MEAN-FIELD MODELS IN THE THEORY OF RANDOM MEDIA. II

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A study is made of a stationary random medium described by the evolution equation $\partial\psi/\partial t=\varkappa\bar{\Delta}_V\psi+\xi(\mathbf{x})\psi$, where $\bar{\Delta}_V$ is the operator of mean-field diffusion in the volume $V\subset\mathbf{Z}^d$, $\xi(\mathbf{x})$, $\mathbf{x}\in V$, are independent random variables with normal distribution $N(0, \sigma^2)$. A study is made of the asymptotic behavior of the solution $\psi(\mathbf{x}, t)$ and its statistical moments $m_p(\mathbf{x}, t)=\langle\psi^p(\mathbf{x}, t)\rangle$, $p=1, 2,\ldots$, as $t \to \infty$, $|V| \to \infty$. The paper continues the earlier [1].

We continue the study begun in [1]* of random media with mean-field (nonlocal) diffusion specified in a volume $V\subset\mathbf{Z}^d$ by means of the operator

$$\bar{\Delta}_V f(\mathbf{x})=\frac{1}{|V|}\sum_{\mathbf{x}'\in V}(f(\mathbf{x}')-f(\mathbf{x})). \tag{1}$$

In this paper, we consider a stationary random medium described by the evolution equation

$$\frac{\partial\psi}{\partial t}=\varkappa\bar{\Delta}_V\psi+\xi(\mathbf{x})\psi,\qquad \psi(\mathbf{x},0)=\psi_0(\mathbf{x})\geqslant 0. \tag{2}$$

The potential $\xi(\mathbf{x})=\xi(\mathbf{x}, \omega)$ is a collection of independent equally distributed random variables with continuous distribution, and $\varkappa>0$ is the diffusion coefficient.

The aim of the paper is a study in the spirit of Sec. 2 of [1] of the asymptotic behavior of the solution $\psi=\psi(\mathbf{x}, t, \omega)$ of the probem (2) and of its statistical moments $m_p=m_p(\mathbf{x}, t)=\langle\psi^p(\mathbf{x}, t)\rangle$, $p=1, 2,\ldots$, as $t \to \infty$, $|V| \to \infty$.

We recall that $|V|$ and t tend to infinity in accordance with the prescription

$$|V|\sim t^{d/2},\quad t\to\infty \tag{3}$$

(this condition is motivated in [1]). Although it is clear from the form (1) of the operator $\bar{\Delta}_V$ that the topology of the lattice $\mathbf{Z}^d$ plays no role, by means of (3) we have succeeded to some degree in taking into account the dimension of space.

Note also that the condition (3) leads essentially to a series scheme in the solution of the problem (2), namely, for given number $N = |V|$ of sites, the solution $\psi(\mathbf{x}, t)$ is considered on a time interval $t\in[0, t_N]$, $t_N\sim N^{2/d}$.

2. Asymptotics of the Growth of the Solution $\psi(\mathbf{x}, t, \omega)$

We consider the operator $H_V=\varkappa\bar{\Delta}_V+\xi(\mathbf{x})$ on the right-hand side of (1), which acts on the space of functions $L^2(V)$. We recall (see [1]) that the eigenvalues of the operator H_V, expressed in the form $\lambda-\varkappa$, satisfy the equation

$$1=\varkappa|V|^{-1}\sum_{\mathbf{x}\in V}(\lambda-\xi(\mathbf{x}))^{-1}, \tag{4}$$

which has (almost certainly) precisely $N = |V|$ roots $\lambda_1 < \ldots < \lambda_N$, and

$$\xi_{(1)}<\lambda_1<\xi_{(2)}<\lambda_2<\ldots<\xi_{(N-1)}<\lambda_{N-1}<\xi_{(N)}<\lambda_N, \tag{5}$$

where $\xi_{(1)}<\ldots<\xi_{(N)}$ is the variation series of the random variables $\{\xi(\mathbf{x})\}$. The eigenfunction corresponding to λ_i, $\varphi_i(\mathbf{x})$, normalized by the condition $\|\varphi_i\|=1$, has the form

*The sections are numbered continuously through the complete study.

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$$\varphi_i(\mathbf{x})=(\lambda_i-\xi(\mathbf{x}))^{-1}\Big[\sum_{\mathbf{x}'\in V}(\lambda_i-\xi(\mathbf{x}'))^{-2}\Big]^{-1/2}. \tag{6}$$

Note that by virtue of Parseval's equation

$$\sum_{i=1}^{N}\varphi_i^2(\mathbf{x})=\|\delta_{\mathbf{x}}\|^2=1. \tag{7}$$

Further, we shall assume that the initial function $\psi_0(\mathbf{x})$ in (2) is nonzero, i.e., $\|\psi_0\| > 0$.

We expand $\psi(\mathbf{x}, t)$ with respect to the eigenbasis $\{\varphi_i(\mathbf{x})\}$:

$$\psi(\mathbf{x}, t)=\sum_{i=1}^{N}c_i\varphi_i(\mathbf{x})\exp\{(\lambda_i-\varkappa)t\},$$

where c_i are the Fourier coefficients in the expansion of $\psi_0(\mathbf{x})$.

LEMMA 2.1. We set $\rho_N(\mathbf{x})=\psi(\mathbf{x}, t)-c_N\varphi_N(\mathbf{x})\exp\{(\lambda_N-\varkappa)t\}$. Then $|\rho_N(\mathbf{x})|\leqslant\|\psi_0\|\exp\{(\lambda_{N-1}-\varkappa)t\}$.

Proof. By the Cauchy–Schwarz inequality

$$|\rho_N(\mathbf{x})|\exp\{-(\lambda_{N-1}-\varkappa)t\}\leqslant\Big[\sum_{i=1}^{N}c_i^2\sum_{i=1}^{N}\varphi_i^2(\mathbf{x})\Big]^{1/2}=\|\psi_0\|$$

by virtue of (7). □

This lemma shows that to study the asymptotic behavior of $\psi(\mathbf{x}, t)$ it is necessary to estimate c_N, $\varphi_N(\mathbf{x})$, and also to analyze the behavior of the highest eigenvalues λ_N, λ_{N-1} as $N \to \infty$. We make such an analysis for the case of a Gaussian potential ξ.

PROPOSITION 2.1. Let $\xi(\mathbf{x}, \omega)$ have the normal distribution $\mathbf{N}(0, \sigma^2)$. Then

$$t^{-1}\ln\psi(\mathbf{x}, t)=\sigma(d\ln t)^{1/2}-\varkappa+\gamma(\mathbf{x}, t),$$

where with probability 1 (uniformly with respect to x) $\gamma\to 0$ as $t \to \infty$.

We shall frequently require the following

LEMMA 2.2 (see [2], §4.4). With probability 1 in the limit $N \to \infty$

$$\xi_{(N)}=\sigma(2\ln N)^{1/2}+o(1). \tag{8}$$

By virtue of the symmetry, the asymptotic behavior (8) is also valid for $-\xi_{(1)}$. It follows from this that with probability 1 for sufficiently large N

$$\xi_{(N)}-\xi_{(1)}\leqslant 4\sigma(\ln N)^{1/2}. \tag{9}$$

LEMMA 2.3. With probability 1 for sufficiently large N

$$\varkappa N^{-2}\leqslant\varphi_N(\mathbf{x})\leqslant 1, \tag{10}$$

$$\varkappa N^{-2}\|\psi_0\|\leqslant c_N\leqslant\|\psi_0\|. \tag{11}$$

Proof. First $\varphi_N(\mathbf{x})\leqslant\|\varphi_N\|=1$. Further, by virtue of (6)

$$\varphi_N(\mathbf{x})\geqslant N^{-1/2}\frac{\lambda_N-\xi_{(N)}}{\lambda_N-\xi_{(1)}}. \tag{12}$$

It follows from Eq. (4) for $\lambda = \lambda_N$ that

$$\varkappa N^{-1}(\lambda_N-\xi_{(N)})^{-1}\leqslant 1\leqslant\varkappa(\lambda_N-\xi_{(N)})^{-1},$$

whence

$$\varkappa N^{-1}\leqslant\lambda_N-\xi_{(N)}\leqslant\varkappa. \tag{13}$$

Using (9) and (13), we have almost certainly for sufficiently large N

$$\lambda_N-\xi_{(1)}\leqslant\varkappa+4\sigma(\ln N)^{1/2}. \tag{14}$$

Substituting the estimates (13) and (14) in (12), we finally prove (10). Further, by the Cauchy–Schwarz inequality

$$c_N=(\psi_0,\varphi_N)\leqslant\|\psi_0\|\cdot\|\varphi_N\|=\|\psi_0\|.$$

On the other hand, by virtue of (10),

$$c_N\geqslant\varkappa N^{-2}\sum_{\mathbf{x}\in V}\psi_0(\mathbf{x})\geqslant\varkappa N^{-2}\|\psi_0\|,$$

since $\psi_0(\mathbf{x})\geqslant 0$, $\mathbf{x}\in V$. □

COROLLARY. With probability 1 for sufficiently large N

$$2\|\psi_0\|\geqslant\psi(\mathbf{x},t)\exp\{-(\lambda_N-\varkappa)t\}\geqslant\|\psi_0\|N^{-4}[\varkappa^2-N^4\exp\{-(\lambda_N-\lambda_{N-1})]t\}]. \tag{15}$$

The proof follows readily from Lemmas 2.1 and 2.3. □

We now turn to the study of the asymptotic behavior of λ_N,λ_{N-1} as $N\to\infty$.

LEMMA 2.4.

$$\lambda_N-\xi_{(N)}<\varkappa\left(1-\frac{m}{N}\right)[1-\varkappa(\xi_{(N)}-\xi_{(m)})^{-1}]^{-1}, \tag{16}$$

$$\lambda_{N-1}-\xi_{(N-1)}<\varkappa\left(1-\frac{m}{N}\right)[1-\varkappa(\xi_{(N-1)}-\xi_{(m)})^{-1}]^{-1} \tag{17}$$

for all m, $1\leqq m\leqq N-2$, such that the expressions in the square brackets in (16) and (17) are positive.

Proof. By virtue of Eq. (4) we obtain for $\lambda=\lambda_N$

$$1<\varkappa(\lambda_N-\xi_{(m)})^{-1}+\varkappa\left(1-\frac{m}{N}\right)(\lambda_N-\xi_{(N)})^{-1},$$

from which (16) follows. We can prove (17) similarly. □

Note that from the estimates (16) and (17) we shall be able to deduce $\lambda_N\sim\xi_{(N)}$ and $\lambda_{N-1}\sim\xi_{(N-1)}$ if we can choose the number m in such a way that for it (16) and (17) hold and $m/N\to 1$, $\xi_{(N)}-\xi_{(m)}\to\infty$, $\xi_{(N-1)}-\xi_{(m)}\to\infty$. This is our immediate aim.

We require some facts about the almost certain behavior of uniform order statistics $0<u_{(1)}<\ldots<u_{(N)}<1$ as $N\to\infty$.

LEMMA 2.5 [3]. Let k be fixed. Let $a_N\downarrow 0$ and $\sum N^{-1}a_N{}^k<\infty$. Then with probability 1 for sufficiently large N $u_{(k)}\geqslant N^{-1}a_N$.

LEMMA 2.6 [4]. Let k be fixed. If $a_N\uparrow$, $N^{-1}a_N\downarrow$ and $\sum N^{-1}a_N{}^k\exp(-a_N)<\infty$, then with probability 1 for sufficiently large N $u_{(k)}\leqslant N^{-1}a_N$.

LEMMA 2.7 [3]. Let $k=k_N\uparrow\infty$ and $N^{-1}k\sim p_N\downarrow 0$. If $k/\ln_2 N\to\infty$ and $Np_N/\ln_2 N\uparrow$, then with probability 1

$$\liminf_{N\to\infty}\frac{Nu_{(k)}-k}{(2k\ln_2 N)^{1/2}}=-1$$

(here and in what follows $\ln_2=\ln\ln$).

By means of Lemmas 2.5–2.7 we obtain a number of facts about the almost certain behavior of the original (normal) order statistics $\xi_{(1)}<\ldots<\xi_{(N)}$, going over from them in accordance with the formula

$$u_{(i)}=\Phi(\xi_{(i)}/\sigma) \tag{18}$$

to uniform order statistics (here, $\Phi(x)$ is the function of the normal distribution $N(0,1)$). Here, we shall use the well-known relation ([2], §2.3)

$$1-\Phi(x)\sim(2\pi)^{-1/2}x^{-1}\exp(-x^2/2),\quad x\to+\infty. \tag{19}$$

LEMMA 2.8. With probability 1 for sufficiently large N

$$\xi_{(N-1)}>\sigma(\ln N)^{1/2}.$$

Proof. By means of (18), we go over to the equivalent inequality

$$1-u_{(N-1)}<1-\Phi(\sqrt{\ln N}),$$

or (by virtue of the symmetry of $1-u_{(N-1)}$ and $u_{(2)}$)

$$u_{(2)}<1-\Phi(\sqrt{\ln N}). \tag{20}$$

In accordance with (19)

$$1-\Phi(\sqrt{\ln N})\sim C(N\ln N)^{-1/2},$$

and to prove (20) it remains to apply Lemma 2.6. □

Remark. Similarly, using Lemmas 2.5 and 2.6, we can prove Lemma 2.2.

LEMMA 2.9. We set $k=[\sqrt{N}]$, $m=N-k+1$. Then with probability 1 for sufficiently large N

$$\xi_{(m)}<\sigma(\ln N)^{1/2}.$$

The proof is made as in Lemma 2.8, using Lemma 2.7. □

LEMMA 2.10. With probability 1 as $N\to\infty$

$$\lambda_N=\xi_{(N)}+O(N^{-1/2}), \tag{21}$$

$$\lambda_{N-1}=\xi_{(N-1)}+O(N^{-1/2}). \tag{22}$$

Proof. It follows from Lemmas 2.2 and 2.9 that $\xi_{(N)}-\xi_{(m)}\to\infty$ (almost certainly) for $m=N-k+1$, $k=[\sqrt{N}]$. Then from (16) we obtain (21). Similarly, from (17) we obtain (22) by virtue of Lemmas 2.8 and 2.9. □

We now obtain a lower bound of the distance between the two highest order statistics $\xi_{(N-1)}$ and $\xi_{(N)}$.

LEMMA 2.11. With probability 1 for sufficiently large N

$$\xi_{(N)}-\xi_{(N-1)}>\sigma(\ln N)^{-2}.$$

Proof. Applying Lemma 2.5 for $k=1$ to the spacing $u_{(N)}-u_{(N-1)}$ and using (19), we have (almost certainly) for sufficiently large N

$$u_{(N)}-u_{(N-1)}=\Phi(\xi_{(N)}/\sigma)-\Phi(\xi_{(N-1)}/\sigma)>(N\ln^2 N)^{-1}. \tag{23}$$

Note that for all $x_1 < x_2$

$$\Phi(x_2)-\Phi(x_1)\leqslant\Phi'(x_1)(x_2-x_1).$$

Then from (23) we obtain

$$\xi_{(N)}-\xi_{(N-1)}>\sigma(2\pi)^{1/2}\exp(\xi_{(N-1)}^2/2\sigma^2)\cdot(N\ln^2 N)^{-1}, \tag{24}$$

and it remains to show that for sufficiently large N the right-hand side of (24) is larger than $\sigma(\ln N)^{-3}$, i.e.,

$$\xi_{(N-1)}>\sigma(2\ln N-2\ln_2 N)^{1/2},$$

which is proved in exactly the same way as Lemma 2.8. □

Thus, Lemmas 2.10 and 2.11 show that the expression in the square brackets in (15) is equivalent to a constant, since with allowance for (3)

$$(\lambda_N-\lambda_{N-1})t>Ct(\ln N)^{-2}\sim CN^{2/d}(\ln N)^{-2}\to\infty.$$

It then follows from (15) that

$$t^{-1}\ln\psi(\mathbf{x},t)=\lambda_N-\varkappa+o(1).$$

Finally, using (21) and (8), we obtain

$$t^{-1}\ln\psi(\mathbf{x},t)=\sigma(2\ln N)^{1/2}-\varkappa+o(1)=\sigma(d\ln t)^{1/2}-\varkappa+o(1).$$

The proof of Proposition 2.1 is completed. □

Remark. The solution $\psi(\mathbf{x},t)$ can be represented by means of the Kac–Feynman formula (see (26) below) in the form of a functional of an appropriate random walk η_t. In [5], in which the asymptotic behavior of $\psi(\mathbf{x},t)$ in the problem (2) with local Laplacian Δ is studied, it is emphasized that the main contribution to the growth of $\psi(\mathbf{x},t)$ is made, not by the typical trajectory η_t that moves to a distance $R\sim t^{\frac{1}{2}}$ from its initial position, but by a so-called optimal trajectory, corresponding to large walks $R\sim t$. However, in the mean-field model we study we have actually eliminated such walks from consideration by using the condition (3). In fact, this influences only the numerical coefficient of the term $(\ln t)^{\frac{1}{2}}$ in Proposition 2.1.

3. Asymptotic Growth of the Statistical Moments

For simplicity, we shall now assume that $\psi_0(x)\equiv 1$ (note that Propositions 3.1 and 3.2 given below can readily be extended to the case when $\psi_0(x)$ is a random homogeneous field independent of the potential ξ, i.e., when the random variables $\psi_0(x)$, $x\in V$, are symmetrically dependent). We are interested in the logarithmic asymptotic behavior of the statistical moments $m_p(x, t)=\langle\psi^p(x, t)\rangle$, $p=1, 2, \ldots$, as $t \to \infty$. We study initially the first moment $m_1(x, t)$.

PROPOSITION 3.1. We assume that the potential $\xi(x, \omega)$ has normal distribution $N(0, \sigma^2)$. Then in the limit $t \to \infty$

$$\ln m_1(x, t)=\sigma^2 t^2/2-\varkappa t+\alpha(t), \tag{25}$$

where $\alpha(t)=o(t)$. More precisely, in the one-dimensional case ($d=1$) $\alpha(t)=\varkappa t^{1/2}+o(t^{1/2})$, and for $d\geqslant 2$ $\alpha(t)=o(\ln t)$.

Proof. We consider in the volume V a random walk η_t that remains at a given point $x\in V$ for the indicative time with parameter $\varkappa$ and then jumps to one of the points $y\in V$ with equal probability $1/N$ (a transition $x\to x$ is also allowed). We can readily find for η_t the generating operator

$$Af(x)\equiv\lim_{t\downarrow 0} t^{-1}(M_x f(\eta_t)-f(x))=\varkappa\bar{\Delta}_V f(x)$$

(here, M_x is the average over all the trajectories $\eta_\cdot$ that at the time t = 0 leave the point x). Using the Kac–Feynman formula,* we can represent the solution of the problem (2) in the form

$$\psi(x, t)=M_x\exp\left\{\int_0^t \xi(\eta_s)\,ds\right\}\psi_0(\eta_t). \tag{26}$$

Let $\tau(y, t)$ be the total time the random walk η_s, $s \leqq t$, remains at the point $y\in V$; then

$$\int_0^t \xi(\eta_s)\,ds=\sum_{y\in V}\xi(y)\,\tau(y, t).$$

By means of the formula $\langle\exp\zeta\rangle=\exp(\langle\zeta^2\rangle/2)$ (for the normal random variable ζ with zero mean), we obtain from (26)

$$\langle\psi(x, t)\rangle=M_x\Phi_t(\eta_\cdot)\langle\psi_0(\eta_t)\rangle, \tag{27}$$

where

$$\Phi_t(\eta_\cdot)=\exp\left\{\frac{1}{2}\sigma^2\sum_{y\in V}\tau^2(y, t)\right\}. \tag{28}$$

By virtue of the homogeneity of the field ψ_0 the factor $\langle\psi_0(\eta_t)\rangle$ does not depend on the realization η_t, and it can be taken in front of the symbol M_x. Note also that by virtue of the symmetry of the points $x\in V$ the mean value $\langle\psi(x, t)\rangle$ does not depend on x. Therefore, the symbol of the mathematical expectation in (27) can be given without the index **x**, it being assumed that the initial position of the random walk η_t is chosen randomly with uniform distribution: $P\{\eta_0=x\}=1/N$, $x\in V$. Thus,

$$\langle\psi(x, t)\rangle=\langle\psi_0\rangle M\Phi_t(\eta_\cdot), \tag{29}$$

where $\Phi_t(\eta_\cdot)$ is determined by formula (28).

We denote by γ the number of different sites $y\in V$ visited by the random walk η_s, $s \leqq t$, and by μ the number of jumps made in the process (including jumps of the type $y\to y$). Obviously, $\mu + 1 \geqq \gamma$. Further, we denote

$$\tau_{\max}=\max_{y\in V}\tau(y, t)$$

and set $M\Phi_t(\eta_\cdot)=J_1+J_2+J_3$, where

*See [6] (§2.6). One can also readily verify directly that (26) gives the solution of the problem (2).

$$J_1=M\Phi_t(\eta.)I\{\gamma=1\}=\exp(\sigma^2t^2/2)P\{\gamma=1\},\quad J_2=M\Phi_t(\eta.)I\{\gamma\geqslant 2,\ \tau_{\max}<t(1-\varepsilon)\},$$
$$J_3=M\Phi_t(\eta.)I\{\gamma\geqslant 2,\ \tau_{\max}\geqslant t(1-\varepsilon)\},$$

and we choose

$$\varepsilon=(2\varkappa/\sigma^2)t^{-1}\to 0. \tag{30}$$

It is easy to find J_1. In accordance with the total probability formula

$$P\{\gamma=1\}=\sum_{m=0}^{\infty}P\{\mu=m\}P\{\gamma=1\,|\,\mu=m\}=\sum_{m=0}^{\infty}e^{-\varkappa t}\frac{(\varkappa t)^m}{m!}\left(\frac{1}{N}\right)^m=\exp(-\varkappa t+\varkappa t/N),$$

whence

$$J_1=\exp(\sigma^2t^2/2-\varkappa t+\varkappa t/N). \tag{31}$$

Note that in accordance with (31) the term J_1 already makes the main contribution to the asymptotic behavior (25).

Further, since

$$\sum_{\mathbf{y}}\tau^2(\mathbf{y},t)\leqslant\tau_{\max}\sum_{\mathbf{y}}\tau(\mathbf{y},t)=\tau_{\max}t,$$

we have by virtue of (30)

$$J_2\leqslant\exp(\sigma^2t^2(1-\varepsilon)/2)=\exp(\sigma^2t^2/2-\varkappa t), \tag{32}$$

which also agrees with (25).

It now remains to estimate J_3.

<u>LEMMA 3.1.</u> There is a representation of J_3 in the form

$$J_3=\sum_{k=2}^{N}C_N^ke^{-\varkappa t}(\varkappa t)^{-1}\int_0^{\infty}dy\,e^{-y}(\beta y)^k/(k-1)!\,MF(u_1,\ldots,u_k)I\{u_{\max}\geqslant 1-\varepsilon\}, \tag{33}$$

where $\beta=\varkappa t/N$, $u_{\max}=\max(u_1,\ldots,u_k)$,

$$F(u_1,\ldots,u_k)=\prod_{i=1}^{k}\exp(\sigma^2t^2u_i^2/2)f(\beta yu_i),\quad f(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!(n+1)!}, \tag{34}$$

and the random variables u_1, ..., u_k are uniform spacings on [0, 1], i.e., the lengths of the intervals formed when k − 1 random independent points are cast on the interval [0, 1] with uniform distribution.

<u>Proof.</u> We label the sites $\mathbf{x}\in V:\mathbf{x}_1,\ldots,\mathbf{x}_N$, and write τ_i in place of $\tau(\mathbf{x}_i,t)$. Let ν_i be the number of arrivals of the random walk η_s, $s\leqq t$, at the site $\mathbf{x}_i$, and τ_{ij} be the time it remains at the site $\mathbf{x}_i$ after the j-th arrival at $\mathbf{x}_i$ until the subsequent jump (or until the time t if there are no more jumps). Then $\tau_i=\tau_{i1}+\ldots+\tau_{i\nu_i}$.

We transform J_3 by means of the formula for the total mathematical expectation:

$$J_3=\sum_{k=2}^{N}\sum_{m=k-1}^{\infty}P\{\mu=m\}M\Phi_t(\eta.)I\{\gamma=k,\tau_{\max}\geqslant t(1-\varepsilon)\,|\,\mu=m\}=$$
$$\sum_{k=2}^{N}C_N^k\sum_{m=k-1}^{\infty}e^{-\varkappa t}\frac{(\varkappa t)^m}{m!}M\exp\left\{\frac{1}{2}\sigma^2(\tau_1^2+\ldots+\tau_k^2)\right\}\times$$
$$I\{\nu_1>0,\ldots,\nu_k>0,\ \tau_{\max}\geqslant t(1-\varepsilon)\,|\,\nu_1+\ldots+\nu_k=m+1\}. \tag{35}$$

We use the fact (see [7], Chap. 3, §3) that under the condition $\nu_1+\ldots+\nu_k=m+1$ (i.e., for a fixed number of jumps) the joint conditional distribution of the random variables $\{\tau_{ij}\}$, i = 1, ..., k, j = 1, ..., ν_i, is identical to the joint (unconditional) distribution of m + 1 uniform spacings on the interval [0, t]. Since the spacings are symmetrically dependent, the mathematical expectation in (35) depends only on the lengths of the spacings and not on their mutual disposition. Therefore, we can limit ourselves to the trajectories η_s, $s \leq t$, that do not leave each site x_i until they have spent there all the time τ_i, and we can take them into account in (35) with an appropriate polynomial weight. Then the mathematical expectation in (35) takes the form

$$\sum_{\substack{n_1>0,\ldots,n_k>0,\\ n_1+\ldots+n_k=m+1}} \frac{(m+1)!}{n_1!\ldots n_k!}\left(\frac{1}{N}\right)^{m+1} M\exp\left\{\frac{1}{2}\sigma^2t^2(\Delta_1^2+\ldots+\Delta_k^2)\right\}I\{\Delta_{\max}\geqslant 1-\varepsilon\}, \tag{36}$$

where Δ_i is the sum of the lengths $\nu_i = n_i$ of the successive spacings (on the interval [0, 1]). Knowing the density of the Δ_k distribution ([7], Chap. 1, §7)

$$p_{\Delta_k}(x)=\frac{m!}{(n_k-1)!(m-n_k)!}x^{n_k-1}(1-x)^{m-n_k}$$

and using the fact that for fixed $\Delta_k = t_k$ the random variables $\Delta_1,\ldots,\Delta_{k-1}$ have the same distribution as would be obtained if $m - n_k$ random points were cast on the interval $[0, 1 - t_k]$ (see [7], Chaps. 1, 3), it is easy to show that the distribution P_Δ of the vector $(\Delta_1, \ldots, \Delta_k)$ is absolutely continuous with respect to the joint distribution P_u of the spacings $(u_1, \ldots, u_k)$, and

$$\frac{dP_\Delta}{dP_u}(x_1,\ldots,x_k)=\frac{m!}{(n_1-1)!\ldots(n_k-1)!}x_1^{n_1-1}\ldots x_k^{n_k-1}/(k-1)!,$$

$$x_1+\ldots+x_k=1,\quad 0\leqslant x_i\leqslant 1,\quad i=1,\ldots,k.$$

Then

$$M\exp\left\{\frac{1}{2}\sigma^2t^2(\Delta_1^2+\ldots+\Delta_k^2)\right\}I\{\Delta_{\max}\geqslant 1-\varepsilon\}=$$

$$\frac{m!}{(n_1-1)!\ldots(n_k-1)!}\,\frac{1}{(k-1)!}M\exp\left\{\frac{1}{2}\sigma^2t^2(u_1^2+\ldots+u_k^2)\right\}u_1^{n_1-1}\ldots u_k^{n_k-1}I\{u_{\max}\geqslant 1-\varepsilon\}. \tag{37}$$

We substitute (37) in (36) and (35) and express the factor $(m + 1)!$ in terms of the gamma function:

$$(m+1)!=\int_0^\infty e^{-y}y^{m+1}\,dy.$$

Finally, interchanging the positions of the operations of summation and integration, we arrive at the formula

$$J_3=\sum_{k=2}^{N}C_N^k e^{-\varkappa t}(\varkappa t)^{-1}\int_0^\infty dy\, e^{-y}(\beta y)^k/(k-1)!\, M\exp\left\{\frac{1}{2}\sigma^2t^2(u_1^2+\ldots+u_k^2)\right\}I\{u_{\max}\geqslant 1-\varepsilon\}\times$$

$$\sum_{m=k-1}^{\infty}\ \sum_{\substack{n_1>0,\ldots,n_k>0,\\ n_1+\ldots+n_k=m+1}}\ \prod_{i=1}^{k}\frac{(\beta yu_i)^{n_i-1}}{n_i!(n_i-1)!}, \tag{38}$$

where $\beta=\varkappa t/N$. Since the terms in the double sum in (38) factorize, the summation can be carried out over each of the indices $n_1, \ldots, n_k$ independently. As a result, this double sum takes the form $\prod_i f(\beta yu_i)$, where the function f is determined by Eq. (34), and (33) is proved. □

LEMMA 3.2. For the function $f(z)$ defined in (34), the inequality $f(z) \leq \exp(2\sqrt{z})$ holds for $z \geq 0$.

Proof. The function $f(z)$ can be expressed in terms of the Bessel function ([8], Eq. 8.447(2))

$$I_1(z)=\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}\left(\frac{z}{2}\right)^{1+2n},$$

namely,

$$f(z)=\frac{1}{\sqrt{z}}I_1(2\sqrt{z}),\quad z>0. \tag{39}$$

We use the integral representation ([8], Eq. 3.534(1))

$$I_1(z)=(2z/\pi)\int_0^1 (1-x^2)^{1/2}\operatorname{ch}(xz)\,dx.$$

Then for $z \geq 0$

$$I_1(z)\leqslant(2z/\pi)\int_0^1 (1-x^2)^{1/2}\operatorname{ch} z\,dx=(z/2)\operatorname{ch} z,$$

from which, by virtue of (39), the inequality of the lemma follows. □

LEMMA 3.3. For the mathematical expectation in (33) we have the estimate

$$MF(u_1,\ldots,u_k)I\{u_{\max}\geqslant 1-\varepsilon\}\leqslant\exp\{\sigma^2t^2/2+2(\beta y)^{1/2}(1+(N\varepsilon)^{1/2})\}\cdot k!(\sigma^2t^2/2)^{-(k-1)}. \tag{40}$$

Proof. Since the random variables $u_1, \ldots, u_k$ are symmetrically dependent and occur symmetrically in the mathematical expectation (40),

$$MF(u_1,\ldots u_k)I\{u_{\max}\geqslant 1-\varepsilon\}=kMF(u_1,\ldots,u_k)\times$$
$$I\{u_{\max}\geqslant 1-\varepsilon,\ u_{\max}=u_1\}=kMF(u_1,\ldots,u_k)I\{u_1\geqslant 1-\varepsilon\} \tag{41}$$

(since $1-\varepsilon > 1/2$; see (30)). We estimate $F(u_1, \ldots, u_k)$, using Lemma 3.2 and Jensen's inequality for the function $x^{\frac{1}{2}}$:

$$F(u_1,\ldots,u_k)\leqslant\exp\{{}^1/{}_2\sigma^2t^2(u_1{}^2+\ldots+u_k{}^2)+2(\beta y)^{1/2}(u_1{}^{1/2}+\ldots$$
$$+u_k{}^{1/2})\}\leqslant\exp\{\sigma^2t^2u_1/2+2(\beta yu_1)^{1/2}+2(\beta y(k-1)(1-u_1))^{1/2}\}\leqslant\exp\{\sigma^2t^2u_1/2+2(\beta y)^{1/2}+2(\beta yN\varepsilon)^{1/2}\}.$$

Substituting in (41) and calculating

$$M\exp\{\sigma^2t^2u_1/2\}I\{u_1\geqslant 1-\varepsilon\}=(k-1)\exp\{\sigma^2t^2/2\}\int_0^\varepsilon x^{k-2}\exp\{-\sigma^2t^2x/2\}\,dx\leqslant$$
$$(k-1)!\exp\{\sigma^2t^2/2\}(\sigma^2t^2/2)^{-(k-1)},$$

we obtain (40). □

We complete the proof of Proposition 3.1. We apply the estimate of Lemma 3.3 to (33). Summing over k, we obtain

$$\sum_{k=2}^N kC_N{}^k(2\beta y\sigma^{-2}t^{-2})^k\leqslant N(1+2\beta y\sigma^{-2}t^{-2})^N\leqslant N\exp\{2\beta Ny\sigma^{-2}t^{-2}\}=N\exp\{\varepsilon y\}$$

(recall that $\beta=\varkappa t/N$, $\varepsilon=(2\varkappa/\sigma^2)t^{-1}$, see (30)), whence

$$J_3\leqslant CtN\exp\{\sigma^2t^2/2-\varkappa t\}\int_0^\infty \exp\{-ay+2by^{1/2}\}\,dy,$$

where

$$a=1-\varepsilon\to 1,\quad b=\beta^{1/2}+(\beta N\varepsilon)^{1/2}=(\varkappa t/N)^{1/2}+\sqrt{2}\varkappa/\sigma. \tag{42}$$

Calculating the integral by means of the substitution $z = y^{\frac{1}{2}} - b/a$, we obtain the estimate

$$\int_0^\infty \exp\{-ay+2by^{1/2}\}\,dy\leqslant a^{-1}\{1+2b(\pi/a)^{1/2}\exp(b^2/a)\}.$$

Then, by virtue of (42)

$$J_3\leqslant C_1N\exp\{\sigma^2t^2/2-\varkappa t+o(t)\}. \tag{43}$$

Thus, combining the estimates (31), (32), and (43), we can obtain (25). If we take into account in these estimates the connection (3), we can determine the form of $\alpha(t)$ in (25) more accurately. □

We turn to the higher (mixed) moments $m_p(\mathbf{X},t)=\langle\psi(\mathbf{x}_1,t)\ldots\psi(\mathbf{x}_p,t)\rangle$, where $\mathbf{X}=(\mathbf{x}_1,\ldots,\mathbf{x}_p)$, $\mathbf{x}_i\in V$ (some of the sites $\mathbf{x}_i$ may be coincident).

PROPOSITION 3.2. Suppose the potential $\xi(x, \omega)$ has normal distribution $N(0, \sigma^2)$. Then in the limit $t \to \infty$

$$\ln m_p(\mathbf{X}, t) = \sigma^2 p^2 t^2/2 - \varkappa p t + \alpha_p(t), \tag{44}$$

where $\alpha_p(t) = o(t)$. More precisely, for $d=1$ $\alpha_p(t) = O(t^{1/2})$, while for $d \geqslant 2$ $\alpha_p(t) = o(\ln t)$.

Proof. Introducing p independent copies $\eta_t^{(1)}, \ldots, \eta_t^{(p)}$ of the random walk η_t, we can, as in the case of (28)–(29), write $m_p(\mathbf{X}, t)$ in the form

$$m_p(\mathbf{X}, t) = M_{\mathbf{X}} \exp\left\{\frac{1}{2}\sigma^2 \sum_{y \in V}\left[\sum_{i=1}^{p} \tau_i(y, t)\right]^2\right\}, \tag{45}$$

where $\tau_i(y, t)$ is the total time that the random walk $\eta_s^{(i)}$, $s \leqslant t$, remains at the point y. Considering in (45) only the paths $\eta_\cdot^{(1)}, \ldots, \eta_\cdot^{(p)}$ that during time t do not leave their initial position, and taking into account their statistical weight $\exp(-p\varkappa t)$, we obtain for $m_p(\mathbf{X}, t)$ the lower bound

$$m_p(\mathbf{X}, t) \geqslant \exp\{\sigma^2 p^2 t^2/2 - p\varkappa t\}. \tag{46}$$

On the other hand, applying the elementary inequality

$$(a_1 + \ldots + a_p)^2 \leqslant p(a_1^2 + \ldots + a_p^2)$$

to the argument of the exponential in (45), we have

$$m_p(\mathbf{X}, t) \leqslant M_{\mathbf{X}} \exp\left\{\frac{1}{2}\sigma^2 p \sum_{y \in V} \sum_{i=1}^{p} \tau_i^2(y, t)\right\} = \left[M \exp\left\{\frac{1}{2}(\sigma\sqrt{p})^2 \sum_{y \in V} \tau^2(y, t)\right\}\right]^p = [\tilde{m}_1(t)]^p$$

(because of the independence of the processes $\eta_\cdot^{(1)}, \ldots, \eta_\cdot^{(p)}$). In accordance with (28)–(29), $\tilde{m}_1(t)$ is none other than the first moment $\langle\psi\rangle$ in the model (1)–(2) with σ replaced by $\sigma\sqrt{p}$. Using for $\tilde{m}_1$ Proposition 3.1, and taking into account the estimate (46), we obtain (44). □

Remark. The argument that we used in the proof of Proposition 3.2, which reduces the study of the asymptotic behavior of the higher moments m_p to consideration of the first moment, has a general nature and can be used in the realistic model with local Laplacian Δ.

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