AVERAGED DESCRIPTION OF WAVE BEAMS IN LINEAR AND NONLINEAR MEDIA (THE METHOD OF MOMENTS)

S. N. Vlasov, V. A. Petrishchev, and V. I. Talanov

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For a monochromatic wave beam the moments of the transverse energy-flux density distribution are introduced into consideration. It is shown that in a linear medium a polynomial representation in z holds for moments of any order. In a medium having cubic nonlinearity polynomials in z of corresponding order can be used to represent the moments of zero and first order, and the centrifugal moments of the second order. Examples of the application of the average description of wave beams are considered.

As is well known, one of the effective methods of solving problems in transport theory is the method of moments [1, 2]. In this method, the problem of finding a certain distribution f(t) is replaced by that of determining the moments $R_n = \int_{-\infty}^{\infty} t^n f(t)dt$ of this distribution. The effectiveness of the method of moments in problems of transport theory is related to the relative simplicity of the equations for R_n. Knowledge of all the moments allows known methods to be used to reconstruct the form of the function f(t) [3].

However, even in those cases when it is found to be impossible to determine all the moments of the desired distribution, information on the first several moments proves useful [4, 5]. This is especially important in nonlinear problems in which to find the entire distribution often necessitates numerical methods. In the present paper we shall speak of the use of the method of moments in analyzing problems of linear and nonlinear quasioptics.

1. The Connection between the Averaged Description of the Field and the Energy and Momentum Conservation Laws

In three-dimensional space (x_1, x_2, x_3) let there be a certain set of quantities comprising the scalar w(t, **r**), the vector **S**(t, **r**), and the tensor $T_{\alpha\beta}(t, \mathbf{r})$ which satisfy the relations

$$\frac{\partial w}{\partial t} = -\operatorname{div} \boldsymbol{S}; \tag{1a}$$

$$\frac{1}{c^2}\frac{\partial \boldsymbol{S}}{\partial t} = \operatorname{div} \hat{T}; \tag{1b}$$

$$\frac{\partial}{\partial t}\sum_{\alpha=1}^{3} T_{\alpha\alpha} = -\operatorname{div} \boldsymbol{Q}, \tag{1.1c}$$

where Q is a certain vector. Let us introduce the moments of the quantity w(t, **r**) into consideration: the zero-order moment

$$W(t) = \int_V w dv, \tag{1.2}$$

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the first-order moment

$$r_c(t) = \frac{1}{W}\int_V r\,w(t, r)\,dv \tag{1.3}$$

and the centrifugal second-order moment

$$I(t) \equiv a_{\text{eff}}^2 = \frac{1}{W}\int_V r^2 w\,dv. \tag{1.4}$$

In (2)-(4) the integral is taken over the entire infinite space. If the quantity w is given the meaning of energy density, then the first two equations of (1.1) will describe the laws of conservation of energy and momentum for the field. Thus, in the case of an electromagnetic field in vacuo $w = (1/8\pi)(E^2 + H^2)$; $S = (c/4\pi)[EH]$ is the density of the energy flux; $T_{\alpha\beta} = T^M_{\alpha\beta} = (1/4\pi)\{E_\alpha E_\beta + H_\alpha H_\beta - (1/2)\delta_{\alpha\beta}(E^2 + H^2)\}$ is the Maxwellian tension tensor; $\sum_{\alpha=1}^{3} T_{\alpha\alpha} = w$ and $Q = S$. In a homogeneous medium with permittivity and permeability ε, μ:

$$w = (ED + BH)/8\pi, \qquad S = \frac{c}{4\pi}[EH],$$

$$T_{\alpha\beta} = \frac{1}{\varepsilon\mu}\frac{1}{4\pi}\left\{E_\alpha D_\beta + H_\alpha B_\beta - \frac{1}{2}\delta_{\alpha\beta}(E_\alpha D_\beta + H_\alpha B_\beta)\right\} \equiv \frac{1}{\varepsilon\mu}T^M_{\alpha\beta},$$

$$\sum_{\alpha=1}^{3} T_{\alpha\alpha} = \frac{1}{\varepsilon\mu}w, \quad Q = \frac{S}{\varepsilon\mu}.$$

By virtue of Eqs. (1.1) the following relationships are fulfilled for the field localized in a certain region of space:

$$W(t) = \text{const} = W(0); \tag{1.5a}$$

$$\frac{dr_c(t)}{dt} = \text{const} = \left.\frac{dr_c}{dt}\right|_{t=0}; \tag{1.5b}$$

$$\frac{d^2 I(t)}{dt^2} = \text{const} = \left.\frac{d^2 I}{dt^2}\right|_{t=0}. \tag{1.5c}$$

Here (1.5a) derives from (1.1a), (1.5a) derives from (1.1a) and (1.1b), and (1.5c) derives from (1.1a)-(1.1c).

Thus, the distribution moments indicated above are represented in t by polynomials of the appropriate order:

$$W = W_0, \quad r_c = r_{c0} + v_c t, \quad I = I_0 + Bt + At^2 \tag{1.6}$$

having coefficients determined by the initial conditions

$$v_c = \left.\frac{1}{W_0}\int_V S\,dv\right|_{t=0},$$

$$A = -\left.\frac{c^2}{W_0}\int_V \sum_1^3 T_{\alpha\alpha}\,dv\right|_{t=0}, \quad B = \left.\frac{2}{W_0}\int_V r\,S\,dv\right|_{t=0}.$$

The relations in Eq. (6) have a simple physical meaning: the energy W of the field is conserved, the "energy center" r_c moves along a straight line with a constant velocity v_c, and the square of the effective radius of the bunch a_{eff}^2 varies according to a parabolic law (for $t \to \infty$, $a_{\text{eff}} \sim t$) – the motion of the bunch is similar to the motion of a cloud of noninteracting particles in the absence of external forces (in this case r_c is defined as the center of mass).

2. Averaged Description of Stationary Wave Beams in a Linear Medium

Let us now consider a wave beam $\mathscr{E} = Ee^{-ikz}$ that is stationary in time and is described by the quasioptic equation (in the coordinates $\mathbf{kr}$)

$$\frac{\partial E}{\partial z} = \frac{1}{2i}\Delta_{\perp} E. \tag{2.1}$$

We shall be interested in the dependence of the moments of the intensity $|E|^2$ on z:

$$r_{mn} = \int_S x^m y^n |E|^2 \, dx\, dy. \tag{2.2}$$

Since it follows from (2.1) that the relationships

$$\frac{\partial |E|^2}{\partial z} = -\operatorname{div}_{\perp} \boldsymbol{S}_{\perp}; \tag{2.3}$$

$$\frac{\partial \boldsymbol{S}_{\perp}}{\partial z} = \operatorname{div}_{\perp} \hat{T}; \tag{2.4}$$

$$\frac{\partial}{\partial z}\sum_{i=1}^{2} T_{ii} = \operatorname{div}_{\perp}\boldsymbol{Q} \tag{2.5}$$

hold, where

$$\boldsymbol{S}_{\perp} = \frac{E\Delta_{\perp}E^* - \text{c. c.}}{2i}; \tag{2.6}$$

$$T_{ik} = \frac{1}{4}\Delta_{\perp}|E|^2\delta_{ik} - \frac{1}{2}\left(\frac{\partial E}{\partial x_i}\frac{\partial E^*}{\partial x_k} + \frac{\partial E^*}{\partial x_i}\frac{\partial E}{\partial x_k}\right); \tag{2.7}$$

$$\boldsymbol{Q} = -\frac{1}{4i}\{E\nabla_{\perp}\Delta_{\perp}E^* - \text{c. c.}\}, \tag{2.8}$$

it follows that for the quantities

$$r_{00} \equiv W_0,$$

$$\frac{1}{r_{00}}(r_{10}x_0 + r_{01}y_0) \equiv \boldsymbol{r}_{c\perp},$$

$$\frac{1}{r_{00}}(r_{20} + r_{02}) \equiv a^2\text{eff}$$

equations analogous to (1.2)-(1.4) are applicable with the substitution of z for t and S for V:

$$r_{00}(z) = r_{00}(0); \tag{2.9}$$

$$r_{c\perp}(z) = r_{c\perp}(0) + \alpha z; \tag{2.10}$$

$$a^2_{\text{eff}}(z) = a^2_{\text{eff}}(0) + Bz + Az^2, \tag{2.11}$$

where

$$\alpha = \frac{1}{r_{00}}\int_S S_{\perp}\, ds = \frac{1}{r_{00}}\int_S E_0^2\nabla_{\perp}\varphi\, ds\Big|_{z=0}; \tag{2.12}$$

$$B = \frac{2}{r_{00}}\int_S \boldsymbol{r}_{\perp}\boldsymbol{S}_{\perp}\, ds \equiv \frac{2}{r_{00}}\int_S (\boldsymbol{r}_{\perp}\nabla_{\perp}\varphi)E_0^2 ds\Big|_{z=0}; \tag{2.13}$$

$$A = \frac{1}{r_{00}}\int_S |\nabla_{\perp}E|^2 ds \equiv \frac{1}{r_{00}}\int_S [(\nabla_{\perp}E_0)^2 + E_0^2(\nabla_{\perp}\varphi)^2]\, ds\Big|_{z=0}, \tag{2.14}$$

$$E = E_0 e^{-i\varphi}.$$

Thus, the total energy flux of the beam is conserved, the intensity center lies on one straight line* (a consequence of the law of conservation of transverse beam momentum), and the square of the effective beam length varies according to a parabolic law.

It can be shown that the polynomial (in z) representation of the intensity moments of the field described by Eq. (2.1) also holds for the subsequent orders r_{mn}:

$$r_{mn}(z) = P_{m+n}(z), \tag{2.15}$$

where $P_{m+n}(z)$ is a polynomial of degree (m + n). This can be verified most simply using the integral Fresnel transformation equivalent to Eq. (2.1):

$$E(\mathbf{r}, z) = \frac{i}{2\pi z}\int_S E(\mathbf{r}'_\perp, 0)\exp\left(-i\frac{|\mathbf{r}_\perp - \mathbf{r}'_\perp|^2}{2z}\right) d^2\mathbf{r}'_\perp. \tag{2.16}$$

Let us consider the focusing of a Gaussian beam: $E = e_0 \exp[-r^2/2a_0^2 + i(r^2/2F)]$. For this beam $a_{\text{eff}} = a$, and the expression

$$a^2_{\text{eff}}(z) \equiv a^2(z) = \frac{1}{a_0^2} z^2 + \left(1 - \frac{z}{F}\right)^2 a_0^2 \tag{2.17}$$

describes the variation of the real width a of the beam. Thus, the wave beams in the average description behave analogously to Gaussian beams having a corresponding width and divergence. It can be shown that for collimated beams ($\varphi(0) = 0$) for a stipulated value of $a^2_{\text{eff}}(0)$ Gaussian distributions realize the minimal diffraction divergence of the effective cross section.

As our other example, let us consider the passage of an arbitrary beam through a quadratic phase corrector $\varphi_0 = -r^2/2F$. Assuming in (2.9)-(2.14) that $\varphi = \varphi_i + \varphi_0$, where φ_i is the phase of the incident beam, we find that

$$\begin{gathered}(\mathbf{r}_{c\perp})_{\text{out}} = (\mathbf{r}_{c\perp})_{\text{in}}, \quad \boldsymbol{\alpha}_{\text{out}} = \boldsymbol{\alpha}_{\text{in}} - \frac{(\mathbf{r}_{c\perp})_{\text{in}}}{F}, \\ (a_{\text{eff}})_{\text{out}} = (a_{\text{eff}})_{\text{in}}, \\ B_{\text{out}} = B_{\text{in}} - \frac{2(a^2_{\text{eff}})_{\text{in}}}{F}, \quad A_{\text{out}} = A_{\text{in}} - \frac{B_{\text{in}}}{F} + \frac{(a^2_{\text{eff}})_{\text{in}}}{F^2}.\end{gathered} \tag{2.18}$$

The indicated relationships, together with (2.9)-(2.14), allow arbitrary beams in the system of quadratic phase correctors to be considered. Note that for nonquadratic correctors the transformations of the coefficients that determine the change in the effective width of the beam and in its direction will contain the input-distribution moments of order higher than the second.

3. Description of Partially Coherent Fields and Fields in Statistically Inhomogeneous Media

Let us indicate still another way of obtaining the moments of the intensity distribution in quasioptics, which is also suitable for the description of partially coherent fields. Let us introduce a mutual-coherence function [5]:

$$B(\mathbf{r}, \boldsymbol{\rho}, z) = \left\langle E\left(\mathbf{r} + \frac{\boldsymbol{\rho}}{2}, z\right) E^*\left(\mathbf{r} - \frac{\boldsymbol{\rho}}{2}, z\right)\right\rangle, \tag{3.1}$$

where $\mathbf{r}(x, y)$, $\boldsymbol{\rho}(\xi, \eta)$ are radius-vectors in the plane z = const. The equation for the function $B(\mathbf{r}, \boldsymbol{\rho}, z)$

$$i\frac{\partial}{\partial z} B(\mathbf{r}, \boldsymbol{\rho}, z) = \nabla_r \nabla_\rho B(\mathbf{r}, \boldsymbol{\rho}, z), \tag{3.2}$$

which derives from (2.1), can be solved by a method analogous to that developed in [2]. For the moments of the mutual coherence function

$$r_{mn}(\boldsymbol{\rho}) = \int_S x^m y^n B(\mathbf{r}, \boldsymbol{\rho}, z)\, d^2\mathbf{r} \tag{3.3}$$

* Attention was already directed to this fact in [6].

it is not difficult to derive from (3.2) recurrence relations:

$$i \frac{\partial r_{mn}}{\partial z} = -m \frac{\partial}{\partial \xi} r_{m-1,\, n} - n \frac{\partial}{\partial \eta} r_{m,\, n-1}. \tag{3.4}$$

From (3.4) it follows that

$$r_{00}(\rho, z) = r_{00}(\rho, 0) = \text{const}; \tag{3.5}$$

$$r_{01}(\rho, z) = r_{01}(\rho, 0) - \frac{1}{i} z \frac{\partial r_{00}(\rho, 0)}{\partial \eta},$$
$$r_{10}(\rho, z) = r_{10}(\rho, 0) - \frac{1}{i} z \frac{\partial r_{00}(\rho, 0)}{\partial \xi}; \tag{3.6}$$

$$r_{02}(\rho, z) = r_{02}(\rho, 0) - \frac{2}{i} z \frac{\partial r_{01}(\rho, 0)}{\partial \eta} - \frac{\partial^2 r_{00}(\rho, 0)}{\partial \eta^2} z^2,$$
$$r_{20}(\rho, z) = r_{20}(\rho, 0) - \frac{2}{i} z \frac{\partial r_{10}(\rho, 0)}{\partial \xi} - \frac{\partial^2 r_{00}(\rho, 0)}{\partial \xi^2} z^2 \tag{3.7}$$

etc. Thus, the polynomial representation for the intensity moments is a particular case (for $\rho = 0$) of the polynomial representation for the moments of the mutual coherence function $B(\mathbf{r}, \rho, z)$. The results (3.4)-(3.7) can naturally be generalized for the case of the propagation of partially coherent wave beams in statistically inhomogeneous media. In the known approximation of [5, 7] this case can be described by the equation

$$i \frac{\partial B}{\partial z} = \nabla_r \nabla_\rho B - \frac{i}{4} d_\varepsilon(\rho) B \tag{3.8}$$

for the mutual coherence function (3.1). In (3.8)

$$d_\varepsilon(\rho) = \int_0^\infty D_\varepsilon\left(\sqrt{\rho^2 + z^2}\right) dz - \int_0^\infty D_\varepsilon\left(\sqrt{z^2}\right) dz,$$

$$D_\varepsilon(\rho^{(3)}) = \left\langle \left[\varepsilon\left(R + \frac{\rho^{(3)}}{2}\right) - \varepsilon\left(R - \frac{\rho^{(3)}}{2}\right) \right] \right\rangle \equiv D_\varepsilon(\rho^{(3)})$$

is the structural function of locally homogeneous and isotropic fluctuations of the permittivity ε (R and $\rho^{(3)}$ are radius-vectors in three-dimensional space x, y, z: $\rho^{(3)} = \sqrt{\rho^2 + z^2}$).

From (3.8) one can obtain a recurrence equation which generalizes (3.4) and describes the variation of the moments of the mutual-coherence function B:

$$i\left(\frac{\partial}{\partial z} + \frac{1}{4} d_\varepsilon(\rho)\right) r_{mn} = -m \frac{\partial}{\partial \xi} r_{m-1,\, n} - n \frac{\partial}{\partial \eta} r_{m,\, n-1}. \tag{3.9}$$

Hence it follows that

$$r_{00}(\rho, z) = r_{00}(\rho, 0) \exp\left(-\frac{1}{4} d_\varepsilon(\rho) z\right),$$

$$r_{01}(\rho, z) = \left[r_{01}(\rho, 0) - \frac{1}{i} \frac{\partial r_{00}(\rho, 0)}{\partial \eta} z + \frac{1}{i} \frac{\partial d_\varepsilon}{\partial \eta} r_{00}(\rho, 0) \frac{z^2}{8} \right] \exp[-(1/4)\, d_\varepsilon(\rho)\, z],$$

$$r_{02}(\rho, z) = \left[(r_{02}(\rho, z))_{\text{reg}} + \frac{1}{i} \frac{\partial d_\varepsilon}{\partial \eta} r_{01}(\rho, 0) \frac{z^2}{4} + \left(\frac{1}{12} \frac{\partial^2 d_\varepsilon}{\partial \eta^2} \right.\right.$$
$$\left.\left. \times r_{00}(\rho, z) + \frac{1}{4} \frac{\partial d_\varepsilon}{\partial \eta} \frac{\partial r_{00}(\rho, 0)}{\partial \eta} \right) z^3 - \left(\frac{\partial d_\varepsilon}{\partial \eta}\right)^2 r_{00}(\rho, 0) \frac{z^4}{64} \right] \exp[-(1/4)\, d_\varepsilon(\rho) z]. \tag{3.10}$$

In particular,

$$a^2_{\text{eff}}(z) = (a^2_{\text{eff}}(z))_{\text{reg}} + \frac{1}{12} [\Delta_\rho d_\varepsilon(\rho)]_{\rho=0} z^3. \tag{3.11}$$

As is evident from (3.11), a random irregularity of the medium introduces an additive contribution to the broadening of the effective beam cross section [4, 5].

4. Wave Beams in a Nonlinear Medium Having a Permittivity $\varepsilon = \varepsilon_0(1 + \varepsilon'|E|^2)$

Let us consider wave beams in a cubic medium which are described by the equation

$$\frac{\partial E}{\partial z} = \frac{1}{2i}(\Delta_\perp E + \varepsilon'|E|^2)E). \tag{4.1}$$

By direct differentiation of the intensity moments with respect to z using Eq. (4.1) it can be shown that wave beams in media having a permittivity $\varepsilon = \varepsilon_0(1 + \varepsilon'|E|^2)$ may be described by Eqs. (2.3)-(2.5) in which

$$T_{ik} = T_{ik}^L + \frac{1}{4}\varepsilon'|E|^4\delta_{ik},$$

$$\mathbf{Q} = \mathbf{Q}^L - \frac{\varepsilon'}{2i}|E|^2(E\nabla_\perp E^* - \text{c.c.}),$$

T_{ik}^L and $\mathbf{Q}^L$ are the corresponding expressions for the linear medium.

As a result of this Eqs. (2.9)-(2.11) are valid as previously, the only difference being that

$$A \equiv A^{NL} = A^L - \frac{\varepsilon'}{2r_{00}}\int_S |E|^4\,ds. \tag{4.2}$$

Let us consider certain consequences which derive from Eqs. (2.9)-(2.11) taking into account (4.2).* In a nonlinear medium the intensity center of the beam propagates along a straight line; this straight line is the same as in a linear medium. This conclusion is valid not only for a cubic medium but also for an arbitrary dependence $\varepsilon(|E|^2)$. Hence, the transverse displacements of a beam having an asymmetrical amplitude profile [8] are local in character, while as a whole the beam propagates in a straight line.

From (2.11) and (4.2) it follows that any collimated beam, beginning with a certain critical power P_{cr}, will, on the average, "collapse" in a nonlinear medium: for $A^{NL} < 0$, $d^2a_{eff}^2/dz^2 < 0$. The critical power is determined from the condition $A^{NL} = 0$:

$$\int_S \left[|\nabla_\perp E|^2 - \frac{\varepsilon'}{2}|E|^4\right] ds = 0. \tag{4.3}$$

Assuming $E = E_0 f(\mathbf{r}_\perp)$, we obtain from (4.3)

$$E_{0\,cr}^2 = \frac{2}{\varepsilon'}\frac{\int_S (\nabla_\perp f)^2\,ds}{\int_S f^4\,ds} \tag{4.4}$$

or

$$P_{cr} = \frac{cn}{4\pi k_0^2\varepsilon'}\frac{\int_S (\nabla_\perp f)^2 ds \int_S f^2\,ds}{\int_S f^4\,ds}, \tag{4.5}$$

where $k_0 = (\omega/c)\sqrt{\varepsilon_0}$. The critical power (4.5) is determined solely by the parameters of the medium and the profile of the transverse distribution $f(\mathbf{r}_\perp)$; it is independent of its amplitude or width. Thus, for a Gaussian beam $f \sim \exp(-r^2/2a^2)$:

$$P_{cr}^G = \frac{cn}{2\varepsilon' k_0^2}. \tag{4.6}$$

Let us determine the profile for which the critical power is minimal. Varying P_{cr} with respect to $f(\mathbf{r}_\perp)$, we find that this profile must satisfy the equation

* In view of the arbitrariness of the origin of the coordinate z in (2.11) the coefficient A is the integral of the original equations. For arbitrary nonlinearity an analogous invariant was determined in [14].

$$\Delta_\perp f + 2\frac{\int_S (\nabla f)^2 ds}{\int_S f^4 ds} f^3 - \frac{\int_S (\nabla f)^2 ds}{\int_S f^2 ds} f = 0,$$

which by means of the substitution

$$r_{\text{new}} = r_\perp \left(\frac{\int_S (\nabla_\perp f)^2 ds}{\int_S f^2 ds} \right)^{1/2},$$

$$f_{\text{new}} = f\left(2\int_S f^2 ds / \int_S f^4 ds\right)^{1/2}$$

can be reduced to the form

$$\Delta_\perp^{\text{new}} f_{\text{new}} + f_{\text{new}}^3 - f_{\text{new}} = 0. \tag{4.7}$$

The simplest solution of this equation corresponds to an axisymmetric beam which is a solution of Eq. (4.1) independent of z. Thus, a beam the profile of which coincides with the profile of a uniform (with respect to z) beam has the least critical power: $P_{\text{cr min}} = P_{\text{uni}}$. The quantity

$$P_{\text{uni}} = \frac{cn}{4\pi k_0^2 \varepsilon'} \; 5.7637 \tag{4.8}$$

was determined in [9] by numerical integration of Eq. (4.7). However, the stationary properties of the functional $P_{\text{cr}}[f]$ allow P_{uni} to be calculated with comparatively high accuracy by standard variational methods. As is evident from (4.6), a Gaussian beam already yields an approximation that is adequate in practice for P_{uni}: $P_{\text{cr}}^G = 1.09\, P_{\text{uni}}$. It should be noted that the value obtained for the critical power of a Gaussian beam is somewhat higher than the value determined as a result of numerical calculations in [10]: $P_{\text{cr}}^G = 1.015\, P_{\text{uni}}$. This is probably because in the first case the critical power determines the "collapse" threshold of the beam as a whole ($d^2 a_{\text{eff}}^2/dz^2 < 0$), while in the second case it determines the threshold of local "collapse" when a focal point having an infinite intensity is formed on the beam axis. In a certain small power interval $P_{\text{uni}} < P < 1.09\, P_{\text{uni}}$ the formation of the focal point is accompanied by an increase in the effective width of the beam.

Let us write Eq. (2.11) for the effective beam width by introducing into it the critical power (4.5):

$$a_{\text{eff}}^2 = A_0^L\left(1 - \frac{P}{P_{\text{cr}}}\right) z^2 + \int_S (r_\perp + \nabla_\perp \varphi z)^2 E_0^2 ds \Big/ \int_S E_0^2 ds, \tag{4.9}$$

where

$$A_0^L = \int_S (\nabla_\perp E_0)^2 ds \Big/ \int_S E_0^2 ds \tag{4.10}$$

is a coefficient characterizing the diffraction divergence of a collimated beam having the same amplitude profile in a linear medium. In particular, for a focused beam ($\varphi = -r^2/2F$)

$$a_{\text{eff}}^2(z) = A_0^L\left(1 - \frac{P}{P_{\text{cr}}}\right) z^2 + \left(1 - \frac{z}{F}\right)^2 a_{\text{eff}}^2(0). \tag{4.11}$$

From (4.11) it is evident that the critical self-focusing power for a focused beam (F > 0) is the same as that for a collimated beam: for $P = P_{\text{cr}}$ the wave beam on the average has the shape of a cone $a_{\text{eff}} = |1 - z/F|\, a_{\text{eff}}(0)$ with its vertex at the point z = F. This result has been noted already in [11]. It is not difficult to verify the proposition that the effective width of the beam (2.11) satisfies the equation

$$\frac{d^2 a_{\text{eff}}}{dz^2} = \frac{A a_{\text{eff}}^2(0) - B^2/4}{a_{\text{eff}}^3}. \tag{4.12a}$$

In particular, for a focused beam

$$\frac{d^2 a_{\text{eff}}}{dz^2} = \frac{A_0^L (1 - P/P_{\text{cr}})\, a_{\text{eff}}^2(0)}{a_{\text{eff}}^3}. \tag{4.12b}$$

If $E = E_0 \exp(-r^2/2a^2)$, then $a_{\text{eff}}(0) = a_0$, $A^L = 1/a_0^2$,

$$\frac{d^2 a_{\text{eff}}}{dz^2} = \frac{1 - P/P_{\text{cr}}}{a_{\text{eff}}^3}.$$

The latter equation is derived in the theory of so-called aberrationless self-focusing [12], understanding by a_{eff} the width of the Gaussian beam. The analysis performed refines the meaning of the aberrationless approximation: the derived equation describes the effective width of the self-focusing beam, which may differ noticeably from the actual width determined from some fixed level. At the same time, in (4.12b) the idea of critical power is also refined: $P_{\text{cr}} = 4\,\overline{P}_{\text{cr}}$, where $\overline{P}_{\text{cr}}$ is the critical self-focusing power obtained in the near-axial approximation [12]. Note that the equation for the width of a Gaussian beam, which can be derived from the action functional for Eq. (4.1) by the variational method [13], coincides completely with (4.12b) and thus describes the effective beam width rather than the true width.

From (4.11), for $P > P_{\text{cr}}$, the self-focusing length may be determined as the distance to the point where $a_{\text{eff}} = 0$; for a collimated beam:

$$z_{\text{sf}}^* = \frac{a_{\text{eff}}(0)}{[A_0^L (P/P_{\text{cr}} - 1)]^{1/2}}. \tag{4.13}$$

For a Gaussian beam

$$z_{\text{sf}}^* = \frac{a_0^2}{(P/P_{\text{cr}} - 1)^{1/2}}. \tag{4.14}$$

It is of interest to compare this expression with the value of the self-focusing length determined from numerical calculations as the distance to the focal point [10]:

$$z_{\text{sf}} = \frac{0.366\, a_0^2}{\left\{\left(\sqrt{P/P_{\text{cr}}^G} - 0.825\right)^2 - 0.03\right\}^{1/2}}. \tag{4.15}$$

For $P \gg P_{\text{cr}}^G$

$$z_{\text{sf}}^* = \frac{a_0^2}{(P/P_{\text{cr}})^{1/2}}, \quad z_{\text{sf}} = \frac{a_0^2}{2\sqrt{2}(P/P_{\text{cr}})^{1/2}}, \tag{4.16}$$

i.e., z_{sf} is less than z_{sf}^* by almost a factor of 3. For $P \sim P_{\text{cr}}^G$, $z_{\text{sf}} \approx 0.85 a_0^2/(P/P_{\text{cr}} - 1)^{1/2}$, which is very close to z_{sf}^*. Thus, the determination of z_{sf} from the averaged description (and likewise from the aberrationless approximation with a refined critical power) yields a negligible error only for $P \sim P_{\text{cr}}$. With increasing power, z_{sf} and z_{sf}^* begin to diverge greatly. The points z_{sf} and z_{sf}^* could be interpreted, respectively, as the points corresponding to local (partial) and absolute collapse. However, at the local-collapse point we are dealing with a singularity of the solution, as a consequence of which the averaged description is valid only up to this point.† Under these conditions the quantity z_{sf}^* characterizes the general tendency of beam behavior on the segment $0 < z < z_{\text{sf}}$. Substituting $z = z_{\text{sf}}$ from (4.16) into the expression for the Gaussian beam, it is not difficult to verify that for $P \gg P_{\text{cr}}$ the effective cross section toward the point z_{sf} decreases by 12.5%. Note that for $P \gg P_{\text{cr}}$ the effective broadening of the beam in a linear medium at distances of $z \sim z_{\text{sf}}$ amounts to a quantity of the order of $(1/8)(P_{\text{cr}}^G/P) \ll 1$, so that nonlinear focusing narrows the beam only insignificantly in comparison with its width in a linear medium. Note that, by virtue of the invariance of Eq. (4.1) relative to the focusing transformation [11], this conclusion also applies to the focused beams: toward local-collapse points the effective beam width varies slightly in comparison with the case of a linear medium.

The possibilities of the average description are not restricted to the cases considered here. In the quasioptics approximation the polynomial representation of the intensity moments also holds for pulses in dispersive media. Among the nonlinear problems the description of fields in statistically inhomogeneous media having a cubic nonlinearity should be mentioned. This problem was considered in [15].

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*As numerical calculations show, the invariance of the quantity A^{NL} (4.2) is violated in transition through local-collapse points, which is evidence of the singularity of the solution at these points.

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