Set-Valued Analysis **2**: 357–367, 1994.

Young Measures, Weak and Strong Convergence and the Visintin–Balder Theorem

MICHEL VALADIER
Département de Mathématiques, case 051, Université Montpellier II, 34095 Montpellier Cedex 5, France

(Received: 1 December 1993)

Abstract. This paper is concerned with sequences in L^1 which converge weakly. Young's measures theory permits us to give sufficient conditions insuring the strong convergence and to understand the behaviour of the sequences which do not converge strongly.

Mathematics Subject Classifications (1991). 46E27, 46E30, 28A20, 49J45.

Key words: Young's measures theory, convergences, Visintin–Balder Theorem.

1. Introduction

The development of Young's measures theory has a long story. Obviously it goes back to L. C. Young [48–50]. The first aim was to give a description of the limits of minimizing sequences in the calculus of variations and, further, in the optimal control theory (see J. Warga [47] and A. Ghouila-Houri [30]. See the 20th problem of Hilbert quoted in I. Ekeland and R. Temam's book [27, Comments to chapters 9 and 10]). More recently, H. Berliocchi and J. M. Lasry [17] extended the theory so as to make it work without compactness and, with some economical applications in view, were daring enough to consider affine functions taking the value $+\infty$. Then, E. J. Balder [3, 5, 8] gave the parametric version of the Prohorov theorem and lower semicontinuity theorems which make the theory very efficient and applicable.

Most of the properties of weakly convergent sequences in L^1 which are not strongly convergent must have been understood by C. Olech [36] and L. Tartar [39, 40] many years ago. Particularly, L. Tartar showed the usefulness of Young measures in this question. But the Visintin theorem [46] brought a new result and its proof using Young measures, due to E. J. Balder [6], allows many extensions. For the use of Young measures in PDE and Mechanics, see L. C. Evans [29], M. Chipot and D. Kinderlehrer [24] and D. Kinderlehrer and P. Pedregal [33, 34].

In Section 3 we recall some properties of weakly convergent sequences.

In Section 4 we show that some frightening results of Measure theory used in Young measures theory are rather natural. The weak convergence of measures corresponds, when the spaces are intervals $[a, b]$ and $[c, d]$, to what could be convergence of black and white photographs on the rectangle $[a, b] \times [c, d]$. The disintegration of a bounded positive measure on $[a, b] \times [c, d]$ is roughly speaking what happens to the image before TV transmission: it is scanned (then the television set builds the image line after line).

In Section 5 we give some examples of Young measures, specially of limit Young measures which are nonassociated to functions.

In Section 6 we give some theorems about Young measures connected to our problem. We refer to our course [43], but most of the results come from one of the Balder's numerous papers [3–11].

In Section 7 we give the Visintin and Balder theorems and give some ideas on their proofs.

In Section 8 we show that the hypotheses of the Visintin and Balder theorems are only sufficient conditions and we give a variant of Girardi's criterion.

2. Notations and Preliminaries

In the following Ω denotes an open subset of $\mathbb{R}^N$, μ the Lebesgue measure on Ω. All résults extend to abstract measure spaces. Let $L^1(\Omega, \mu; \mathbb{R}^d)$ denote the Banach space of integrable functions. For a sequence in $L^1(\Omega, \mu; \mathbb{R}^d)$, $u^n \rightharpoonup u$, means u^n converges to u weakly, that is

$$\forall p \in L^\infty, \int_\Omega \langle p, u^n - u \rangle \to 0.$$

Let δ_x denote the Dirac measure at x, S^{d-1} the unit sphere of $\mathbb{R}^d$.

DEFINITION. Let C be a closed convex subset of a normed linear space and $y \in C$. The point y is a *denting point* of C if

$$\forall \varepsilon > 0, y \notin \overline{\text{co}}(C \backslash B(y, \varepsilon)).$$

NOTATIONS. $\partial_{\text{ext}} C$ for the set of extreme points of C, $\partial_{\text{dent}} C$ for the set of denting points of C.

A denting point is always an extreme point. In finite dimension an extreme point is a denting point [42, Lemme 1 p. 5.4].

3. Weak and Strong Convergence

It is well-known that in L^1 strong convergence implies weak convergence but the converse does not hold.

EXAMPLE. Let $\Omega = [0, 1]$, $d = 1$ and $u^n(x) = \sin\ nx$. Then u^n converges weakly but not strongly to $u \equiv 0$ since $\| u^n \|_{L^1} \to 2/\pi$.

The purpose of this paper is to review some theorems which bring light on this phenomenon. The following observations are very useful.

If $u^n \rightharpoonup u$, one can say:

(1) $(u^n)_n$ is norm bounded in L^1 (this holds because the index set is N, this is not valid for generalized sequences). It is a consequence of the Banach–Steinhaus theorem.
(2) $(u^n)_n$ is uniformly integrable. This is stronger than (1). It is a consequence of the Dunford–Pettis theorem.
(3) One knows (Lebesgue–Vitali theorem) that, if $(u^n)_n$ is uniformly integrable (and $\mu(\Omega)$ is finite), its strong convergence is equivalent to its convergence in measure:
$$\forall \varepsilon > 0, \mu(\{x \in \Omega : \| u^n(x) - u(x) \| \geq \varepsilon\}) \to 0.$$

4. Definition of Young Measures

In this section and the two following, we intend to present some basic results of Young measures trying to avoid measure theory technicalities.

DEFINITION. A *Young measure* on $\Omega \times \mathbb{R}^d$ is a positive measure τ on $\Omega \times \mathbb{R}^d$ such that for any Borel set $A \subset \Omega$, $\tau(A \times \mathbb{R}^d) = \mu(A)$. For any measurable function $u : \Omega \to \mathbb{R}^d$, the Young measure *associated to* u is the (unique) Young measure carried by the graph of u. (Another definition of ν is: for any Borel set B, $\nu(A \times B) = \mu(A \cap u^{-1}(B))$.)

There exist Young measures not associated to functions.

The Young measure ν associated to u represents the amount of chalk (or ink) laid down when drawing the graph of u with the 'law'
$$\nu(A \times \mathbb{R}^d) = \mu(A). \qquad \text{(YML)}$$
The following figure depicts this law for the function sin x (Figure 1). The limit Young measure obtained when the frequency tends to $+\infty$ will be given in the next section.

We want to present disintegration very simply. Let us think of $\Omega \times \mathbb{R}^d$ as a rectangle $[a, b] \times [c, d]$. A Young measure τ can represent a black and white photograph. Above any x there is a conditional distribution τ_x (which is a probability measure on $\mathbb{R}^d$). In some sense this corresponds to the scanning of the image before TV transmission (exchange vertical and horizontal). Then the television set builds the image line after line in accordance to the formula:
$$\tau = \int_\Omega [\delta_x \otimes \tau_x] \, dx!$$

The measure τ and the family $(\tau_x)_{x \in \Omega}$ are two ways of description of the same image. The second way does not imply the existence of stochastic events

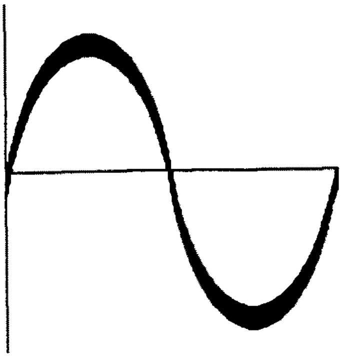

Fig. 1. Thickness according to (YML)

or of a player having a random strategy. For a short proof of the existence of disintegration see L. C. Evans [29]. See also [41, 43]. The family $(\tau_x)_{x\in\Omega}$ is very useful: the barycenter of τ_x will be used later.

The notion of weak convergence of Young measures (for a precise definition see the references) is essential. It is called the *narrow convergence*. If $\Omega \times \mathbb{R}^d$ was exactly $[a, b] \times [c, d]$, this notion would be nothing else but convergence of images (recall that a sequence $(\lambda^n)_n$ of measures on a compact metric space K converges weakly to λ if for all real continuous function p on K, $\int_K p\, d\lambda^n \to \int_K p\, d\lambda$).

5. Examples of Young Measures

In this section we give some examples of Young measures, specially of Young measures associated to functions which converge to a Young measure non associated to a function. In the following examples, $\Omega = [0, 1]$ and μ is the Lebesgue measure. They all are particular cases of the following. Let u^1 be a periodic function on $\mathbb{R}$ with period 1 and $u^n(x) = u^1(nx)$. Then (as soon as u^1 is measurable) the Young measures ν^n converge to a limit τ whose disintegration τ_x is constant and verifies for any real bounded measurable function on $\mathbb{R}$

$$\int_{\mathbb{R}} p\, d\tau_x = \int_{[0,1]} p(u^1(x))\, dx.$$

EXAMPLE 1. The Rademacher functions on $[0, 1]$,

$$u^n(x) = +1 \text{ if } x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$$

for all even k, $u^n(x) = -1$ otherwise, have a limit τ which has not a density. Its disintegration is given by

$$\tau_x = \frac{1}{2}(\delta_1 + \delta_{-1}).$$

EXAMPLE 2. The functions, $u^n(x) = nx$ (mod. 1) on [0, 1], have as limit the Lebesgue measure on the square $[0, 1]^2$.

EXAMPLE 3. Let $\Omega = [0, 1]$, $d = 1$ and $u^n(x) = \sin(nx)$. One can prove that the Young measure ν^n associated to u^n converges to τ where τ is carried by $\Omega\times]-1, 1[$ and has the density

$$(x, y) \mapsto \frac{1}{\pi\sqrt{1-y^2}}.$$

6. Some Properties of Young Measures Connected to Oscillations

In this section we give some properties of Young measures connected to our problem. The Young measure associated to u^n (resp. u) is denoted by ν^n (resp. ν). A general Young measure is denoted by τ.

PROPOSITION 1. *If ν^n and ν are Young measures associated to the measurable functions u^n and u, then*

$$\nu^n \longrightarrow \nu \ narrowly \ \Leftrightarrow u^n \longrightarrow u \ in\ measure.$$

Reference. [43, Prop. 6].

THEOREM 2. *Suppose $u^n \rightharpoonup u$ in $L^1(\Omega, \mu; \mathbb{R}^d)$.*

(1) *There exist a subsequence $(n_k)_k$ and a Young measure τ such that $\nu^{n_k} \longrightarrow \tau$. Then* a.e. *the disintegration τ_x has a barycenter* $\mathrm{bar}(\tau_x)$, $u(x) = \mathrm{bar}(\tau_x)$ *and*
$$\| u^{n_k} - u \|_{L^1} \to \int_{\Omega\times\mathbb{R}^d} \| y - u(x) \| \, \tau(d(x, y)).$$
Moreover, if τ_x is a.e. *a Dirac measure, then $\tau = \nu$ and $u^{n_k} \longrightarrow u$ strongly.*

(2) *If u^n does not converge strongly, there exist a subsequence $(n_k)_k$ and a Young measure τ as in (1) above such that τ is not associated to a function.*

(3) *$u^n \longrightarrow u$ strongly $\Leftrightarrow \nu^n \longrightarrow \nu$ narrowly.*

References. For the first part of (1) [43, Th. 19]. The convergence of $\| u^{n_k} - u \|_{L^1}$ has been noticed by E. J. Balder [3, 10] and follows from [43, Th.17] applied to the integrand on $\Omega \times \mathbb{R}^d$, $(x, y) \mapsto \| y - u(x) \|$. Finally (3) follows from Proposition 1 and the Lebesgue–Vitali theorem.

When $u^n \not\to u$ and $u^n \not\rightharpoonup u$, a limit measure τ as in Part 2 of Th. 2 contains information about the asymptotic osillatory behaviour of the subsequence $(u^{n_k})_k$. This is specially meaningful in Mechanics when the energy function is not quasi-convex in the sense of C. B. Morrey [35]. The material may appear in two phases (or more). Papers by J. L. Ericksen [28], D. Kinderlehrer and others [24, 33, 34] treat crystals. Similar phenomenons were already studied in Control Theory (when some

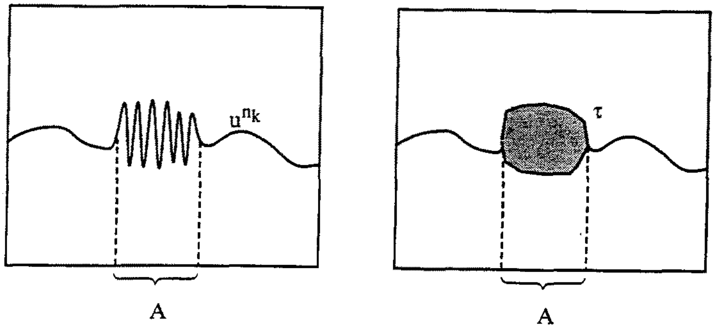

Fig. 2.

convexity is lacking) under the name of Relaxation. For example when controlling a rocket, it may happen that an optimal control would need a velocity between the full power and zero power of the engines. But this may be technically impossible. Approximated optimal controls are obtained by high frequency switches between full and zero power (even if the control space is the compact connected convex space [0, 1] nonexistence may happen when the system is governed by a nonlinear differential equation). Recent works by J. P. Raymond, A. Cellina [23], E. J. Balder are devoted to special cases of existence without convexity.

If $u^n \not\rightharpoonup u$, $u^n \not\to u$ and $\nu^{n_k} \to \tau$ with τ nonassociated to a function, there exists a measurable set A with $\mu(A) > 0$ such that $\forall x \in A$, τ_x is not a Dirac measure. Then one can expect that above A, in some sense (it would be valuable to give this a precise meaning), the functions u^{n_k} oscillate with 'frequencies tending to $+\infty$' (see Figure 2).

7. Visintin and Balder Theorems

THEOREM 3. (Visintin). *Let for any $x \in \Omega$, $\Gamma(x)$ be a closed convex subset of $\mathbb{R}^d$, u^n ($n \in \mathbb{N}$) and u functions in $L^1(\Omega, \mu; \mathbb{R}^d)$ such that $u^n \rightharpoonup u$ weakly and $\forall n$,* a.e. *$u^n(x) \in \Gamma(x)$. Then*

(i) a.e. $u(x) \in \Gamma(x)$,
(ii) *moreover if* a.e. $u(x) \in \partial_{\text{ext}}\Gamma(x)$, *then $u^n \to u$ strongly.*

Ideas of the proof (without Young measures). First (i) follows from the fact that the strongly closed convex set $\{v \in L^1(\Omega, \mu; \mathbb{R}^d)$: a.e. $v(x) \in \Gamma(x)\}$ is weakly closed. Other possibility: invoke the Mazur lemma.

For (ii) suppose $\mu(\Omega) < +\infty$ and imagine first the worse situation. That is the whole sequence verifies for some $\varepsilon > 0$:

$$\forall x \in \Omega, \| u^n(x) - u(x) \| \geq \varepsilon.$$

Then, from $u(x) \in \partial_{\text{ext}}\Gamma(x) = \partial_{\text{dent}}\Gamma(x)$ and the Hahn–Banach theorem, there exist $p(x) \in S^{d-1}$ and $\alpha(x)$ such that

$$(p(x), u^n(x) - u(x)) \geq \alpha(x) > 0. \tag{*}$$

Thanks to a measurable choice theorem one may assume $p \in L^\infty$. By integration of (*) one gets a contradiction with $u^n \rightharpoonup u$. A more refined discussion is required in the general case.

Let us consider now the general case. Suppose to simplify $u \equiv 0$. If u^n does not converge in measure to u, $\exists \varepsilon > 0, \exists \eta > 0$, such that for infinitely many n,

$$\mu(\{x \in \Omega : \| u^n(x) \| \geq \varepsilon\}) \geq \eta, \tag{**}$$

so we may suppose that all the sequence verifies (**). Let

$$v^n = 1_{\Omega_n} u^n, \quad w^n = u^n - v^n,$$

where

$$\Omega_n = \{x \in \Omega : \| u^n(x) \| \geq \varepsilon\}.$$

Since $(v^n, w^n)_n$ is uniformly integrable we may suppose that it converges weakly to (v, w). But $v^n + w^n = u^n$ converges to 0. Hence a.e. $1/2(v(x) + w(x)) = 0$. Moreover since v and w are still selectors of Γ and $0 \in \partial_{\text{ext}}\Gamma(x)$, one has $v = w = 0$.

Now, when $v^n(x) \neq 0$, $v^n(x) \notin \overline{\text{co}}(\Gamma(x) \backslash B(0, \varepsilon))$. It remains to continue with a more refined construction than above.

REMARK. Still more sophisticated truncation methods (initiated by T. Rzeżuchowski [38]) are used in the paper with A. Amrani and C. Castaing [1, 2]. See also H. Benabdellah's papers [14–16].

NOTATION. For a sequence $(y_n)_n$ in $\mathbb{R}^d$, $\text{Ls}(y_n)$ denotes the set of limit points of the sequence. It is the Painlevé–Kuratowski limit sup of the singletons $\{y_n\}$.

THEOREM 4. (Balder). *Let u^n $(n \in \mathbb{N})$ and u functions in $L^1(\Omega, \mu; \mathbb{R}^d)$ such that $u^n \rightharpoonup u$ weakly. Then*

(j) a.e. $u(x) \in \overline{\text{co}}(\text{Ls}(u^n(x)))$
(jj) *moreover if* a.e. $u(x) \in \partial_{\text{ext}}\overline{\text{co}}(\text{Ls}(u^n(x)))$, *then $u^n \to u$ strongly.*

COMPARISON. The Balder theorem is stronger than Visintin's one, since, as soon as $C_2 \subset C_1$, $y \in \partial_{\text{ext}}C_1$ and $y \in C_2$, then $y \in \partial_{\text{ext}}C_2$. Here $\overline{\text{co}}(\text{Ls}(u^n(x)))$ is contained in $\Gamma(x)$, so $u(x) \in \partial_{\text{ext}}\Gamma(x) \Rightarrow u(x) \in \partial_{\text{ext}}\overline{\text{co}}(\text{Ls}(u^n(x)))$.

Ideas of the proof. First (j) is easy to prove with Young measures because if τ is the Young measure given by Part 1 of Th. 2, τ_x is a.e. carried by $\text{Ls}(u^n(x))$ (see

[43, Prop. 15, p. 166]). A proof of (j) without Young measures has been given in [2, Th. 8, p. 176]. For (jj), Balder's idea is that if $\nu^{nk} \rightharpoonup \tau$, since τ_x has as barycenter the extreme point $u(x)$ of $\overline{\text{co}}(\text{Ls}(u^n(x)))$, then $\tau_x = \delta_{u(x)}$, hence τ is equal to ν the Young measure associated to u. So, by Part 1 of Th. 2, $u^n \to u$ strongly.

A consequence of the Visintin theorem. One can recover from Visintin's theorem in L^1 the following result. If $p \in]1, \infty[$, if $\mathbb{R}^d$ is equipped with a strictly convex norm, if $u^n \rightharpoonup u$ in L^p, if $\| u \|_{L^p} \geq \limsup \| u^n \|_{L^p}$, then $u^n \to u$ strongly. The proof does not use uniform convexity arguments. See [46, Th. 3], [42, Cor. 11], [14–16].

8. More Discussion

The Balder theorem gives a sufficient condition, not a necessary condition.

EXAMPLE. Let $\Omega = [0, 1]$, $d = 1$ and for $k \in \mathbb{N}$, $p \in \{0, ..., 2^k - 1\}$,

$$v^n = 1_{[\frac{p}{2^k}, \frac{p+1}{2^k}]} \quad \text{if } n = 2^k + p.$$

Then $\| v^n \|_{L^1} = 2^{-k}$ tends to 0 and it is easy to see, and classical (this is the most usual example of a sequence converging in measure but not a.e.), that for any x in $\Omega, \text{Ls}(v^n(x)) = \{0, 1\}$. Then if $u^n = v^n$ if n is even, $-v^n$ if n is odd, $\| u^n \|_{L^1} \to 0$ and $\text{Ls}(u^n(x)) = \{-1, 0, 1\}$. So, with $u \equiv 0$,

$$\forall x \in \Omega, u(x) \notin \partial_{\text{ext}} \overline{\text{co}}(\text{Ls}(u^n(x))).$$

COMMENT. If $u^n \to u$ there exists a subsequence such that $u^{n_k}(x) \to u(x)$ a.e. Then $\text{Ls}(u^{n_k}(x)) = \{u(x)\}$, hence Balder's condition is satisfied for such a subsequence: $u(x) \in \partial_{\text{ext}} \overline{\text{co}}(\text{Ls}(u^{n_k}(x)))$. But this condition is not necessary for the whole sequence.

Even for subsequences the Visintin condition is not a necessary condition.

EXAMPLE. Let r^n be the Rademacher functions, $a_n \in]0, \infty[$, $a_n \to 0$, and $u^n = a_n r^n$.

Then, for any subsequence, $0 \in \text{int}[\overline{\text{co}}\{u^{n_k}(x) : k \in \mathbb{N}]$ because

$$\sup_{k \in \mathbb{N}} u^{n_k}(x) > 0 \text{ a.e.}$$

(Consider the Lebesgue measure on [0, 1] as a probability. The events $\{u^{nk} \leq 0\}$ have probability 1/2 and are independent so have a negligible intersection) and, symmetrically,

$$\inf_{k \in \mathbb{N}} u^{n_k}(x) < 0 \text{ a.e.}$$

The following criterion is not easy to handle but gives a necessary and sufficient condition. One is expecting for other criterions.

THEOREM 5. (Girardi–Balder–Valadier). *Suppose* $u^n \rightharpoonup u$ *in* $L^1(\Omega, \mu; \mathbb{R}^d)$. *Then* $u^n \to u$ *strongly if and only if the following criterion is verified:* $\forall \varepsilon > 0$, $\forall A \subset \Omega$ *with* $\mu(A) > 0$, $\exists N \in \mathbb{N}$, $\exists B \subset A$ *with* $\mu(B) > 0$, *such that* $\forall n \geq N$,

$$\frac{1}{\mu(B)} \int_B \| u^n(x) - \frac{1}{\mu(B)} \int_B u^n \, \mathrm{d}\mu \| \, \mathrm{d}x < \varepsilon.$$

References. M. Girardi [31, 32], E.J. Balder [10], M. Valadier [45]. For further results, see B. Bernoussi [18], E.J. Balder, M. Girardi and V. Jalby [12].

References

1. Amrani, A., Castaing, C. and Valadier, M.: Convergence forte dans L^1 impliquée par la convergence faible. Méthodes de troncature, *C. R. Acad. Sci. Paris Sér. I* **314** (1992), 37–40.
2. Amrani, A., Castaing, C. and Valadier, M.: Méthodes de troncature appliquées à des problèmes de convergence faible ou forte dans L^1, *Arch. Rational Mech. Anal.* **117** (1992), 167–191.
3. Balder, E. J.: A general approach to lower semicontinuity and lower closure in optimal control theory, *SIAM J. Control Optim.* **22** (1984), 570–598.
4. Balder, E. J.: A general denseness result for relaxed control theory, *Bull. Austral. Math. Soc.* **30** (1984), 463–475.
5. Balder, E. J.: An extension of Prohorov's theorem for transition probabilities with applications to infinite-dimensional lower closure problems, *Rend. Circ. Mat. Palermo* Serie II, **34** (1985), 427–447.
6. Balder, E. J.: On weak convergence implying strong convergence in L_1-spaces, *Bull. Austral. Math. Soc.* **33** (1986), 363–368.
7. Balder, E. J.: Generalized equilibrium results for games with incomplete information, *Math. Oper. Res.* **13** (2) (1988), 265–276.
8. Balder, E. J.: On Prohorov's theorem for transition probabilities, University of Utrecht, Department of Mathematics, Preprint 516, 1988, Publié dans *Sém. Anal. Convexe Montpellier*, exposé 9, 1989.
9. Balder, E. J.: On equivalence of strong and weak convergence in L_1-spaces under extreme point conditions, *Israel J. Math.* **75** (1991), 21–47.
10. Balder, E. J.: From weak to strong L_1-convergence by an oscillation restriction criterion of BMO type, University of Utrecht, Department of Mathematics, Preprint, May 1991.
11. Balder, E. J.: On weak convergence implying strong convergence under extremal conditions, *J. Math. Anal. Appl.* **163** (1992), 147–156.
12. Balder, E. J., Girardi, M. and Jalby, V.: From weak to strong types of L^1_E-convergence by the Bocce criterion, University of Utrecht, Department of Mathematics, Prepring 826, to appear in *Studia Math.*.
13. Ball, J. M.: A version of the fundamental theorem for Young measures, in M. Rascle, D. Serre and M. Slemrod (eds.) *Partial Differential Equations and Continuum Models of Phase Transitions*, Lecture Notes in Physics **344**, Springer-Verlag, Berlin, 1989, pp. 207–215.
14. Benabdellah, H.: Contribution aux Problèmes de Convergence Fort-Faible, à la Géométrie des Espaces de Banach et aux Inclusions Différentielles, Thèse de Doctorat, Montpellier, 1991.
15. Benabdellah, H.: Extrémalité, stricte convexité et convergence dans L^1_E, *Sém. Anal. Convexe Montpellier*, exposé 4, 1991.
16. Benabdellah, H.: Extrémalité et entaillabilité sur des convexes fermés non nécessairement bornés d'un espace de Banach. Caractérisation dans le cas des espaces intégraux, *Sém. Anal. Convexe Montpellier*, exposé 5, 1991.

17. Berliocchi, H. and Lasry, J. M.: Intégrandes normales et mesures paramétrées en calcul des variarions, *Bull. Math. Soc. France* **101** (1973), 129–184.
18. Bernoussi, B.: Oscillations et compacité en mesure, *Sém. Anal. Convexe Montpellier*, exposé 11, 1992.
19. Castaing, C.: Compacité dans l'espace des mesures de probabilité de transition, *Atti Sem. Mat. Fis. Univ. Modena* **34** (1985–86), 337–351.
20. Castaing, C.: Quelques résultats de convergence des suites adaptées, *Acta Math. Vietnam.* **14** (1989), 51–66 (first published in *Sém. Anal. Convexe Montpellier*, exposé 2, 1987.
21. Castaing, C.: Convergence faible et sections extrémales, *Sém. Anal. Convexe Montpellier*, exposé 2, 1988.
22. Castaing, C. and Valadier, M.: *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Math. **580**, Springer-Verlag, Berlin, 1977.
23. Cellina, A.: Lecture in these Conferences (Marseille, June 1992).
24. Chipot, M. and Kinderlehrer, D.: Equilibrium configurations of crystals, *Arch. Rational Mech. Anal.* **103** (1988), 237–277.
25. Dacorogna, B.: *Weak Continuity and Weak Lower Semicontinuity of Non-Linear Functionals*, Lecture Notes in Mathematics **922**, Springer-Verlag, Berlin, 1982.
26. Dacorogna, B.: *Direct Methods in the Calculus of Variations*, Springer-Verlag, Berlin, 1989.
27. Ekeland, I. and Temam, R.: *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, 1976 (French edition: Dunod Gauthier-Villars, Paris, 1974).
28. Ericksen, J. L.: Some phase transitions in crystals, *Arch. Rational Mech. Anal.* **73** (1980), 99–124.
29. Evans, L. C.: *Weak convergence methods for nonlinear partial differential equations*, C.B.M.S. **74**, Amer. Math. Soc., Providence, 1990.
30. Ghouila-Houri, A.: Sur la généralisation de la notion de commande d'un système guidable, *Rev. Fr. Inf. Rech. Op.* **4** (1967), 7–32.
31. Girardi, M.: Compactness in L_1, Dunford-Pettis operators, geometry of Banach spaces, *Proc. Amer. Math. Soc.* **111** (1991), 767–777.
32. Girardi, M.: Weak vs. norm compactness in L_1: the Bocce criterion, *Studia Math.* **98** (1991), 95–97.
33. Kinderlehrer, D. and Pedregal, P.: Characterization of Young measures generated by gradients, *Arch. Rational Mech. Anal.* **115** (1991), 329–365.
34. Kinderlehrer, D. and Pedregal, P.: Caractérisation des mesures de Young associées à un gradient, *C. R. Acad. Sci. Paris Sér. I* **313** (1991), 765–770.
35. Morrey, C. B.: *Multiple Integrals in the Calculus of Variations*, Springer-Verlag, Berlin, 1966.
36. Olech, C.: Existence theory in optimal control, in *Control Theory and Topics in Functional Analysis*, International Atomic Energy Agency, Vienna, 1976, pp. 291–328.
37. Rzeżuchowski, T.: Strong convergence of selections implied by weak, *Bull. Austral. Math. Soc.* **39** (1989), 201–214.
38. Rzeżuchowski, T.: Impact of Dentability on Weak Convergence in L^1, *Bolletino UMI* **7** (1992), 71–80.
39. Tartar, L.: Une nouvelle méthode de résolution d'équations aux dérivées partielles non linéaires, in P. Bénilan and J. Robert (eds.) *Journées d'analyse non linéaire*, Lecture Notes in Math. **665**, Springer-Verlag, Berlin, 1978, pp. 228–241.
40. Tartar, L.: Compensated compactness and applications to partial differential equations, in R. J. Knops (ed.) *Nonlinear Analysis and Mechanics: Heriot–Watt Symposium, Vol. IV*, Research Notes in Math. **39**, Pitman, London, 1979, pp. 136–212.
41. Valadier, M.: Désintégration d'une mesure sur un produit, *C. R. Acad. Sci. Paris Sér. A* **276** (1973), 33–35.
42. Valadier, M.: Différents cas où, grâce à une propriété d'extrémalité, une suite de fonctions intégrables faiblement convergente converge fortement, *Sém. Anal. Convexe Montpellier*, exposé 5, 1989.
43. Valadier, M.: Young measures, in A. Cellina (ed.) *Methods of Nonconvex Analysis*, Lecture Notes in Math. **1446**, Springer-Verlag, Berlin, 1990, pp. 152–188.
44. Valadier, M.: Application des mesures de Young aux suites uniformément intégrables dans un Banach séparable, *Sém. Anal. Convexe Montpellier*, exposé 3, 1990.

45. Valadier, M.: Oscillations et compacité forte dans L^1, *Sém. Anal. Convexe Montpellier*, exposé 7, 1991.
46. Visintin, A.: Strong convergence results related to strict convexity, *Comm. Partial Differential Equations* **9** (1984), 439–466.
47. Warga, J.: *Optimal Control of Differential and Functional Equations*, Academic Press, New York, 1972.
48. Young, L. C.: On approximation by polygons in the Calculus of variations, *Proc. Royal Soc.* **141** (1933), 325–341.
49. Young, L. C.: Generalized curves and the existence of an attained absolute minimum in the Calculus of Variations, *Comptes Rendus de la Société des Sc. et des lettres de Varsovie* cl. III **30** (1937), 212–234.
50. Young, L. C.: *Lectures on the Calculus of Variations and Optimal Control Theory*, Saunders, New York, 1969.