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INTEGRATION OF THE STATIONARY PROBLEM FOR A CLASSICAL SPIN CHAIN

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A general solution is found to the stationary problem for an anisotropic discrete Heisenberg chain with classical spins and also its natural generalization — the discrete analog of the n-dimensional Neumann problem. Explicit expressions are obtained for the solutions in terms of θ functions, and also expressions for the energy of these solutions. The integration is based on the interpretation of the solutions in terms of a Bloch eigenfunction of a finite-gap difference Schrödinger operator.

Several recent physics studies have been devoted to discrete chains with classical spins [1-3]. There are several reasons for the interest in such chains; in particular, Pokrovskii and Khokhlachev [4] have shown that the problem of the wave functions of the Hamiltonian in the Heisenberg model leads to the integration of such chains. Particular solutions and integrals of the corresponding anisotropic chain with classical spins have been found by Granovskii and Zhedanov [2,3], who have shown that the obtained system is a natural discrete analog of the classical Neumann system [5].

In the present paper, we investigate the general solution of this problem and its n-dimensional generalization in the framework of finite-gap theory of the difference Schrödinger operator.

1. We consider a discrete chain at the sites of which there are variables S_k $(k\in\mathbb{Z})$, $S_k\in\mathbb{R}^{n+1}$, $|S_k|=1$ with interaction energy

$$H=-\sum_{k\in\mathbb{Z}}(S_k, JS_{k+1}), \tag{1}$$

where $J=\mathrm{diag}(J_1,\ldots, J_{n+1})$, and $0<J_1<\ldots<J_{n+1}$. For n = 2, we have the anisotropic Heisenberg chain with classical spin [3]. The stationary states are found from the equation

$$S_{k+1}+S_{k-1}=\lambda_k J^{-1}S_k, \tag{2}$$

which is obtained from (1) by variation with respect to S_k subject to the constraint $|S_k|=1$. In contrast to the continuum case, the multiplier λ_k is not determined uniquely by the constraint condition; one can readily show that there are two possibilities: either $\lambda_k = 0$, or

$$\lambda_k=2(S_{k-1}, J^{-1}S_k)/|J^{-1}S_k|^2. \tag{3}$$

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We shall consider only the second case, which corresponds to the correct continuum limit [1,3,].

Thus, we consider the system of nonlinear difference equations (2), (3). We define the function

$$\Phi(\lambda, S_k, S_{k+1}) = \sum_{\alpha=1}^{n+1} \frac{(S_k^{\alpha})^2}{I_\alpha^2-\lambda} + \frac{1}{2}\sum_{\beta\neq\alpha} \frac{(IS_k \wedge S_{k+1})^2_{\alpha\beta}}{(I_\beta^2-\lambda)(I_\delta^2-\lambda)}, \tag{4}$$

where $(x\wedge y)_{\alpha\beta}=x^\alpha y^\beta - x^\beta y^\alpha$, $I=J^{-1}$, $I_\alpha=J_\alpha^{-1}$.

PROPOSITION 1. By virtue of (2), (3) the equation $\Phi(\lambda, S_k, S_{k+1})=\Phi(\lambda, S_{k+1}, S_{k+2})$ holds, and, therefore, the functions $F_\alpha(S_k, S_{k+1}) = \operatorname{res}\Phi|_{\lambda=I_\alpha^2}$ do not depend on the number k and are integrals of the system (2), (3).

The form of the integrals

$$F_\alpha = (S_k^\alpha)^2 + \sum_{\beta\neq\alpha} \frac{(IS_k\wedge S_{k+1})^2_{\alpha\beta}}{I_\beta^2 - I_\alpha^2} \tag{5}$$

clearly resembles the integrals of the Neumann problem [5] and for n = 2 was obtained by Granovskii and Zhedanov [3].

2. We show that there is a discrete analog of the Moser–Trubovits isomorphism [5,6], which establishes a correspondence between the solutions of Eqs. (2) and (3) and finite-gap difference Schrödinger operators. In the simplest "soliton" case, this fact was noted in [3]. We recall the necessary facts from the finite-gap theory of the difference Schrödinger operator, which was begun in the work of Novikov, Tanaka, and Date and developed by Krichever (see [7,8]). We shall be interested in the class of operators of the form

$$(L\psi)_k = c_{k+1}\psi_{k+1} + c_k\psi_{k-1}, \tag{6}$$

which was used by Manakov to integrate the difference Korteweg–de Vries equation and in the framework of finite-gap theory was identified by Novikov [7]. In addition, we shall consider only operators with odd number of spectral bands. Such operators have the following algebraic-geometric description (see [8]).

We consider a hyperelliptic curve Γ of the form $y^2=R(z)$, $R(z)=\prod_{i=0}^{2n}(z^2-z_i^2)$ and a nonsingular divisor D of degree 2n invariant with respect to involution of the curve τ: (y, z) → (−y, −z) (in the definition of τ in the review [8] there is an error in the sign). On Γ, we define functions ψ_m having at the points of the divisor D simple poles and in the neighborhood of the infinitely distant points $P_\pm$ the asymptotic behavior $\psi_m \sim \alpha_m^{\pm 1} z^{\pm m}$. Such functions are determined up to the sign. They are Bloch eigenfunctions for the difference operator

$$(L\psi)_k = c_{k+1}\psi_{k+1} + c_k\psi_{k-1} = z\psi_k, \quad c_k = \alpha_{k-1}\alpha_k^{-1}, \tag{7}$$

for which $\pm z_i$ are the ends of the bands of the spectrum.

The following relation holds:

$$\psi_k(P)\,\psi_k(\sigma(P)) = P_k(z)/P(z), \tag{8}$$

where

$$\sigma:(y,z)\to(-y,z),\ P(z)=\prod_{i=1}^{n}(z^2-\gamma_i^2),\quad P_k(z)=\prod_{i=1}^{n}(z^2-\gamma_i^2(k)),$$

and $\pm\gamma_i$ and $\pm\gamma_i(k)$ are, respectively, the projections of the divisor of the poles D and the divisors of the zeros of the function ψ_k onto the z plane. For reality of c_k, it is necessary to require, besides reality of z_i, that $\pm\gamma_i$ lie separately in finite forbidden bands (where R(z) > 0). Using the freedom in the choice of the sign of ψ_k, we can assume that α_k and, hence, c_k too, are positive.

Now suppose the polynomial R(z) has the form

$$R(z)=\prod_{i=1}^{n}(z^2-E_i^2)\prod_{\alpha=1}^{n+1}(z^2-I_\alpha^2), \tag{9}$$

so that I_α is a set of ends of bands such that between nearest ones there is just one gap (forbidden band). We consider the sequence of vectors $\mathbf{p}_k$ formed from normalized Bloch functions at the ends of the bands,

$$p_k^\alpha=b_\alpha\psi_k(I_\alpha),\quad b_\alpha=\sqrt{P(I_\alpha)\Big/\prod_{\beta\neq\alpha}(I_\alpha^2-I_\beta^2)}. \tag{10}$$

LEMMA. The sequence $\mathbf{p}_k$ satisfies the equations

$$c_{k+1}\mathbf{p}_{k+1}+c_k\mathbf{p}_{k-1}=J^{-1}\mathbf{p}_k,\quad |\mathbf{p}_k|=1. \tag{11}$$

The proof of the lemma is completely analogous to the proof of the continuum case [6].

We consider a transformation of the unit sphere $\mathbf{S}=j(\mathbf{p})$: $\mathbf{S}=J\mathbf{p}/|J\mathbf{p}|$.

THEOREM. Let $\mathbf{p}_k$ be a sequence of vectors formed from the normalized Bloch eigenfunctions of the finite-gap operator (7), (9) at the ends of the bands (10). Then $\mathbf{S}_k=j(\mathbf{p}_k)$ is a general solution of the system (2), (3).

The proof uses the following directly verified identity:

$$|J\mathbf{p}_k|^2=\det J^2P_k(0). \tag{12}$$

Substituting $\mathbf{p}_k=j^{-1}(\mathbf{S}_k)=J^{-1}\mathbf{S}_k|J\mathbf{p}_k|$ in (11), we obtain

$$c_{k+1}|J\mathbf{p}_{k+1}|\mathbf{S}_{k+1}+c_k|J\mathbf{p}_{k-1}|\mathbf{S}_{k-1}=|J\mathbf{p}_k|J^{-1}\mathbf{S}_k.$$

It follows from Eq. (7) for z = 0 and the identities (8) and (12) that $c_{k+1}|J\mathbf{p}_{k+1}|=c_k|J\mathbf{p}_{k-1}|$, whence $\mathbf{S}_{k+1}+\mathbf{S}_{k-1}=\lambda_kJ^{-1}\mathbf{S}_k$, $\lambda_k\neq0$.

The reciprocal correspondence is established by means of the relations

$$\sum_{\alpha=1}^{n+1}\frac{I_\alpha^2(S_k^\alpha)^2}{\lambda^2-I_\alpha^2}=-\frac{P_k(\lambda)}{Q(\lambda)}\frac{Q(0)}{P_k(0)}, \tag{13}$$

$$\sum_{\alpha=1}^{n+1}\frac{I_\alpha^2F_\alpha}{\lambda^2-I_\alpha^2}=-\frac{T(\lambda)}{Q(\lambda)}\frac{Q(0)}{T(0)},\quad T(\lambda)=\prod_{i=1}^{n}(\lambda^2-E_i^2),\quad Q(\lambda)=\prod_{\alpha=1}^{n+1}(\lambda^2-I_\alpha^2). \tag{14}$$

Note that formulas (13) and (14) can be regarded as the discrete analog of Dubrovin's equations for the zeros of the Bloch function. Another way of obtaining such relations has been noted by Novikov [6].

3. To obtain explicit expressions for the solutions, we can, for example, use the review of [7]; however, the expressions given there refer to a general operator of second order and are expressed in terms of a θ function of the curve Γ of the kind 2n. In our symmetric case, everything reduces to θ functions of the kind n. Since such formulas are known only for the case n = 1 [9], we give their explicit form in the general case, using the relations of [10].

On Γ we choose a basis of cycles a_i, b_i $(i=1,\ldots,2n)$ such that a_i hang above the gaps and $\tau(a_i)=a_{i+n}$, $\tau(b_i)=b_{i+n}$ $(i=1,\ldots,n)$. We determine the basis of holomorphic forms $\omega_i=f_i(z)y^{-1}dz$ normalized by the conditions $\oint_{a_i}\omega_j=\delta_{ij}$, and the basis of τ-invariant forms $\alpha_i=\omega_i+\omega_{i+n}$ $(i=1,\ldots,n)$, and also the matrix $B_{ij}=\oint_{b_i}\alpha_j$ $(i,j=1,\ldots,n)$.

The matrix B is the Riemann matrix of the curve Γ_0, the factor of the curve Γ with respect to the action of τ; from it, one can construct in the standard manner (see, for example, [7]) the θ function θ(u). Further, suppose

$$\Omega=(z^{2n}+\mu_1z^{2n-2}+\ldots+\mu_n)y^{-1}dz \tag{15}$$

is an Abelian differential of the third kind normalized by the conditions $\oint_{a_i}\Omega=0$, **U** is the vector of its b periods: $U_k = \frac{1}{2\pi i}\oint_{b_k}\Omega \quad (k=1,\ldots,n)$, and the mapping $\mathbf{A}(P)$ is determined by the equation

$$A(P)^i = \int_{P_+}^{P} \alpha_i \quad (i=1,\ldots,n), \quad P\in\Gamma.$$

PROPOSITION 2. The Bloch function of the finite-gap operator and the general solution corresponding to it of the stationary problem (2), (3) have the form

$$\psi_m(P) = \varkappa_m \exp\left(m\int_{P_0}^{P}\Omega\right)\frac{\theta(\mathbf{A}(P)+m\mathbf{U}+\zeta)}{\theta(\mathbf{A}(P)+\zeta)}, \tag{16}$$

$$S_m{}^{\alpha} = \beta_m b_\alpha J_\alpha \frac{\theta[\nu_\alpha](m\mathbf{U}+\zeta)}{\theta[\nu_\alpha](\zeta)}, \tag{17}$$

where P_0 is the largest point of the spectrum, β_m is determined from the condition $|\mathbf{S}_m|=1$, and ν_α are the characteristics corresponding to the semiperiods $\mathbf{A}(I_\alpha)$.

The integrals in the definition of $\mathbf{A}(P)$ and in the exponential are made consistent by means of the natural path from P_0 to P_+, and the constant $\varkappa_m$ is found from the relation

$$\varkappa_m^{-2} = \frac{\theta(m\mathbf{U}+\zeta)\,\theta((m-2)\mathbf{U}+\zeta)}{\theta(\zeta)\,\theta(-2\mathbf{U}+\zeta)},$$

which follows from the Riemann relation $\mathbf{A}(P_-)=-2\mathbf{U}$. For completeness, we also give the formula for the corresponding coefficients c_m:

$$c_m{}^2 = \frac{\theta((m-1)\mathbf{U}+\zeta)\,\theta((m-2)\mathbf{U}+\zeta)}{\theta(m\mathbf{U}+\zeta)\,\theta((m-3)\mathbf{U}+\zeta)}\beta^{-1},$$

where β is determined from the condition

$$\int_{P_0}^{P}\Omega = \ln z + \ln\beta + O\left(\frac{1}{z}\right), \quad P\sim P_+, \quad z=z(P).$$

When some of the roots of the polynomial R(z) coincide, there is a degeneracy of the θ functions, which corresponds to a decrease in the kind of the curve. Thus, the solutions found in [2] and expressed in terms of elliptic functions correspond to $E_1^2 = E_2^2$.

4. We consider the question of the energy of the obtained solutions:

$$E = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}(\mathbf{S}_{k+i}, J\mathbf{S}_{k+i+1}) = \overline{(\mathbf{S}_k, J\mathbf{S}_{k+1})}.$$

PROPOSITION 3. The energy of the solution (16) is

$$E = \mu_n r \det J^2,$$

where μ_n is the coefficient in the formula for the differential Ω (15), and $r=(\mathbf{S}_k, J^{-1}\mathbf{S}_{k+1})$ is one of the integrals of the problem and it is related to the ends of the spectral bands by the relation

$$r^2 = \prod_{i=1}^{n} E_i^{-2}\prod_{\alpha=1}^{n+1} I_\alpha{}^2.$$

The proof follows from the chain of equations

$$E = {}^1/_2\overline{((\mathbf{S}_k, J\mathbf{S}_{k+1})+(\mathbf{S}_k, J\mathbf{S}_{k-1}))} = {}^1/_2\overline{\lambda_k} = \overline{|J^{-1}\mathbf{S}_k|^{-2}}r = r\overline{|J\mathbf{p}_k|^2} = r\overline{P_k(0)}\det J^2 = \mu_n r\det J^2,$$

since $\Omega = \overline{P_k(z)y^{-1}dz}$. The expression for the integral r is obtained by means of the relation (14):

$$r^2 = \sum_{\alpha=1}^{n+1} I_\alpha{}^2 F_\alpha = -Q(0)/T(0) = \prod_{i=1}^{n} E_i^{-2}\prod_{\alpha=1}^{n+1} I_\alpha{}^2.$$

The coefficient μ_n is found from the system of linear equations for μ_i:

$$\oint_{a_i} \Omega = 0 \Leftrightarrow \sum_{j=1}^{n} K_{ij}\mu_j + K_{i0} = 0, \quad K_{ij} = \oint_{a_i} \frac{z^{2(n-j)}\, dz}{\sqrt{R(z)}}, \quad i = 1, \ldots, n.$$

5. We end by discussing some open questions. The considered problem (2), (3) has all the features of an integrable (even algebraically) system, i.e., it is an example of what one may naturally call an integrable map. It would be interesting to understand the mechanism of its integrability in the sense of a certain analog of Liouville's theorem.* It would seem that hitherto only one example of such a system has been found — the ellipsoidal billiard table. One can show that for it there is also a connection with a certain class of difference operators, this making it possible to obtain formulas in terms of θ functions. Another open question is the correspondence between the obtained solutions and the wave functions of the quantum Hamiltonian.

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*After this paper had been sent to press, I succeeded in clarifying this question and in finding new integrable systems of such type.