17. L. Bianchi, Lezioni di Geometria Differenziale, 4th Edition, Bologna (1922-1923).
18. A. R. Forsyth, Theory of Differential Equations, Vols. 5-6, Dover Publications, New York (1959).
19. F. Lund, Ann. Phys. (N.Y.), **115**, 251 (1978).
20. V. K. Abalakin, E. P. Aksenov, E. A. Grebennikov, V. G. Demin, and Yu. A. Ryabov, Handbook of Celestial Mechanics and Astrodynamics [in Russian], Nauka, Moscow (1976).
21. A. N. Leznov and M. V. Savel'ev, Fiz. Elem. Chastits At. Yadra, **11**, 40 (1980).
22. A. N. Leznov and M. V. Savel'ev, Fiz. Elem. Chastits At. Yadra, **12**, 125 (1981).

FORMULATION OF GAUGE THEORIES OF GENERAL FORM.
II. GAUGE-INVARIANT RENORMALIZABILITY AND RENORMALIZATION STRUCTURE

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Gauge-invariant renormalizability is established for a gauge theory of general form in linear gauges, namely, both the renormalized and the unrenormalized action satisfy the Zinn-Justin equation; the Ward identities (in the Zinn-Justin form) remain the same on renormalization. For theories with closed algebra it is shown under the assumption that the locality hypothesis is valid that the renormalization of the action, which is complicated off the mass shell, reduces on the mass shell to the addition of gauge-invariant structures and a multiplicative renormalization of the fields. At the same time, the gauge-invariant structures that vanish on the classical equations of motion can be ignored.

1. The present paper is a direct continuation of [1] (Part I). Under some natural assumptions, we prove here the gauge-invariant renormalizability of gauge theories of general form as formulated in Part I, and we establish the structure of the renormalization of such theories both off and on the mass shell. Staying within the framework of perturbation theory, or rather a loop expansion, we restrict ourselves to theories that are index renormalizable.

By gauge-invariant renormalizability we understand renormalizability with conservation of the Ward identities, which express the symmetry in a quantum theory. Specifically, we have in mind the Ward identity for the generating functional Γ of the vertex functions in the Zinn-Justin form [2] (see §2).

This form, which completely expresses the gauge content of the theory, is remarkable above all by virtue of its invariance with respect to the gauge algebra, which makes it possible to pose the question of gauge-invariant renormalizability in general form. The answer to the question is affirmative, and the form of the Ward identity itself is invariant with respect to renormalizations (§3).

In speaking of the renormalization structure, we mean the structure of the modified renormalized action. From the point of view of the modified action, gauge-invariant renormalizability signifies continuous (with respect to the parameter η of the loop expansion) deformation of the original modified action S, $S \to S(\eta)$, with conservation of the Zinn-Justin equation for this deformation, $(S(\eta), S(\eta)) = 0$ (see (1.6)*) (§3). Accordingly, gauge-invariant renormalizability does not in general signify conservation of the original gauge algebra but rather a deformation of it associated with the renormalization of the fields.

For theories with an originally closed algebra, the structure of the renormalization off the mass shell consists of adding to the original action gauge-invariant counterterms with a subsequent canonical transformation of the variables, i.e., the fields and sources (in the terminology of [3], the fields and antifields). At the same time, it is possible (for theories that are not index renormalizable, with dimensional coupling constants) that the canonical transformation mixes the original fields and the ghost fields, with the consequence that the renormalization renders the gauge algebra open. This leads to a modification of the Faddeev–Popov rules even for theories with an originally closed algebra (§4).

* In referring to the equations of [1], we shall denote Eq. (6) of [1], for example, by (1.6).

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Although the renormalized action is local (in each order of the loop expansion), locality of the canonical transformation mentioned above cannot be proved in the framework of the most general treatment. If locality of the canonical transformation is assumed, and this is the gist of the locality hypothesis discussed below, then on the mass shell, i.e., for the S matrix, the modified action admits an important simplification, namely, the canonical transformation of the variables reduces to a multiplicative renormalization of the fields (§4).

The following proposition, which holds irrespective of the locality hypothesis and apparently has long been suspected by many scientists is especially important: Suppose that in some order of the loop expansion there appears a gauge-invariant local counterterm which vanishes on the classical equations of motion; then it can be omitted (ignored altogether!) on the mass shell in not only the given but also in the higher approximations (§4).

For theories with originally open algebras, the renormalization structure includes a preliminary canonical transformation to a closed algebra. The existence of this additional transformation prevented us finding significant simplifications of the on-shell renormalization structure. True, there is still the possibility of ignoring for the S matrix the counterterms that vanish on the classical equations of motion (§5).

We now discuss in more detail the assumptions under which the main results of the paper are obtained.

Of course, we assume the absence of anomalies, i.e., we assume the existence of a regularization that preserves the gauge algebra. Moreover, the regularization is assumed to be such (of dimensional type) that the singular (containing $\delta(0)$) local measures of integration and the Jacobians of the local changes of variables are equal to unity. The problem of allowance for the measure in other regularizations can be solved on the basis of [3].

For simplicity, we restrict ourselves to linear gauges.

Finally, for the case of closed algebras, we adopt the locality hypothesis, which is formulated as the assumption of locality of the representation of general form of the solution of the Ward identity for the counterterms and ultimately reduces to the assumption of locality of the canonical transformation that occurs in the renormalization. It should be noted that in all the cases when the Ward identity for the counterterms could be solved explicitly (Yang–Mills theories [4], two-dimensional chiral theories [5], gravitation R^2), this hypothesis has been confirmed.

2. This section is preparatory in nature. It is devoted to the formulation of the Ward identity for the generating functional Γ of the vertex functions in the Zinn-Justin form [2] and the formulation of a lemma needed to establish the renormalization structure.

We begin with the derivation of the Ward identity. According to [3], the generating functional Z of the Green's functions is constructed as follows (see also [1]):

$$Z(J,\theta,\bar{\theta})=\int d\Phi dc d\bar{c}\exp\left\{\frac{i}{\eta}(S_{\psi,e}+J_i\Phi^i+\theta_\alpha c^\alpha+\bar{\theta}_\alpha\bar{c}^\alpha)\right\}=\exp\frac{i}{\eta}W,$$

where $S_{\psi,e} = S_\psi(\Phi, c, K, L) - Q_\psi(N)$ and the index ψ means that instead of K, L, N we must make the substitution $K_i\to\delta\psi/\delta\Phi^i$, $L_\alpha\to\delta\psi/\delta c^\alpha$, $N_\alpha\to\delta\psi/\delta\bar{c}^\alpha$; $Q_\psi(N)$ is the gauge term which lifts the degeneracy of the modified action S_ψ, and ψ is a gauge fermion, so that $\delta\psi/\delta\bar{c}^\alpha$ is the subsidiary, or gauge, condition. Compared with [3], we have omitted the local measure of integration (see §1). Note that our method of introducing the gauge condition is not the most general,* though, first, it contains all the usual gauges and, second, it is quite sufficient for our purposes. Compared with [1], we have introduced the loop expansion parameter η and the additional source N_α with Grassmann parity P_α and ghost number 0.

In what follows, it is convenient to retain the sources K, L, N, for which it is sufficient to consider a gauge fermion ψ of the form

$$\psi(\Phi, c, \bar{c}, K, L, N)=K_i\Phi^i+L_\alpha c^\alpha+N_\alpha\bar{c}^\alpha+\psi(\Phi, c, \bar{c}).$$

We emphasize that since the S matrix does not depend on the choice of ψ it is also independent of

* It corresponds to the case when the gauge fermion ψ in [3] has the form $\psi=\bar{c}^\alpha f_\alpha(\pi)+\psi(\Phi, c, \bar{c})$, i.e., depends in a special manner on π.

the sources K, L, N. Accordingly, in the calculation of the S matrix there is no need to set $K = L = N = 0$.

As a consequence of the Zinn-Justin equation (1.6) for S, the effective action $S_{\psi,e}$ satisfies the equation (which we shall also call the Zinn-Justin equation; cf. [2])

$$S_{\psi,e,i}\frac{\delta S_{\psi,e}}{\delta K_i}+\frac{\delta S_{\psi,e}}{\delta c^\alpha}\frac{\delta S_{\psi,e}}{\delta L_\alpha}+\frac{\delta S_{\psi,e}}{\delta \bar{c}^\alpha}\frac{\delta S_{\psi,e}}{\delta N_\alpha}=0. \tag{1}$$

We introduce in the usual manner the generating functional of the vertex functions:

$$\Gamma(\Phi, c, \bar{c}, K, L, N)=W-J_i\Phi^i-\theta_\alpha c^\alpha-\bar{\theta}_\alpha\bar{c}^\alpha, \qquad \Phi^i=\delta W/\delta K_i, \quad c^\alpha=\delta W/\delta\theta_\alpha, \quad \bar{c}^\alpha=\delta W/\delta\bar{\theta}_\alpha.$$

Averaging Eq. (1) functionally, we obtain in the standard manner the Ward identity for Γ in the form

$$\Gamma_{,i}\frac{\delta\Gamma}{\delta K_i}+\frac{\delta\Gamma}{\delta c^\alpha}\frac{\delta\Gamma}{\delta L_\alpha}+\frac{\delta\Gamma}{\delta\bar{c}^\alpha}\frac{\delta\Gamma}{\delta N_\alpha}=0, \tag{2}$$

which exactly repeats the Zinn-Justin equation (1) for $S_{\psi,e}$. In the derivation, we have ignored the terms $\delta^2 S_\psi/\delta\Phi^i\delta K_i$, $\delta^2 S_\psi/\delta c^\alpha\delta L_\alpha$, $\delta^2 Q/\delta\bar{c}^\alpha\delta N_\alpha$, which are porportional to $\delta(0)$ (see §1). Allowance for the additional measure of integration [3] makes it possible to derive the Ward identity (2) without neglecting these terms; it does not lead to fundamental changes in any of the arguments, and for simplicity we shall systematically omit this measure, appealing to an appropriate regularization.

We note in passing that the use of Eq. (1) makes it possible to prove in a simple manner the gauge invariance of the S matrix, i.e., that it is independent of the choice of ψ. Indeed, consider the variation ΔZ due to a change $\Delta\psi$ of the gauge fermion:

$$\Delta Z=\int d\Phi dc d\bar{c}\left(\Delta\psi_{,i}\frac{\delta}{\delta K_i}+\frac{\delta\Delta\psi}{\delta c^\alpha}\frac{\delta}{\delta L_\alpha}+\frac{\delta\Delta\psi}{\delta\bar{c}^\alpha}\frac{\delta}{\delta N_\alpha}\right)\exp\left\{\frac{i}{\eta}\left(S_{\psi,e}+J\Phi+\theta c+\bar{\theta}\bar{c}\right)\right\}.$$

Integrating by parts and using (1) (and again omitting the local contributions proportional to $\delta(0)$), we find that the modification of Z is due to the modification of only the terms with the sources J, θ, $\bar{\theta}$:

$$\Delta Z=i\left[J_i\Delta\psi\frac{\delta}{\delta K_i}(-)^{P_i}+\theta_\alpha\Delta\psi\frac{\delta}{\delta L_\alpha}(-)^{P_\alpha+1}+\bar{\theta}_\alpha\Delta\psi\frac{\delta}{\delta N_\alpha}(-)^{P_\alpha+1}\right]Z,$$

where in $\Delta\psi(\Phi, c, \bar{c})$ it is necessary to make the substitution $\Phi^k\to -i\delta/\delta J_k$, etc. The standard proof of the equivalence theorem [6] shows that on the transition to the S matrix the corresponding terms can (up to a possible multiplicative renormalization of the fields) be ignored. But this means that the S matrix does not depend on the choice of ψ.

Below, for simplicity, we restrict ourselves to the case of linear gauges, when

$$\psi(\Phi, c, \bar{c})=\bar{c}^\alpha t_{\alpha i}\Phi^i, \qquad Q_\psi(N)={}^1/_2 N_\alpha\varkappa^{\alpha\beta}N_\beta,$$

where $t_{\alpha i}$ and $\varkappa^{\alpha\beta}$ do not depend on the fields.

In this case, there are two simplifying circumstances. First, $S_{\psi,e}$ depends on K and $\bar{c}$ only in the combination

$$\tilde{K}_i=K_i+\bar{c}^\alpha t_{\alpha i}, \tag{3}$$

and it satisfies the equation

$$\frac{\delta S_{\psi,e}}{\delta\bar{c}^\alpha}+(-)^{P_\alpha}t_{\alpha i}\frac{\delta S_{\psi,e}}{\delta K_i}=0.$$

But this means that Γ satisfies exactly the same equation, and, therefore, a dependence on K and $\bar{c}$ occurs in it only in the combination $\tilde{K}$ (3). Second, the dependence on N can be readily found explicitly, so that Γ can be represented in the form

$$\Gamma=\mathcal{T}(\Phi, c, \tilde{K}, L)-{}^1/_2(N_\alpha+t_{\alpha i}\Phi^i)\varkappa^{\alpha\beta}(N_\beta+t_{\beta j}\Phi^j). \tag{4}$$

We indicate explicitly in (4) that Γ depends on K and $\bar{c}$ only through $\tilde{K}$ (3); thus, we take into account completely the gauge condition. Below, we shall omit the tilde over $\tilde{K}$ in $\mathcal{T}$ (formally, this corresponds to $\bar{c} = 0$), remembering that in the corresponding final expressions it is necessary to make the substitution $K \to \tilde{K}$ (3) (thereby restoring the dependence on $\bar{c}$).

As a consequence of Eq. (2), $\mathcal{T}(\Phi, c, K, L)$ satisfies the same Zinn-Justin equation (1.6) as S (cf. [2]):

$$\mathcal{T}_{,i}\frac{\delta\mathcal{T}}{\delta K_i}+\frac{\delta\mathcal{T}}{\delta c^\alpha}\frac{\delta\mathcal{T}}{\delta N_\alpha}\equiv\frac{1}{2}(\mathcal{T},\mathcal{T})=0. \tag{5}$$

In establishing the renormalization structure, a key part is played by the equation

$$\Omega(S)A\equiv(S,A)=0,\qquad \Omega(S)=S_{,i}\frac{\delta}{\delta K_i}+\frac{\delta S}{\delta c^\alpha}\frac{\delta}{\delta L_\alpha}+(-)^{P_i}\frac{\delta S}{\delta K_i}\frac{\delta_L}{\delta\Phi^i}+(-)^{P_\alpha+1}\frac{\delta S}{\delta L_\alpha}\frac{\delta_L}{\delta c^\alpha} \tag{6}$$

($\delta_L/\delta\Phi^i$, $\delta_L/\delta c^\alpha$ are the symbols of the left derivatives*), which arises as a result of a small variation of the Ward identity (5). The results obtained below are based to a considerable degree on a certain representation which we now establish of the general solution of this equation for even functional $A = A(\Phi, c, K, L)$ with initial condition

$$A(\Phi,c,0,0)=0. \tag{7}$$

The operator $\Omega(S)$ in (6) has the property

$$\Omega^2(S)=0, \tag{8}$$

which is a consequence of the Zinn-Justin equation (1.6) for S. Equations (6), (7), (8) are a copy of Eqs. (1.13)-(1.15) with the general solution (1.16). Do we need to prove once more (which would be quite sufficient for our purposes) that any solution of Eqs. (6) and (7) can be represented in the form

$$A=\Omega(S)X, \tag{9}$$

which copies (1.16), and also assume that X can be taken to be local (it is obvious that X is defined up to terms of the form $\Omega(S)Y$) if A is a local functional? We note that $\Omega(S)$ can be represented in the form of an expansion with respect to the c fields:

$$\Omega(S)=W(S_0,R)+\sum_{n\geqslant 1}\Omega_n(S),$$

where $\Omega_n(S)$, $n \geq 1$, has homogeneity degree n with respect to the c fields. Then it follows from the lemma proved below that the necessary structure of the solution (9) follows from the already proven and assumed (locality hypothesis (!), [1]) properties of Eqs. (1.13) and (1.14).

We give this simple lemma, which generalizes in an obvious manner the well-known special cases [7, 8].

<u>LEMMA.</u> Suppose the operator $\Lambda(\alpha)$ has the form

$$\Lambda(\alpha)=\sum_{n=0}\alpha^n\Lambda_n$$

and has the property $\Lambda^2(\alpha) = 0$ (hence, $\Lambda_0^2 = 0$ as well). Suppose any solution of the equation

$$\Lambda_0 A=0 \tag{10}$$

with some α-independent linear subsidiary condition on A can be represented in the form

$$A=\Lambda_0 X \tag{11}$$

and $\Lambda(\alpha)X$ satisfies the same subsidiary condition. Then any solution $A(\alpha)$ of the equation

$$\Lambda(\alpha)A(\alpha)=0, \tag{12}$$

represented by a formal series in powers of α and satisfying the same subsidiary condition, has the form

$$A(\alpha)=\Lambda(\alpha)X(\alpha). \tag{13}$$

In addition, if $\Lambda(\alpha)$ and, hence, Λ_0 are local operators and any local solution of Eq. (10) can be represented in the form (11) with local X, then any local solution of Eq. (12) can be represented in the form (13) with local $X(\alpha)$.

For the proof, we substitute the expansion $A(\alpha)=\sum_{n=k}\alpha^n A_n$ in Eq. (12). In the lowest order in α, we

* These derivatives appear here as an exception to the condition adopted in [1], according to which the derivatives with respect to the fields are right derivatives and with respect to the sources left derivatives; we also recall that P_i, $P_\alpha + 1$ are the Grassmann parities of the fields Φ^i and ghosts c^α, respectively.

have $\Lambda_0 A_k = 0$. But A_k satisfies the subsidiary condition, and therefore $A_k = \Lambda_0 X_k$. We represent $A(\alpha)$ in the form

$$A(\alpha) = \Lambda(\alpha)(\alpha^k X_k) + A^{(1)}(\alpha).$$

It is obvious that $A^{(1)}(\alpha)$ satisfies (12) and that its expansion with respect to α begins with the term of order α^{k+1}. In addition, $\Lambda(\alpha)(\alpha^k X_k)$ satisfies the subsidiary condition, and therefore so does $A^{(1)}(\alpha)$. Substituting the expansion $A^{(1)}(\alpha)$ in (12), we obtain $\Lambda_0 A^{(1)}_{k+1} = 0$, whence $A^{(1)}_{k+1} = \Lambda_0 X_{k+1}$. Substituting $A(\alpha)$ in the form

$$A(\alpha) = \Lambda(\alpha)(\alpha^k X_k + \alpha^{k+1} X_{k+1}) + A^{(2)}(\alpha),$$

we see that $A^{(2)}(\alpha)$ satisfies (12) and the subsidiary condition, and its expansion begins with terms of order α^{k+2}, etc.; the proof is then by induction.

In our case $\Lambda(\alpha) = \Omega(S)$, $\Lambda_0 = W(S_0, R)$, the c fields play the part of α, and (7) is the subsidiary condition.

3. In this section, we prove the gauge-invariant renormalizability in the general case.

We consider the single-loop approximation for $\mathcal{T}$:

$$\mathcal{T} = \mathcal{T}_0 + \eta\mathcal{T}_1 + O(\eta^2) \equiv S + \eta\mathcal{T}_1 + O(\eta^2).$$

The single-loop contribution, $\mathcal{T}_1 = \mathcal{T}_{1,\,\mathrm{div}} + \mathcal{T}_{1,\,\mathrm{con}}$ contains the divergent (when the regularization is lifted) term $\mathcal{T}_{1,\mathrm{div}}$, which is a local functional of the type S with divergent coefficients. The complete Ward identity (5) generates in the first order in η the Ward identity for the single-loop divergences

$$\Omega(S)\mathcal{T}_{1,\,\mathrm{div}} = 0. \tag{14}$$

To eliminate the single-loop divergences, it is sufficient to subtract from the original modified action S the counterterm $\mathcal{T}_{1,\,\mathrm{div}}$, i.e., to go over to the renormalized (in the single-loop approximation) action $S^{(1)}(\eta) = S - \eta\mathcal{T}_{1,\,\mathrm{div}}$. By virtue of (1.6) and (14), $S^{(1)}(\eta)$ satisfies a Zinn-Justin equation of the form (1.6) up to terms (which are completely determined) of second order in η:

$$(S^{(1)}, S^{(1)}) = \eta^2(\mathcal{T}_{1,\,\mathrm{div}}, \mathcal{T}_{1,\,\mathrm{div}}) \equiv \eta^2 Q^{(2)}. \tag{15}$$

The generating functional of the vertices constructed from $S^{(1)}$ and renormalized in the single-loop, i.e., linear in η, approximation, $\mathcal{T}^{(1)}(\eta)$, is finite in the single-loop approximation:

$$\mathcal{T}^{(1)}(\eta) = S + \eta\mathcal{T}_{1,\mathrm{con}} + \eta^2\mathcal{T}_2^{(1)} + O(\eta^3),$$

and it satisfies the Ward identity (5) up to terms of second order in η:

$$(\mathcal{T}^{(1)}, \mathcal{T}^{(1)}) = \eta^2 Q^{(2)} + O(\eta^3), \tag{16}$$

the derivation of Eq. (16) being completely analogous to the derivation of Eq. (5) with the right-hand side written down in the necessary η^2 approximation, which for it is identical with the tree approximation.

The two-loop contribution to $\mathcal{T}^{(1)}(\eta)$, like the single-loop approximation, contains a local divergent term: $\mathcal{T}_2^{(1)} = \mathcal{T}^{(1)}_{2,\,\mathrm{div}} + \mathcal{T}^{(1)}_{2,\,\mathrm{con}}$. By virtue of (16), $\mathcal{T}^{(1)}_{2,\,\mathrm{div}}$ satisfies the equation

$$(S, \mathcal{T}^{(1)}_{2,\,\mathrm{div}}) = {}^1/_2 Q^{(2)}. \tag{17}$$

Elimination of the two-loop divergences is achieved by transition to the renormalized (already in the two-loop approximation) action

$$S^{(2)}(\eta) = S^{(1)} - \eta^2\mathcal{T}^{(1)}_{2,\,\mathrm{div}} \equiv S - \eta\mathcal{T}_{1,\,\mathrm{div}} - \eta^2\mathcal{T}^{(1)}_{2,\,\mathrm{div}},$$

which by virtue of (15) and (17) satisfies a Zinn-Justin equation of the form (1.6) up to completely determined terms of third order in η:

$$(S^{(2)}, S^{(2)}) = \eta^3 Q^{(3)} + O(\eta^4).$$

The generating functional of the vertices constructed from $S^{(2)}$ and renormalized in the two-loop, i.e., quadratic in η approximation, $\mathcal{T}^{(2)}(\eta)$, is finite in the two-loop approximation,

$$\mathcal{T}^{(2)}(\eta) = S + \eta\mathcal{T}_{1,\mathrm{con}} + \eta^2\mathcal{T}^{(1)}_{2,\,\mathrm{con}} + \eta^3\mathcal{T}_3^{(2)} + O(\eta^4),$$

and satisfies the Ward identity (5) up to terms of third order in η:

$$(\mathcal{T}^{(2)}, \mathcal{T}^{(2)}) = \eta^3 Q^{(3)} + O(\eta^4) \tag{18}$$

(the comments on (18) are the same as on Eq. (16)). The three-loop contribution $\mathcal{T}_3^{(2)}=\mathcal{T}_{3,\mathrm{div}}^{(2)}+\mathcal{T}_{3,\mathrm{con}}^{(2)}$ to $\mathcal{T}^{(2)}(\eta)$ contains the divergent term $\mathcal{T}_{3,\mathrm{div}}^{(2)}$, which by virtue of (18) satisfies the equation $(S,\mathcal{T}_{3,\mathrm{div}}^{(2)})=\frac{1}{2}Q^{(3)}$, etc. The further arguments are obvious: it remains to apply the method of induction with respect to the number of loops. The final result is as follows: the completely renormalized modified action

$$S_R(\eta)\equiv S^{(\infty)}(\eta)=S-\sum_{n=1}^{\infty}\eta^n\mathcal{T}_{n,\mathrm{div}}^{(n-1)}, \tag{19}$$

which is local in each finite order in η, exactly satisfies the Zinn-Justin equation $(S_R(\eta),S_R(\eta))=0$. Accordingly, the completely renormalized generating functional of the vertices, $\mathcal{T}_R(\eta)\equiv\mathcal{T}^{(\infty)}(\eta)$, exactly satisfies the Ward identity (5): $(\mathcal{T}_R(\eta),\mathcal{T}_R(\eta))=0$. The proof is completed.

4. We see that the gauge-invariant renormalization reduces to a continuous deformation of the original modified action, $S\to S_R(\eta)$, this preserving the Zinn-Justin equation (1.6). In this section, we shall describe the general form of a deformation that leaves the Zinn-Justin equation invariant, and we shall thereby establish the renormalization structure in the general case.

We consider first the case of a closed algebra, when as the renormalized action S (the choice of which for given S_0 and even given R^i_α is, as we know from [1], not unique) we take the minimal construction of the Zinn-Justin equation (1.38). The consideration of this case is key. In what follows in this section, except for a special stipulation, we understand by S its minimal construction.

First of all, we find the general solution of (14) (in the framework of formal series in the c fields). We can represent $\mathcal{T}_{1,\mathrm{div}}$ in the form

$$\mathcal{T}_{1,\mathrm{div}}=\mathcal{T}_{1,\mathrm{div};0}(\Phi)+K_i\mathcal{T}_{1,\mathrm{div};1\alpha}^{\,i}(\Phi)c^\alpha+O(c^2).$$

Considering (14) in the first order in the c fields, we obtain

$$\frac{\delta\mathcal{T}_{1,\mathrm{div};0}(\Phi)}{\delta\Phi^i}R_\alpha{}^i(\Phi)+S_0(\Phi)_{,i}\mathcal{T}_{1,\mathrm{div};1\alpha}^{\,i}(\Phi)=0. \tag{20}$$

This equation shows that $\mathcal{T}_{1,\mathrm{div};0}$ can be represented in the form

$$\mathcal{T}_{1,\mathrm{div};0}(\Phi)=-\Delta S_0^{(1)}(\Phi)-S_0(\Phi)_{,i}X^{(1)i}(\Phi), \tag{21}$$

where $\Delta S_0^{(1)}$ is a gauge-invariant functional:

$$\Delta S_0^{(1)}(\Phi)_{,i}R_\alpha{}^i(\Phi)=0. \tag{21a}$$

To see this, we note, first, that the form (20) and the representation (21) are invariant with respect to possible changes of the variables Φ^i and, second, according to [1] there exist among all possible sets of variables $\{\Phi\}$ sets $\{\Phi^i\}=\{\omega^a,\Phi^\beta\}$ such that in terms of them the generators R^i_α have the form $R^a_\alpha=0$. Now R^β_α is a matrix with an inverse and, accordingly, $S_0(\Phi)=\tilde{S}(\omega)$ does not depend on Φ^β. Rewriting (20) in these variables, we obtain

$$\frac{\delta\tilde{\mathcal{T}}_{1,\mathrm{div};0}}{\delta\Phi^\alpha}+\tilde{S}_0(\omega)_{,a}\mathcal{T}_{1,\mathrm{div};1\beta}^{\,a}(R^{-1})_\alpha{}^\beta=0,$$

from which the representation (21) readily follows.

We now represent $\mathcal{T}_{1,\mathrm{div}}$ in the form

$$\mathcal{T}_{1,\mathrm{div}}=-\Delta S_0^{(1)}(\Phi)-\Omega(S)K_iX^{(1)i}(\Phi)+\Delta\mathcal{T}_{1,\mathrm{div}}.$$

By virtue of (14), (21a), and (8), $\Delta\mathcal{T}_{1,\mathrm{div}}$ by itself satisfies (14) and, in addition, its expansion with respect to the c fields does not contain a zeroth term, so that by virtue of conservation of the ghost number it vanishes for K = L = 0. Therefore, in accordance with Sec. 2 (see the discussion of Eqs. (6) and (7)) $\Delta\mathcal{T}_{1,\mathrm{div}}$ admits a representation of the type (9):

$$\Delta\mathcal{T}_{1,\mathrm{div}}=-\Omega(S)\Delta X_1.$$

Finally, we find that the general solution of (14) can be represented in the form

$$\mathcal{T}_{1,\mathrm{div}}=-\Delta S_{0;1}(\Phi)-\Omega(S)X_1,\quad \Delta S_{0;1}=\Delta S_0^{(1)},\quad X_1=K_iX^{(1)i}+\Delta X_1. \tag{22}$$

We now represent the renormalized modified action S_R (19) in the form $S_R=S_{1R}+A_2$, where

$$S_{1R}(\Phi, c, K, L)=S_0(\Phi')+\eta\Delta S_{0;1}(\Phi')+K_i'R_\alpha{}^i(\Phi')c'^\alpha+{}^1/_2L_\alpha' f_{\beta\gamma}{}^\alpha c'^\beta c'^\gamma,$$

and the variables Φ', c', K', L' are obtained from Φ, c, K, L by a canonical transformation with generating functional $X_{1R}(\Phi, c, K', L')=K_i'\Phi^i+L_\alpha' c^\alpha+\eta X_1(\Phi, c, K', L')$.

It is obvious that the expansion of A_2 in a series in η begins with the terms of order η^2: $A_2=\eta^2T_2+O(\eta^3)$. From the Zinn-Justin equation for S_R there now follows the equation $\Omega(S)T_2 = 0$ of the form (14), whose solution can be represented in the form (22),

$$T_2=\Delta S_{0;2}(\Phi)+\Omega(S)X_2,$$

where X_2 is a functional, and $\Delta S_{0;2}$ satisfies (21a). We represent S_R in the form $S_R=S_{2R}+A_3$, where

$$S_{2R}(\Phi, c, K, L)=S_0(\Phi')+\eta\Delta S_{0;1}(\Phi')+\eta^2\Delta S_{0;2}(\Phi')+K_i'R_\alpha{}^i(\Phi')c'^\alpha+{}^1/_2L_\alpha' f_{\beta\gamma}{}^\alpha c'^\beta c'^\gamma,$$

and the variables Φ', c', K', L' are obtained from Φ, c, K, L by a canonical transformation with generating functional $X_{2R}=X_{1R}+\eta^2X_2(\Phi, c, K', L')$. It is obvious that the expansion of A_3 in a series in η begins with the terms of order η^3. It remains to apply the method of induction with respect to the number of loops in order to obtain the final result: $S_R(\eta)$ can be represented in the form

$$S_R(\eta, \Phi, c, K, L)=S_{0;R}(\Phi')+K_i'R_\alpha{}^i(\Phi')c'^\alpha+{}^1/_2L_\alpha' f_{\beta\gamma}{}^\alpha c'^\beta c'^\gamma, \tag{23}$$

where

$$S_{0;R}=S_0+\eta\Delta S_{0;1}+\eta^2\Delta S_{0;2}+\ldots \tag{23a}$$

is a gauge-invariant functional of the original fields Φ and the variables Φ', c', K', L' are obtained from Φ, c, K, L by a canonical transformation whose generating functional is

$$X_R(\Phi, c, K', L')=K'\Phi+L'c+\eta X_1(\Phi, c, K, L)+\ldots. \tag{23b}$$

The complete renormalized action in such a case has the form

$$S_{R,\Psi,\Theta}(\Phi, c, K, L)=S_R(\Phi, c, \tilde{K}, L)-{}^1/_2(N_\alpha+t_{\alpha i}\Phi^i)\varkappa^{\alpha\beta}(N_\beta+t_{\beta j}\Phi^j),$$

where $\tilde{K}$ is given by (3).

We note that this result was actually already contained in [9]. However it was based essentially on the hypothesis of locality of the representation (22) of the general solution of (14) for the counterterms, i.e., on locality of ΔS_0 and X separately. Moreover, the very representation (22), which is proven here, was a part of the hypothesis in [9]. It should also be noted especially that at the time when the paper [9] was written the given renormalization scheme could actually be applied only to theories of Yang–Mills type, in which the canonical transformation does not mix the fields Φ and the ghosts c, being linear in the c fields. Otherwise, the gauge algebra is open (see below), the action contains the c fields in an essentially nonlinear manner, and a correct formulation of theories of such type does not then exist.

Of course, locality of the representation (22), i.e., locality of $S_{0,R}(\Phi)$ and $X_R(\Phi, c, K', L')$ separately for local $S_R(\eta)$, is extremely desirable. If locality of the representation cannot be ensured, then in local quantum field theory it becomes meaningless to a large degree, since many of its important consequences are lost, in particular, those relating to the deformation of the gauge algebra. Therefore, we defer the discussion of the renormalization structure to Sec. 5, in which we discuss the locality hypothesis in a form sufficient for our purposes. However, it must be emphasized once more that gauge-invariant renormalizability is still true without the locality hypothesis.

It remains to say a few words about the case (admittedly, purely academic) when the original modified action of a theory with a closed algebra is not a minimal Zinn-Justin action. In this case, as we know [1], it differs from such an action by only a canonical change of variables, which, obviously, is simply imposed on the renormalized canonical transformation described above, namely, one must first go over to variables in which the action is minimal, carry out the renormalization procedure described above, and then make the inverse change of variables.

The problem of the renormalization structure of gauge theories with open algebra can be solved in exactly the same way. In this case (see [1]) there exists a canonical transformation to variables in which the gauge algebra is closed and the modified action has the form of the minimal Zinn-Justin construction.

A difference from the case just analyzed is that here the preliminary canonical transformation affects the generators of the gauge transformations, mixing the fields Φ and the ghosts c. Accordingly, though in variables in which the algebra is closed, the renormalized action has the form (23), and $S_{0;R}$ is invariant with respect to the generators of the closed algebra but, in general, not with respect to the generators of the original open algebra. In addition, the generators of the closed algebra and, hence, the representation (23) itself may be nonlocal. If the generators of the closed algebra are local, one can begin directly with the closed algebra, though sometimes there are reasons which still make it more convenient to work with the original generators.

5. In this section, we discuss the sufficient conditions and consequences of locality of the representation (23) of the renormalized action $S_R(\eta)$ for the case of closed algebras. Sufficient conditions are ensured by the locality hypothesis (see [1] and Sec. 2), which is augmented here by a new assumption. This new assumption is that in the representation (21) of the general local solution of Eq. (20) the functionals ΔS_0 and X can be taken to be local. Together with the assumption of locality of the representation (1.16) of the general solution of Eqs. (1.13) and (1.14), from which there follows locality of the representation (9) of the general solution of Eqs. (6) and (7) (see Sec. 2), this is sufficient to ensure locality of the representation (22) of the general solution of the Ward identities (14) for the counterterms.

A direct consequence of the locality hypothesis is locality of the representation (23) of the renormalized action $S_R(\eta)$ in the case of closed algebras. This means that the gauge-invariant term $S_{0,R}(\eta)$, like the canonical transformation Φ, c, K, $L \to \Phi'$, c', K', L', can be assumed local. We assume that all this is true in the framework of the loop expansion, i.e., in each finite order in η.

We discuss in more detail the structure of the renormalization (23) under the condition of its locality, concentrating on the deformation of the gauge algebra on renormalization; this deformation is determined by the first two coefficients of the expansion of S_R with respect to the c fields. It can be shown that in the general case the structure of the gauge algebra changes radically, and the algebra becomes open, although a closed algebra corresponds to the original action.

The renormalization includes two elements: the (usual) addition of gauge-invariant counterterms and the (less usual) canonical transformation of all the variables, from Φ, c, K, L to Φ', c', K', L'. In terms of the new variables, the renormalized action looks like the usual Zinn-Justin action with the same gauge algebra; however, the new variables are, in general, complicated nonlinear (divergent!) functionals of the finite original fields Φ, c, K, L. Accordingly, in terms of the latter, the renormalized action has a complicated form in the general case. The specific features of the theory are in fact contained in the form of the canonical transformation, i.e., the generating functional X_R. It is here evident that dimensional considerations and renormalizability or nonrenormalizability of the theory with respect to the index are decisive. These considerations determine two fundamentally different cases. In the first case, which corresponds to theories that are index renormalizable, the canonical transformation does not mix the fields Φ and the ghosts c: $\Phi' = \Phi'(\Phi)$ does not depend on c. A further detail here is that, depending on the dimensions of the fields Φ, their renormalization may be either multiplicative (for dimensional fields of the Yang–Mills type) or a nonlinear reparametrization of general form (for dimensionless fields such as two-dimensional chiral fields [5]). In the second case, the canonical transformation mixes the fields Φ and the ghosts c: $\Phi' = \Phi'(\Phi, c, \ldots)$ depends essentially on the c fields. From the point of view of the gauge algebra, the difference between these two cases is that in the first the deformation of the algebra is trivial, and leaves it closed, whereas in the second case the algebra becomes open as a result of the deformation. We shall discuss this fact below, illustrating it by taking the examples of Yang–Mills theories and Einsteinian gravitation, which were anlayzed in detail in [4] and [10].

If in the case of four-dimensional Yang–Mills theories with dimensionless coupling constants (which are index renormalizable and essentially exhaust this class of theories) the fields, sources, and derivatives (momenta) are ascribed the natural canonical dimensions, then the renormalized modified action S_R (like S) is a local functional of dimension less than or equal to 4. Then, with allowance for the conservation of the ghost number the generating functional X_R of the canonical transformation can have only the following form:

$$X_R = K_i'(\Lambda^i + t_j{}^i \Phi^j) + L_\alpha' \lambda_\beta{}^\alpha c^\beta,$$

where Λ^i, $t_j{}^i$, $\lambda_\beta{}^\alpha$ do not depend on the fields. Making corresponding calculations, we can calculate all the S_R, but we are interested in $S^i_{R;1\alpha}$ (the coefficients in the term $K_i S^i_{R;1\alpha} c^\alpha$), which are the renormalized generators of the renormalized action $S_{R;0}(\Phi)$ in accordance with the Zinn-Justin equation for S_R. It is readily seen that the deformation of the algebra consists of a transition to an equivalent representation, a shift of the fields, $\Phi^i \to t_j{}^i \Phi^j + \Lambda^i$, and replacement of the original generators by linear combinations of them with matrix λ^α_β. Following the mathematicians, we shall say that such a deformation is trivial. Putting it briefly, triviality of the deformation in renormalizable theories takes the form that the gauge algebra (group) remains the same after the renormalization.

In the case of a Yang–Mills theory of general form, this result was obtained by direct solution of the Ward identities (14) for the counterterms [4] and, thus, the locality hypothesis was in fact proved in this case. It should be emphasized that in the case of a simple gauge group with several Abelian components (of the type U(1)) the Ward identities alone do not ensure triviality of the deformation of the gauge algebra, and it was necessary to invoke essential additional arguments valid in index renormalizable quantum field theories.

In the case of index-nonrenormalizable theories with dimensional coupling constants the situation is different. The important thing is that here, first, there are no restrictions on the dimension of the counterterms; their dimension in a concrete approximation is bounded but it increases with increasing approximation; second, to calculate the dimension of the counterterms the fields must now be ascribed non-canonical dimensions. For example, in gravitation, in which the gravitational field and the fields c, $\bar{c}$ are ascribed dimension 0, and the sources K, L (and the derivative) the dimension 1, the index (degree) of divergence of the diagrams of the n-loop approximation is $2n+2-N_K-N_L-N_{\bar{c}}$, where N_K, N_L, $N_{\bar{c}}$ are the numbers of external K, L, and $\bar{c}$ lines, and the dimension of the counterterms in the n-loop approximation is less than or equal to 2n + 2. As a consequence, the c fields can appear in the renormalized modified action in a power higher than the second, and the sources K and L in a power higher than the first. This fact alone, which is common to index-nonrenormalizable theories, indicates that the renormalized theory must be associated with an open algebra of gauge transformations. We find an expression for the renormalized $S_{R;0}$ and the generators $S_{R;1}$, for which we write down an expansion of X_R with respect to the c fields to the necessary accuracy:

$$X_R=K_i'\tilde{\Phi}^i(\Phi)-{}^1/_2K_i'K_j'\tilde{X}_\alpha{}^{ij}(\Phi)(-)^{P_i}c^\alpha+L_\alpha'X_\beta{}^\alpha(\Phi)c^\beta+\ldots$$

In the approximation in c and K that we need,

$$\Phi'^{\,i}=\tilde{\Phi}^i(\Phi)-K_j'\tilde{X}_\alpha{}^{ij}(\Phi)(-)^{P_i}c^\alpha+\ldots,\quad K_i'=K_j\delta\Phi^j/\delta\tilde{\Phi}^i+\ldots,\ c'^{\,\alpha}=X_\beta{}^\alpha c^\beta+\ldots,$$

and the renormalized modified action S_R has the form

$$S_R=S_{0;R}(\tilde{\Phi}(\Phi))+K_i\left[\frac{\delta\Phi^i}{\delta\tilde{\Phi}^j}R_\beta{}^j(\tilde{\Phi})X_\alpha{}^\beta(\Phi)+S_{0;R}(\tilde{\Phi}(\Phi))_{,j}X_\alpha{}^{ij}(\Phi)\right]c^\alpha+\ldots,$$

$$X_\alpha{}^{ij}(\Phi)=(-)^{P_j(P_l+P_l)}\frac{\delta\Phi^i}{\delta\tilde{\Phi}^l}\frac{\partial\Phi^j}{\delta\tilde{\Phi}^m}\tilde{X}_\alpha{}^{lm}(\Phi)=(-)^{P_iP_j+1}X_\alpha{}^{ji}(\Phi).$$

It can be seen from this that the deformation of the algebra reduces not only to a change of the variables, $\Phi\to\tilde{\Phi}(\Phi)$, and a transition to linear combinations of the original generators by means of the matrix X_α^β (which still keeps the algebra closed), but also includes the addition of trivial generators, $S_{0;R}(\tilde{\Phi}(\Phi))_{,j}X_\alpha{}^{ij}(\Phi)$, which open the algebra. However trivial and even necessary (the gauge symmetry of $S_{R;0}$ is preserved!) this addition may appear, it is necessary to eliminate the divergences. Thus, we must necessarily regard the resulting renormalized theory as a theory with open algebra. Of course, for gauge-invariant renormalizability it is sufficient if the renormalized modified action satisfies the Zinn-Justin equation. The words "open algebra" mean in this context that the renormalized action has a more complicated form than that prescribed by the Faddeev–Popov rules.

We should like to emphasize especially two comments that appear to us very relevant and concern the on-shell renormalization structure. We have seen above that the renormalization of the theory as a whole, i.e., the renormalization of the Green's function off the mass shells, can be a rather refined business even in the case of closed algebras. However, if we restrict ourselves to calculation of the S matrix (calculation of the on-shell Green's functions), the expression for the renormalized action can be appreciably simplified. Indeed, the renormalized modified action S_R (21) is an action of Zinn-Justin type of canonically transformed variables. But in [1] it was shown that the canonical transformation in the modified action reduces on the mass shell to multiplicative renormalization of the field. Therefore, to calculate the S matrix, we need not consider (23) but can restrict ourselves to the effective action

$$S_{R,\Psi,e}=S_{0;R}(Z\Phi)+\bar{c}^\alpha t_{\alpha i}Z_j^{-1i}R_\beta{}^j(Z\Phi)c^\beta-{}^1/_2t_{\alpha i}\Phi^i\varkappa^{\alpha\beta}t_{\beta j}\Phi^j,$$

where $Z\equiv Z_j^i$ is a constant matrix; we have set K = L = N = 0. In other words, to eliminate the divergences from the S matrix it is sufficient to take into account the gauge-invariant counterterms in addition to the multiplicative renormalization of the fields Φ.

When some of the terms in $S_{0;R}(\Phi)$ are proportional to $S_0(\Phi)_{,i}$ or, as one says, disappear on the equations of motion, a further simplification of the renormalized action in the S matrix is possible. Suppose, for example, (to be specific) that $\Delta S_0^{(1)}(\Phi)$ in (21) has the form

$$\Delta S_0^{(1)}(\Phi)=\Delta S_0^{(1,1)}(\Phi)+S_0(\Phi)_{,i}\Lambda^{(1)i}(\Phi) \tag{24}$$

with some local $\Lambda^{(1)i}$, and each term in (24) is gauge invariant. Then, redefining $X^{(1)i}$ by including in it $\Lambda^{(1)i}$, we can assume that in (22) $\Lambda^{(1)i}=0$. In other words, in the construction of $S_{0;R}$ for the S matrix we need include in it only the gauge-invariant structures that do not vanish on the equations of motion.

This last result appears very important and helpful. Only by means of it can we justify the fact, previously assumed without proof, that the counterterms which vanish on the equations of motion make no contribution to the S matrix in not only the approximation in which they appear (which is more obvious) but also not in the higher approximations (which was not obvious).

In particular, if all the gauge-invariant counterterms except, perhaps, those proportional to S_0 vanish on the equations of motion, then to eliminate the divergences from the S matrix we require only ordinary multiplicative renormalizations of the fields and charges of the original action. Such a situation obtains in the single-loop approximation in pure gravitation and in the single- and two-loop approximations in supergravity.

For open algebras, for the reasons given above, we have not succeeded in finding a significant simplification of the on-shell renormalized action, though here too it is possible to omit the gauge-invariant counterterms which vanish on the equations of motion.

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LITERATURE CITED

1. B. L. Voronov and I. V. Tyutin, Teor. Mat. Fiz., **50**, 333 (1982).
2. J. Zinn-Justin, "Renormalization of gauge theories," in: Lecture Notes in Physics, Vol.37, Springer (1975), pp.2-39.
3. I. A. Batalin and G. A. Vilkovisky, Phys. Lett. B, **102**, 27 (1981).
4. B. L. Voronov and I. V. Tyutin, "Gauge-invariant renormalizability of gauge theories," Preprints Nos.137 and 253 [in Russian], P. N. Lebedev Physics Institute, Moscow (1978).
5. B. L. Voronov and I. V. Tyutin, Yad. Fiz., **33**, 1137 (1981).
6. R. É. Kallosh and I. V. Tyutin, Yad. Fiz., **17**, 193 (1973).
7. J. A. Dixon and J. C. Taylor, Preprint (74-74), University of Oxford.
8. S. Joglekar and B. Lee, Ann. Phys. (N.Y.), **97**, 160 (1976).
9. J. A. Dixon, Nucl. Phys. B, **99**, 420 (1975).
10. B. L. Voronov and I. V. Tyutin, Yad. Fiz., **33**, 1710 (1981).