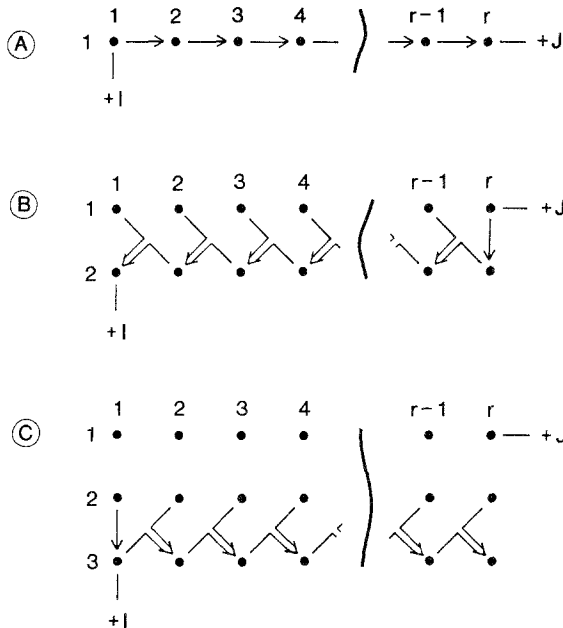


Fig. 3. An alternating sequential procedure for generating a first-order autoregressive process along the first three rows of a net.

the relationship

$$z_{3,j} = \phi''(z_{2,j} + z_{3,j-1}) + a_{3,j}$$

for values of j from 2 to r . Similarly, for any odd numbered row, the first point is initialized by using

$$z_{i,1} = \phi z_{i-1,1} + a_{i,1}$$

and then the remaining points on the row are generated moving from left to right for successive values of j from 2 to r by using the recursive relationship

$$z_{i,j} = \phi''(z_{i-1,j} + z_{i,j-1}) + a_{i,j}$$

The even numbered rows are initiated at the right end of the row by using

$$z_{i,r} = \phi z_{i-1,r} + a_{i,r}$$

and then the remaining points are generated by moving from right to left for successive values of j from 2 to r using the recursive relationship

$$z_{i,r+1-j} = \phi''(z_{i-1,r+1-j} + z_{i,r+2-j}) + a_{i,r+1-j}$$

This process is repeated sequentially in herringbone fashion (Fig. 4) until the entire array is completed.

Typical patterns for artificial series using the two-dimensional spatial first-order AR process (Fig. 5 and 6) generated using the herringbone procedure are

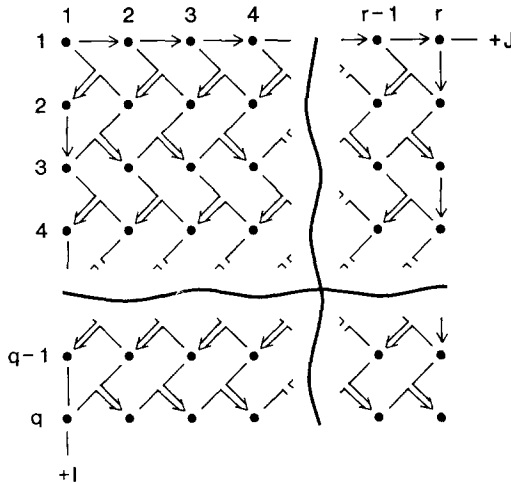


Fig. 4. The sequential procedure for generating a first-order autoregressive process by the herringbone method on a net.

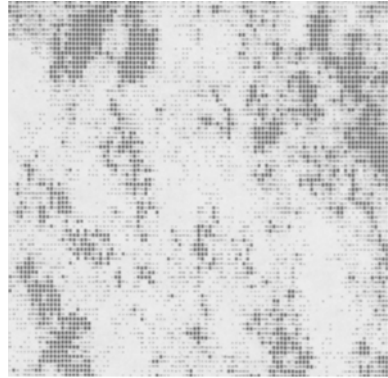


Fig. 5. A contour map of a two-dimensional artificial series generated by the herringbone method. Notice that this map is isotropic when compared with Fig. 2.

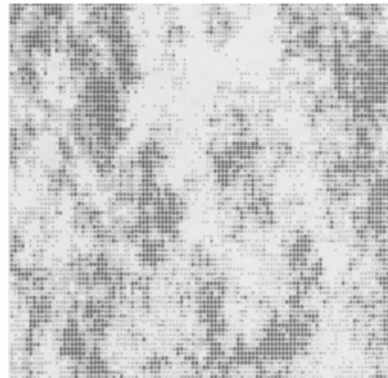


Fig. 6. A contour map of a two-dimensional artificial series generated by the herringbone method but initiated with a different random number seed.

symmetrical with equal correlation along the rows and columns as well as between the NW-SE and NE-SW diagonals.

EXTENSION TO THREE DIMENSIONS

The spatial first-order AR process in three dimensions for an isotropic lattice is obtained as a linear combination of the one-dimensional AR process in the same manner as that for a square net (Aroian and Sharp, 1985)

$$z_{i,j,k} = \phi'''(z_{i-1,j,k} + z_{i,j-1,k} + z_{i,j,k-1}) + a_{i,j,k}$$

As with the formulation in two dimensions, if the generation is carried out always in the same direction starting each row from left to right and then moving upward one layer at a time, the result will have strong correlations along the planar direction defined by the origin and the edge between the front (100) face

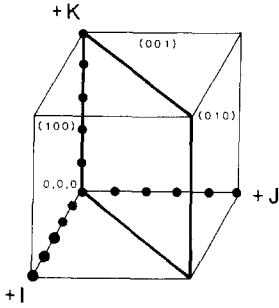


Fig. 7. A sketch showing the planar direction of preferred correlation when a first-order spatial AR process is generated in a unilateral fashion. Lowest correlations will occur in the directions perpendicular to the diagonal plane.

and right-side face (010) of the array (Fig. 7). Highest correlations will occur between points in that plane or in any parallel plane, while the lowest correlation will occur among points in the direction perpendicular to that plane. This stratification in three dimensions can again be rectified by alternating systematically in herringbone fashion among the four body diagonals of a cube (poles to the upper faces of an octahedron).

Even Numbered Layers

For the isometric lattice the procedure would be as follows. First a bottom layer (an odd numbered layer where $k = 1$) would be generated using the herringbone method already described for a square net (Fig. 4). The second layer and any other even numbered layer will begin with the last row and then proceed back toward the first row (Fig. 8). The column from which to start will depend on whether q is odd or even. If q is odd the second layer will be initiated on the right side (last column) while if q is even the second layer (any even layer) will be initiated on the left side (first column). The second layer is initiated by obtaining a starting point with the one-dimensional process along a vertical edge

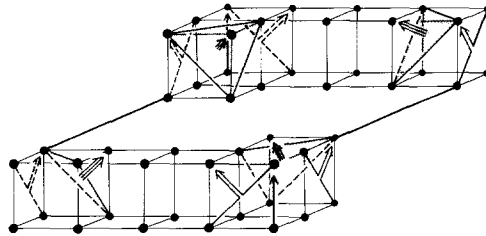


Fig. 8. A perspective drawing showing the orientation of the four directions used to generate an alternating pattern (herringbone) for an isotropic three-dimensional AR process.

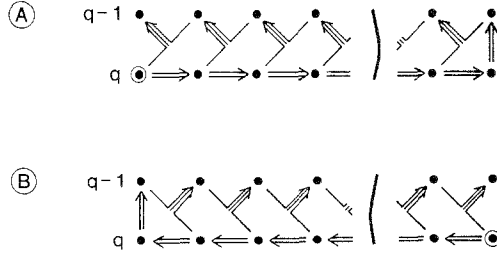


Fig. 9. The initiation of even numbered layers when there are (a) an even number of rows and (b) an odd number of rows. The number of lines forming the arrow indicates whether the series is one-, two-, or three-dimensional.

(Fig. 8) with the relation

$$z_{q,1,k} = \phi z_{q,1,k-1} + a_{q,1,k}$$

when q is even and k is even (Fig. 9a) or

$$z_{q,r,k} = \phi z_{q,r,k-1} + a_{q,r,k}$$

when q is odd and k is even (Figs. 8, 9b). The last row of the second or any even layer is then obtained by using the relationships for a square net along the front face of the lattice, that is

$$z_{q,j,k} = \phi''(z_{q,j-1,k} + z_{q,j,k-1}) + a_{q,j,k}$$

when q is even and k is even (Fig. 9a) or

$$z_{q,r+1-j,k} = \phi''(z_{q,r+2-j,k} + z_{q,r+1-j,k-1}) + a_{q,r+1-j,k}$$

when q is odd and k is even (Fig. 9b) for j ranging from 2 to r .

For the next to the last row of the second or any even layer, the starting point is initialized by applying the relationship for a square net along the side faces of the lattice, that is when $q - 1$ is odd along the right face

$$z_{q-1,r,k} = \phi''(z_{q,r,k} + z_{q-1,r,k-1}) + a_{q-1,r,k}$$

or when $q - 1$ is even along the left face (Fig. 8)

$$z_{q-1,1,k} = \phi''(z_{q,1,k} + z_{q-1,1,k-1}) + a_{q-1,1,k}$$

The remaining points on the next to last row are computed using the full spatial AR recursion relationships for the isometric lattice. That is

$$z_{q-1,r+1-j,k} = \phi'''(z_{q,r+1-j,k} + z_{q-1,r+2-j,k} + z_{q-1,r+1-j,k-1}) + a_{q-1,r+1-j,k}$$

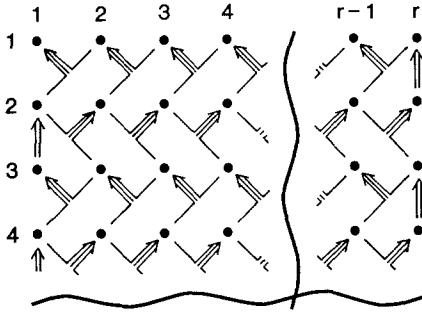


Fig. 10. The completion of any even numbered layer.

when $q - 1$ is odd (Fig. 9a) or

$$z_{q-1,j,k} = \phi'''(z_{q,j,k} + z_{q-1,j-1,k} + z_{q-1,j,k-1}) + a_{q-1,j,k}$$

(Fig. 9b, Fig. 8) when $q - 1$ is even for j ranging from 2 to r . For the remainder of the sheet the required relationships are determined by whether the row is odd or even. For the even rows, the first point in the row is initialized using

$$z_{i,1,k} = \phi''(z_{i+1,1,k} + z_{i,1,k-1}) + a_{i,1,k}$$

and the remaining points are obtained from

$$z_{i,j,k} = \phi'''(z_{i+1,j,k} + z_{i,j-1,k} + z_{i,j,k-1}) + a_{i,j,k}$$

for j ranging from 2 to r . For the odd rows, the last point on the row is initialized using

$$z_{i,r,k} = \phi''(z_{i+1,r,k} + z_{i,r,k-1}) + a_{i,r,k}$$

and the remaining points are obtained from

$$z_{i,r+1-j,k} = \phi'''(z_{i+1,r+1-j,k} + z_{i,r+2-j,k} + z_{i,r+1-j,k-1}) + a_{i,r+1-j,k}$$

for j ranging from 2 to r (Fig. 10).

If the above procedure has been carried out correctly, then the last point calculated will be at the origin of the layer with $i = 1, j = 1$, and $k = s$, where s is even.

Odd Numbered Layers

For the odd numbered layers the first point is initialized using the one-dimensional series along the vertical edge containing the K axis (Fig. 8), that is

$$z_{i,j,k} = \phi z_{i,j,k-1} + a_{i,j,k}$$

The first row is then obtained by using the relationships for a square net along the back face of the lattice (Figs. 8, 11), that is

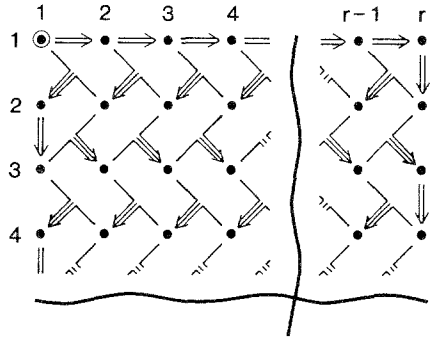


Fig. 11. The initiation of an odd numbered layer for a first-order AR process in three dimensions.

$$z_{1,j,k} = \phi''(z_{1,j-1,k} + z_{1,j,k-1}) + a_{1,j,k}$$

for j ranging from 2 to r .

The initial point on the second row and similarly for any other even numbered row is obtained using the two-dimensional process along the face on the right of the lattice (Figs. 8, 11) so that

$$z_{i,r,k} = \phi''(z_{i-1,r,k} + z_{i,r,k-1}) + a_{i,r,k}$$

Then the remaining points on any odd numbered row are obtained using the three-dimensional process

$$z_{i,r+1-j,k} = \phi'''(z_{i-1,r+1-j,k} + z_{i,r+2-j,k} + z_{i,r+1-j,k-1}) + a_{i,r+1-j,k}$$

for j ranging from 2 to r (Fig. 8, 11).

The initial point on any odd row except the first is obtained using the two-dimensional process along the left face of the lattice such that

$$z_{i,1,k} = \phi''(z_{i-1,1,k} + z_{i,1,k-1}) + a_{i,1,k}$$

Then the remaining points on any odd numbered row are obtained using the three-dimensional process

$$z_{i,j,k} = \phi'''(z_{i-1,j,k} + z_{i,j-1,k} + z_{i,j,k-1}) + a_{i,j,k}$$

for j ranging from 2 to r (Fig. 8, 11). This alternation is continued until the sheet is finished.

Example

By applying the herringbone method to successive layers, an extensive array can be obtained which behaves as an isotropic AR process in three dimensions. As an illustration, an artificial series $81 \times 101 \times 10$ was generated using the recursive relation described in detail above. Then every third layer was plotted (Fig. 12) to illustrate how the series progressively changes in all three dimensions.

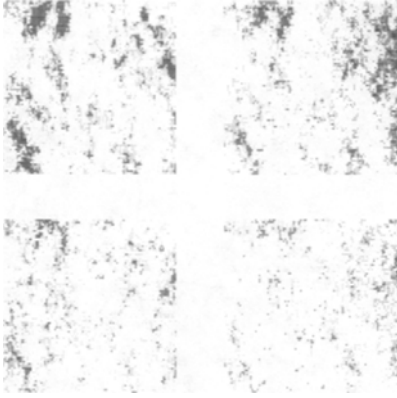


Fig. 12. A contour map of a three-dimensional artificial series generated by the herringbone method. Four sections are shown at a spacing three units apart to illustrate how the series changes in three dimensions.

SUMMARY

The generation of an artificial AR series in two and three dimensions having isotropic symmetry has been induced from a one-sided recursive process by an alternation procedure called the herringbone method. This procedure in two dimensions consists of alternating between the two diagonals of a square from row to row and in three dimensions consists of alternating among the four body diagonals of a cube from row to row and layer to layer. The recursive relationships are both simple and fast and permit the generation of very large arrays with minimal computer time.

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