EQUATIONS FOR TWO-POINT CORRELATION FUNCTIONS ON COMPACT RIEMANN SURFACES

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The general structure of the regularized scalar Green's function on compact Riemann surfaces is investigated. Equations that relate the second (and higher) derivatives of the scalar propagator to the first derivatives are obtained.

1. Introduction

Some recent papers, based on various approaches, have derived and investigated several differential equations for two-dimensional correlation functions on Riemann surfaces [1,2]. Such objects are of interest because they are among the basic elements of string perturbation theory [3]. Nominally, it is possible to identify two directions, which lead to numerous differential relations. One of the ways to obtain interesting mathematical information about the n-point correlation functions is to study the Ward identities in two-dimensional quantum field theory [2] (see also [4]). The other method is based on the principles of conformal theory [1].

In the present paper, combining the "physical" and "conformal" approaches, we derive new equations relating the first and second derivatives of a scalar Green's function on compact Riemann surfaces. The resulting relations are a generalization of the identities

$$(\partial_z G)^2=\partial_z^2 G, \quad \partial_z G\partial_w G=\partial_z\partial_w G, \tag{1}$$

which hold for the scalar two-point function on the plane:

$$G(z, w)=-\ln|z-w|^2. \tag{2}$$

Mathematical aspects of the scalar Green's function on compact Riemann surfaces of arbitrary genus have been analyzed by many authors [5–7]. However, the structure of the scalar propagator in the limit of coincident points (see, for example, [6]) has not been fully investigated. Therefore, the second section of this paper is devoted to consideration of the regularized Green's function (in the limit of coincident points). Sections 3 and 4 contain a derivation of the basic equations. In the concluding section, the results and possibilities for using them are discussed. The notation and helpful identities are given in the Appendix.

2. Regularized Scalar Green's Functions

Let M be a compact Riemann surface of genus h with local complex coordinates z, $\bar{z}$ and metric $ds^2 = g_{z\bar{z}}dzd\bar{z}$. We define on this surface the scalar Green's function $G(x, y)$,* which satisfies the equation

$$\Delta_0 G(z, y)=4\pi\delta(z, y)-4\pi/V, \quad \delta(z,y)\equiv\frac{1}{\sqrt{g}}\delta(z-y), \quad V=\int d^2z\sqrt{g}, \tag{3}$$

where Δ_0 is the scalar Laplacian (see the Appendix), and $1/V$ is the square of the zero mode of the operator Δ_0. The solution of Eq. (3) can be found up to an arbitrary constant, which can be fixed by the additional condition [6]

$$\int d^2y\sqrt{g}\,G(z,y)=0. \tag{4}$$

From Eqs. (3) and (4) we obtain an integral representation for $G(z, y)$:

*We do not indicate among the arguments of the Green's function the explicit dependence on $\bar{z}$ and $\bar{y}$, which is understood.

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$$G(z,y)=4\pi\int_0^\infty dt\{\mathcal{U}(z,y;t)-1/V\}, \tag{5}$$

where $\mathcal{U}(z, y; t)=\exp[-t\Delta_0]\delta(z, y)$ is the heat conduction kernel associated with the Laplacian Δ_0. Note also that since Δ_0 is a self-adjoint operator $G(z, y) = G(y, z)$.

We now consider the kernel $\mathcal{U}_z^{(-)}(z, y; t)^y$ associated with the Laplacian $\Delta_1^{(-)}$ (see (A.2)). By virtue of the identity $\nabla^z\Delta_1^{(-)}=\Delta_0\nabla^z$

$$\nabla^z\mathcal{U}_z^{(-)}(z, y; t)^y=-\nabla^y\mathcal{U}(z, y; t). \tag{6}$$

Using the relation (6), we can readily prove the equation [6]

$$\partial_z\partial_{\bar{y}}G(z, y)=2\pi\delta(z-y)-2\pi\mathcal{P}_{z\bar{y}},\quad \mathcal{P}_{z\bar{y}}\equiv\frac{1}{2}\sum_{I,J=1}^{h} w_z{}^I(\operatorname{Im}\Omega)_{IJ}{}^{-1}\bar{w}_{\bar{y}}{}^J. \tag{7}$$

Here, w_z^I, $I = 1, \ldots, h$, form a canonical basis of holomorphic Abelian differentials, $\partial_{\bar{z}}w_z^I = 0$, which are zero modes of the operator $\Delta_1^{(-)}$, $\mathcal{P}_{z\bar{y}}$ is the projector onto the space of Abelian differentials, and $\Omega_{IJ} = \Omega_{JI}$ is the matrix of periods of the compact Riemann surface (see, for example, [8]).

In what follows, we shall need the law of transformation of the scalar Green's function with respect to an arbitrary transformation of the metric:

$$ds'^2=2g_{z\bar{z}}e^{2\sigma}|dz+\Lambda_{\bar{z}}{}^z d\bar{z}|^2,$$

where σ and $\Lambda_{\bar{z}}{}^z$ are infinitesimally small. The variation with respect to the Weyl dilatation has the form [6]

$$\delta_\sigma G(z,y) = -\frac{2}{V}\int d^2w\sqrt{g}\,\sigma(w)[G(z,w)+G(y,w)]. \tag{8}$$

To find $\delta_\Lambda G(z, y)$, it is necessary to use the transformation properties of the covariant derivatives $\nabla^z_{(n)}$, $\nabla_z{}^{(n)}$ [3]:

$$\delta_\Lambda\nabla^z_{(n)}=-\Lambda^{zz}\nabla_z^{(n)}-n(\nabla_z\Lambda^{zz}),\quad \delta_\Lambda\nabla_z^{(n)}=-\Lambda_{zz}\nabla^z_{(n)}+n(\nabla^z\Lambda_{zz}). \tag{9}$$

As a consequence, we obtain

$$\delta_\Lambda G(z,y) = \frac{1}{2\pi}\int d^2w\Lambda_{\bar{w}}{}^w\partial_w G(z,w)\,\partial_w G(y,w)+\text{h. c.} \tag{10}$$

From Eqs. (9) and (10), we also obtain the law of transformation of the projector $\mathcal{P}_{z\bar{y}}$:

$$\delta_\Lambda\mathcal{P}_{z\bar{y}}=\int d^2w\Lambda_{\bar{w}}{}^w(\nabla_z\nabla_w G(z,w))\mathcal{P}_{w\bar{y}}+\int d^2w\Lambda_w{}^{\bar{w}}(\nabla_{\bar{y}}\nabla_{\bar{w}}G(y,w))\mathcal{P}_{z\bar{w}}. \tag{11}$$

We now study the regularized propagator. Following Polyakov [9], we regularize $G(z, y)$ in accordance with the rule

$$G(z,y;\varepsilon)=4\pi\int_\varepsilon^\infty dt[\mathcal{U}(z,y;t)-1/V], \tag{12}$$

where ε is a small cutoff parameter. We investigate the regularized Green's function in the limit of coincident points. In fact, the law of transformation of the (regularized) scalar Green's function with respect to the Weyl dilatation was found in [6]. However, its structure in the limit $z \to y$ was not fully analyzed.

We have

$$G(z;\varepsilon)\equiv\lim_{z\to y}G(z,y;\varepsilon)=A+\frac{1}{4\pi}\int d^2w\sqrt{g}\,G(z,w)N(w;\varepsilon), \tag{13}$$

where

$$A\equiv\int d^2w\sqrt{g}\,G(w;\varepsilon)/V=-\ln\varepsilon+\ldots,\quad N(z;\varepsilon)\equiv\Delta_0 G(z;\varepsilon)=2(\Delta_0-\nabla_z\nabla^w-\nabla^z\nabla_w)G(z,w;\varepsilon)|_{z=w}. \tag{14}$$

Further, using the integral representation for the propagator (12), and also Eq. (14), we readily obtain

$$N(z;\varepsilon)=8\pi[\mathcal{U}(z,z;\varepsilon)-1/V]-4\pi[\mathcal{U}_z^{(-)}(z,z;\varepsilon)^z+\mathcal{U}_{\bar z}^{(+)}(z,z;\varepsilon)^{\bar z})+4\pi g^{z\bar z}\mathcal{P}_{z\bar z}, \tag{15}$$

where $\mathcal{U}^{(\mp)}$ are the heat kernels associated with the Laplacians $\Delta_{\pm1}^{(\mp)}$. Since ε is small, we can use the expansion (A.4) for the kernels. As a result

$$N(z;\varepsilon\to0)=2\tilde R_g(z)\equiv 2R_g(z)+4\pi g^{z\bar z}\sum_{I,J=1}^{h} w_z{}^I(\mathrm{Im}\,\Omega)_{IJ}{}^{-1}\bar w_{\bar z}{}^J, \tag{16}$$

where $R_g(z)$ is the scalar curvature.

We see that the nontrivial structure of the function $G(z;\varepsilon)$ is determined by the scalar $\tilde R_g$. It is interesting to note that by virtue of the Gauss–Bonet theorem, $\int d^2z\sqrt{g}\,R_g=4\pi(1-h)$, the quantity $\sqrt{g}\tilde R_g$ is completely an invariant which does not depend on the topology of the compact Riemann surface. Indeed,

$$\int d^2z\sqrt{g}\,\tilde R_g=4\pi \tag{17}$$

for any h. One can show that on a surface of arbitrary genus there exists a metric $\tilde g_{\mu\nu}=e^{-2\sigma}g_{\mu\nu}$ for which

$$\tilde R_{\tilde g}=1,\quad \tilde V=4\pi. \tag{18}$$

It remains to determine the constant A, which can depend on the Weyl factor of the metric and the complex structure of the surface. For this, we note that the variation $\delta_\sigma G(z;\varepsilon)$ as a result of the Weyl transformation $\delta g_{\mu\nu}=2\sigma g_{\mu\nu}$ satisfies the equation [10]

$$\int d^2z\sqrt{g}\,\delta_\sigma G(z;\varepsilon)=2\int d^2z\sqrt{g}\,\sigma+O(\varepsilon), \tag{19}$$

which can be readily obtained by means of the proper-time technique. On the other hand, the variation of the function $G(z;\varepsilon)$ can be found by using the expressions (13) and (16):

$$\delta_\sigma G(z;\varepsilon)=\delta_\sigma A+2\sigma(z)-\frac{4}{V}\int d^2y\sqrt{g}\,\sigma(y)\left\{\frac{1}{2}+G(z,y)+\frac{1}{4\pi}\int d^2w\sqrt{g}\,G(w,y)\tilde R_g(w)\right\}+O(\varepsilon). \tag{20}$$

The system of equations (19)–(20) is sufficient to find the functional A:

$$A=-\ln\varepsilon+\frac{1}{2\pi}\int d^2y\sqrt{\hat g}\left[\frac{1}{2}\hat g^{\mu\nu}\partial_\mu\rho\partial_\nu\rho+\rho\tilde R_{\hat g}\right]-$$

$$\frac{1}{16\pi^2}\int d^2w\sqrt{g}\int d^2y\sqrt{g}\tilde R_g(w)G(w,y)\tilde R_g(y)+\Psi(m_i). \tag{21}$$

Here, $\hat g_{\mu\nu}=e^{-2\sigma}g_{\mu\nu}$ is a metric of constant curvature,

$$R_{\hat g}=\begin{cases}1, & h=0,\\ 0, & h=1,\\ -1, & h\geqslant 2,\end{cases} \tag{22}$$

and $\Psi(m_i)$ is a function on the Teichmüller space. It follows from the positivity of $G(z;\varepsilon)$ that $\Psi(m_i)\geqq 0$. Further, since the complex structure is uniquely determined up to diffeomorphisms by the matrix of periods (Torelli's theorem) [11],

$$\Psi(m_i)=\Psi(\Omega_{IJ})\equiv\Psi(\Omega). \tag{23}$$

Thus, the regularized scalar Green's function in the limit coincident points has the structure

$$G(z;\varepsilon)=-\ln\varepsilon+\frac{1}{2\pi}\int d^2y\sqrt{\hat g}\left[\frac{1}{2}\hat g^{\mu\nu}\partial_\mu\rho\partial_\nu\rho+\rho\tilde R_{\hat g}\right]-$$

$$\frac{1}{16\pi^2}\int d^2w\sqrt{g}\int d^2y\sqrt{g}\,\tilde R_g(w)G(w,y)\tilde R_g(y)+\frac{1}{2\pi}\int d^2y\sqrt{g}G(z,y)\tilde R_g(y)+\Psi(\Omega)+O(\varepsilon). \tag{24}$$

Using (24), we readily find the variation of $G(z;\varepsilon)$ under the Weyl transformation

$$\delta_\sigma G(z;\varepsilon)=2\sigma(z)-\frac{4}{V}\int d^2y\sqrt{g}\,\sigma(y)G(z,y)+O(\varepsilon),$$

which agrees with [6]. Equation (24) is one of the main results of this paper.

3. First Equation for the Green's Function

In this section, we obtain a generalization of the first equation in (1) for the Green's function on compact Riemann surfaces.

We consider the theory of a scalar field on the Riemann surface M with action

$$I=\frac{1}{4\pi}\int d^2z[\partial_zX\,\partial_{\bar z}X-\Lambda_{\bar z}^{z}(\partial_zX)^2-\Lambda_z^{\bar z}(\partial_{\bar z}X)^2+4\pi g_{z\bar z}\Phi X^2], \tag{25}$$

where $\Lambda_{\bar z}^{z}(z)$, $\Phi(z)$ are background two-dimensional fields, the first infinitesimally small.

We calculate the $\Lambda-\Phi$ contribution to the effective action $W(\Lambda, \Phi)$, defined as

$$\exp(-W)=\int \mathscr{D}X\exp(-I).$$

In the framework of perturbation theory, this can be done in two ways.

1. We choose as free action I_0

$$I_0=\frac{1}{4\pi}\int d^2z\,\partial_zX\,\partial_{\bar z}X.$$

Then the $\Lambda-\Phi$ contribution is given by diagram (a) in Fig. 1.

2. We define another free action I_0':

$$I_0'=\frac{1}{4\pi}\int d^2z[\partial_zX\,\partial_{\bar z}X-\Lambda_{\bar z}^{z}(\partial_zX)^2-\Lambda_z^{\bar z}(\partial_{\bar z}X)^2].$$

In this case, all the $\Lambda-\Phi$ terms are contained in diagrams (b), where the Green's function G' is constructed from the new metric

$$ds'^2=2g_{z\bar z}|dz+\Lambda_{\bar z}^{z}d\bar z|^2. \tag{26}$$

In both methods, the same rule is used to regularize the propagators (12).

It can be seen from the expression (26) that the dependence of $G'(z;\varepsilon)$ on Λ is determined by the transformation properties of the function $G(z;\varepsilon)$ with respect to infinitesimally small Beltrami transformations. The variation of the last term in (13) and (24) can be readily found by means of Eqs. (10), (11), and (A.8). In turn, the variation $\delta_\Lambda A$ can be determined by using the condition of covariance

$$\nabla^z\frac{\delta A}{\delta g^{zz}}+\nabla^{\bar z}\frac{\delta A}{\delta g^{\bar z\bar z}}=0,$$

and also the law of Weyl transformation

$$\delta_\sigma A=\frac{2}{V}\int d^2y\sqrt{g}\,\sigma(y)\left[1+\frac{1}{2\pi}\int d^2w\sqrt{g}\,G(w,y)\tilde R_g(w)\right].$$

The above is sufficient to find the terms in the effective action in which we are interested. From requirement of equality of the $\Lambda-\Phi$ contributions calculated by methods 1 and 2 there arises an equation for the scalar Green's function:

$$(\partial_zG(z,w))^2=\nabla_z^2G(z,w)+\frac{1}{2\pi}\partial_zG(z,w)\int d^2y\sqrt{g}\,\nabla_zG(z,y)\tilde R_g(y)-$$
$$2\int d^2y\,\partial_zG(z,y)\partial_yG(w,y)\mathscr{P}_{z\bar y}+\Psi_{zz}+\frac{1}{8\pi^2V}\int d^2v\sqrt{g}\,\nabla_zG_z^{(+)}(z,v)^v\int d^2y\sqrt{g}\,\partial_vG(v,y)\tilde R_g(y). \tag{27}$$

Here, Ψ_{zz} is the quadratic differential, $\partial_{\bar z}\Psi_{zz}=0$, related to the function $\Psi(\Omega)$ by

$$\Psi_{zz}=\Psi_{IJ}\cdot\delta\Omega_{IJ}/\delta g^{zz},\quad \Psi_{IJ}=\delta\Psi(\Omega)/\delta\Omega_{IJ}.$$

Using the property [11]

$$\delta\Omega_{IJ}/\delta g^{zz}=\frac{i}{2}w_z^Iw_z^J,$$

we obtain an expression for Ψ_{zz} in terms of the Abelian differentials:

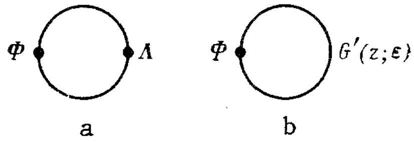

Fig. 1

$$\Psi_{zz}=\frac{i}{2}w_z{}^I\Psi_{IJ}w_z{}^J. \tag{28}$$

The Green's function $G_z{}^{(+)}(z,v)^v$ in (27) satisfies the equation

$$\Delta_1{}^{(+)}G_z{}^{(+)}(z,\ v)^v=4\pi\delta(z,\ v). \tag{29}$$

Equation (29) is correct when h ≧ 2, since in this case the Laplacian $\Delta_1{}^{(+)}$ does not have zero modes — conformal Killing vectors. For h = 0, 1 it is necessary to take into account in the final term of Eq. (27) the zero modes of the Laplacian (29). However, for standard metrics on the sphere and torus the given term is zero (see below).

Our relation (27) is interesting in that it relates the second derivatives of the Green's function to the first. At the same time, by successive differentiation of Eq. (27) it is possible to obtain identities for derivatives of any order. A second remarkable fact is that Eq. (27) is covariant with respect to the Weyl transformation. This is proved by direct variation of diagrams (a) and (b) in Fig. 1. This transformation property makes it possible to go over in formula (27) from the metric $g_{\mu\nu}$ to any other conformally equivalent metric. The identity (27) takes its simplest form for the metric $\tilde{g}_{\mu\nu}$ when $\tilde{R}_{\tilde{g}}$ = 1. In this case

$$(\partial_z\tilde{G}(z,w))^2=\tilde{\nabla}_z{}^2\tilde{G}(z,w)-2\int d^2y\,\partial_z\tilde{G}(z,y)\,\partial_y\tilde{G}(w,y)\mathscr{P}_{z\bar{y}}+\Psi_{zz}. \tag{30}$$

The validity of this equation for Riemann surfaces of genus h = 0, 1 can be shown directly. Let us consider the sphere (h = 0). Abelian differentials on the sphere are zero. As a consequence, the relation (27) takes the form

$$(\partial_zG(z,\ w))^2=\nabla_z{}^2G(z,\ w),\quad h=0. \tag{31}$$

It is readily verified that the Green's function

$$G(z,w)=-\ln\frac{|z-w|^2}{(1+|z|^2)(1+|w|^2)},$$

corresponding to the standard choice of the metric on the sphere (see, for example [3]), satisfies Eq. (31).

In the case of a torus (h = 1), $ds^2 = 2dzd\bar{z}$, $V = 2\tau_2$, $w_z = 1$. Therefore, Eq. (27) can be written in the form

$$(\partial_zG(z-w))^2=\partial_z{}^2G(z-w)+\frac{1}{\tau_2}\int d^2y\,\partial_zG(z-y)\,\partial_yG(w-y)+\text{const},\qquad h=1. \tag{32}$$

The scalar Green's function on the torus can be expressed in terms of the Riemann θ_1 function [3,10]:

$$G(z-w)=-\ln\left|\frac{\theta_1(z-w;\tau)}{\theta_1{}'(0,\tau)}\right|^2-\frac{\pi}{2\tau_2}(z-\bar{z}-w+\bar{w})^2. \tag{33}$$

To prove Eq. (32), we note that the Weierstrass ρ function, $\rho(z-w)\equiv\partial_z^2G(z-w)$, is holomorphic when z ≠ w and has the Laurent expansion

$$\rho(z-w)=\frac{1}{(z-w)^2}+\text{regular terms}.$$

On the other hand, it is easy to show that the combination

$$(\partial_zG(z-w))^2-\frac{1}{\tau_2}\int d^2y\,\partial_zG(z-y)\,\partial_yG(w-y) \tag{34}$$

is also holomorphic when $z \neq w$ and possesses a pole the same as for $\rho(z - w)$. This means that the functions $\rho(z - w)$ and (34) differ only by a constant, in complete agreement with our equation (32).

Note that from the identity (32) we obtain a new integral representation for the Weierstrass function, though in terms of the function θ_1.

4. Second Equation for the Green's Function

Unfortunately, the method presented in the previous section is ineffective for generalization of the second equation in (1). This may be due to the symmetry of the second relation with respect to interchange of z and w. However, the important property of conformal covariance inherent in the first equation of (27) suggests an alternative way of obtaining a second identity. Namely, it is necessary to augment the original "flat" expression (1) in such a way that the resulting equation possesses conformal covariance.

Being guided by this principle, and also the existing symmetry with respect to the arguments, we arrive at the equation

$$\partial_z G(z,w)\partial_w G(z,w) = \partial_z\partial_w G(z,w) - \frac{1}{4\pi}\int d^2y\sqrt{g}\,[\partial_w G(w,y)\partial_z G(z,w) + (w \leftrightarrow z) - \partial_z G(z,y)\partial_w G(w,y)]\tilde{R}_g(y) + \int d^2y\Big\{\partial_z\partial_y G(z,y)\mathscr{P}_{w\bar{y}}\Big[G(w,y) - \frac{1}{4\pi}\int d^2v\sqrt{g}\,G(y,v)\tilde{R}_g(v)\Big] + (z \leftrightarrow w)\Big\} + \varphi_{zw}, \tag{35}$$

where φ_{zw} is a certain function that depends only on the complex structure of the Riemann surface. Using the expressions (8) and (A.7), we can readily verify that Eq. (35) is indeed covariant with respect to Weyl transformations.

It should be noted that the covariance of the first relation (27) cannot be established by direct variation of each term in (27), since the last term can be rewritten in terms of the Green's function G(z, w) only in the form of an infinite series in R_g. This difficulty does not arise in the investigation of the transformation properties of the diagrams that generate the identity (27).

Thus, the generalization of Eqs. (1) to the case of compact Riemann surfaces of arbitrary genus can be achieved by two alternative methods, a fact that, in our view, is interesting in its own right.

The function φ_{zw} on the right-hand side of Eq. (35) can be related to the function $\Psi(\Omega)$ (23). Namely, using the results of the previous section, we can readily show that

$$\varphi_{zz} = i w_z{}^I \Psi_{IJ} w_z{}^J. \tag{36}$$

An open question for us is still the function $\Psi(\Omega)$. It is possible that additional information is contained in the modular properties of the two-dimensional propagators.

5. Conclusions

The main results of our paper are the expressions (24), (27), and (35). We have investigated the general structure of the regularized scalar Green's function in the framework of proper-time regularization. As is shown in [10], this regularization is well suited to compact surfaces and is free of infrared divergences. Not yet fully determined is the function $\Psi(\Omega)$, which depends on the Teichmüller parameters $m_1, \ldots, m_{6h-6}$. We suppose that an explicit expression for $\Psi(\Omega)$ can be fixed by modular invariance. It follows from (24) that the regularized propagator in the limit of coincident points is a constant for Riemann surfaces with the topology of the sphere and torus. Note that the technique considered in Sec. 2 is a generalization of the method developed in our paper [12] for noncompact surfaces.

Further, on the basis of the two different approaches we have derived Eqs. (27) and (35), which relate the second (and higher) derivatives of the Green's function to the first derivatives. The first of them has been obtained from the condition of consistency (25) of two-dimensional quantum field theory. The source of the second is the principle of conformal covariance.

We hope that our results will be helpful in the calculation of loop corrections to string equations of motion in the framework of the σ-model approach [13].

Appendix

Our notation agrees with [3]. In particular, the covariant derivatives are defined in accordance with the rule

$$\nabla_z^{(n)}: T^n \to T^{n+1}, \quad \nabla_{(n)}^z: T^n \to T^{n-1}, \quad \nabla_z^{(n)}(T_n (dz)^n) = (g_{z\bar z})^n \frac{\partial}{\partial z}((g^{z\bar z})^n T_n)(dz)^{n+1},$$
$$\nabla_{(n)}^z (T_n (dz)^n) = g^{z\bar z} \frac{\partial}{\partial z} T_n (dz)^{n-1}. \tag{A.1}$$

Here, T^n is the space of tensor fields with n subscripts z. There are two types of Laplacian:

$$\Delta_n^{(+)} = -2\nabla^z_{(n+1)}\nabla_z^{(n)}, \quad \Delta_n^{(-)} = -2\nabla_z^{(n-1)}\nabla^z_{(n)}, \quad \Delta_0 \equiv \Delta_0^{(+)} = \Delta_0^{(-)}. \tag{A.2}$$

We introduce the heat kernels associated with the operators $\Delta_n^{(\pm)}$:

$$\mathcal{U}_n^{(\pm)}(z, z'; \varepsilon)_{-n} = e^{-\varepsilon\Delta_n^{(\pm)}} 1_n(z, z')_{-n}. \tag{A.3}$$

For small ε, we have in the limit of coincident points the expansion

$$\mathcal{U}_n^{(\pm)}(z, z; \varepsilon)_{-n} = \frac{1}{4\pi\varepsilon} + \frac{1 \pm 3n}{12\pi} R_g + O(\varepsilon), \tag{A.4}$$

where R_g is the scalar curvature.

Let A_I and B_I, I = 1, ..., h, be a canonical basis for the first homology group (see [8]). Then in the space of holomorphic differentials there exists a basis such that

$$\oint_{A_I} w^J = \delta_{IJ}, \quad \oint_{B_I} w^J = \Omega_{IJ},$$

where Ω_{IJ} is the matrix of periods. As a consequence, the basis Abelian differentials w_z^I satisfy the relation [8]

$$\int d^2z \, w_z{}^I \bar w_{\bar z}{}^J = 2(\mathrm{Im}\,\Omega)_{IJ}. \tag{A.5}$$

Accordingly, the operator $\mathcal{P}_{z\bar y}$ (7) is a projector onto the space of holomorphic differentials and

$$\int d^2z \mathcal{P}_{z\bar z} = h. \tag{A.6}$$

Under the Weyl transformation $dg_{\mu\nu} = 2\sigma g_{\mu\nu}$, the scalar curvature varies in accordance with the law

$$\delta_\sigma R_g = -2\sigma R_g + \Delta_0 \sigma. \tag{A.7}$$

Under the Beltrami transformation, the transformation rule has the form

$$\delta_\Lambda R_g = \nabla_z{}^2 \Delta_{\bar z}{}^z + \text{h.c.} \tag{A.8}$$

LITERATURE CITED

1. A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. B, 241, 333 (1984); V. G. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. B, 247, 83 (1984); S. Mathur, S. Mukhi, and A. Sen., Nucl. Phys. B, 312, 15 (1989).
2. T. Eguchi and H. Ooguri, Nucl. Phys. B, 282, 308 (1986).
3. E. D'Hoker and D. H. Phong, Rev. Mod. Phys., 60, 917 (1988).
4. K.-J. Hamada and M. Takao, Nuc. Phys. B, 313, 80 (1989).
5. L. Alvarez-Haume, G. Moore, and C. Vafa, Commun. Math. Phys., 106, 1 (1986); H. Sonoda, Phys. Lett. B, 178, 390 (1986); V. G. Knizhnik, Phys. Lett. B, 180, 247 (1986).
6. E. Verlinde and H. Verlinde, Nucl. Phys. B, 288, 357 (1987).
7. M. I. Dugan and H. Sonoda, Nucl. Phys. B, 289, 227 (1987).
8. D. Mumford, Lectures on Theta Functions [Russian translation], Mir, Moscow (1988).

9. A. M. Polyakov, Phys. Lett. B, 103, 207 (1981).
10. S. Ranjibar-Daemi, A. Salam, and I. A. Strathdee, Int. J. Mol. Phys. A, 2, 667 (1987).
11. V. G. Knizhnik, "Multiloop amplitudes in the theory of quantum strings and complex geometry. Explicit expressions for the measure in terms of theta functions," Preprint 87-61R [in Russian], Institute of Theoretical Physics, Kiev (1987).
12. S. M. Kuzenko and O. A. Solov'ev, Yad. Fiz., 51, 585 (1990).
13. H. Ooguri and N. Sakai, Nucl. Phys. B, 312, 435 (1988).

RANDOM WALKS IN ONE-DIMENSIONAL QUASICRYSTALS

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It is shown that in some one-dimensional quasicrystals a random walk has nonclassical limiting behavior.

The asymptotic properties of random walks in one-dimensional disordered media is a topical problem of research. It has been established that the grain boundaries of some alloys have a one-dimensional quasiperiodic structure (see [1]). Diffusion taking place along the grain boundary of such alloys (for example, oxidation) can be approximated by a random walk in one-dimensional quasicrystals. The classical behavior for this case is when the diffusing particle after an interval of time t has moved away from the initial position by an amount x(t) close to a Gaussian random variable with mean at and variance $\sigma^2 t$ ($\sigma^2 > 0$). The constants a and σ^2 are called the coefficients of linear drift and diffusion, respectively. In the case of classical behavior of the particle, even if there is no linear drift (i.e., $a=0$), the excursion x(t) has the order $\sqrt{t}$ as $t \to +\infty$. The aim of the present note is to describe quasicrystals in which the opposite situation obtains — the behavior of the particle is like the random walk in a random medium described by Sinai [2]. In this case, x(t) has the order $(\ln t)^\delta$, where the constant $\delta > 0$ is found from the parameters of the quasicrystal.

We consider a random walk on a one-dimensional lattice $Z_+ = \{0, 1, 2, \ldots\}$ with reflecting screen on the left. The randomly walking particle can pass in unit time from the vertex $n \in Z_+$ only to the neighboring vertices. We denote the probability of transition from n to n + 1 by p(n), and from n to n − 1 by q(n) = 1 − p(n). The reflection from the screen on the left means that p(0) = 1. We denote the position of the particle at time $t \in \{0, 1, \ldots\}$ by x(t). We assume that x(0) = 0. The behavior of such random walks completely depends on the sequence of numbers p = {p(n), n = 1, 2, ...}. If it possesses a quasiperiodic structure, then we shall call the constructed random process a random walk in a one-dimensional quasicrystal.

Suppose natural numbers k and ℓ are given. We define the one-dimensional quasicrystal p by a sequence of words $\{A_n\}_{n=1}^{+\infty}$, which consist of two symbols A and B. Each successive word in the sequence is formed from the two preceding words in accordance with the rule

$$A_{m+2} = \underbrace{A_{m+1} \ldots A_{m+1}}_{k \text{ times}} \underbrace{A_m \ldots A_m}_{l \text{ times}}.$$

The initial words are A_1 = B and A_2, consisting of k symbols A. In this sequence, each word is the start of the next one. Allowing m to tend to +∞, we ultimately obtain an infinite word that is a sequence of symbols of two types. We write it in a chain and number the symbols:

$$B_1, B_2, \ldots, B_n, \ldots. \tag{1}$$

For arbitrary positive numbers α and β, we define the transition probabilities p(n) of the random walk in the quasicrystal in accordance with the formula

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