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Convergence of the Efficient Sets

DINH THE LUC[1], R. LUCCHETTI[2], and C. MALIVERTI[3]
[1] *Institute of Mathematics, POBox 631 boho, 10000 Hanoï, Vietnam;*
[2] *Dipartimento di Matematica, Università di Milano, Via Saldini 50, 20133 Milano, Italy;*
[3] *Département de Mathématiques, U.R.A. 1586, 123 Avenue Albert Thomas, 87060 Limoges, France.*

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Abstract. Let $A_n, n = 1, 2, \ldots$ be nonempty subsets of a linear metric space E and C_n , $n = 1, 2, \ldots$ convex cones of E . We consider the efficient sets Min(A_n, C_n) and the aim of this paper is to show that under suitable conditions, the convergence of A_n and C_n to A and C respectively, implies the convergence of Min(A_n, C_n) to Min(A, C) . Several illustrative examples are given which clarify the results.

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1. Introduction

Throughout this paper E denotes a metric linear space. We recall [6] that given a nonempty set $A \subset E$, and a proper convex cone $C \subset E$, the set of efficient points of A is defined by:

$$\text{Min}(A, C) = \{a \in A : a \in a' + C \text{ implies } a' \in a + C\}$$

In the case where int C is nonempty, the set of weakly efficient points is defined by

$$\text{WMin}(A, C) = \{a \in A : \text{ there is no } a' \in A \text{ with } a \in a' + intC\}$$

We say that C is pointed if $C \cap -C = \{0\}$. In this case

$$\text{Min}(A, C) = \{a \in A : a \in a' + C \text{ implies } a' = a\}$$

The question of how Min(A, C) and WMin(A, C) change under perturbations of A and C, has long been one of the most important and attractive topics in the theory of vector optimization and its applications. Today there exist a number of papers devoted to this question. Naccache [10], Tanino and Sawaragi [13] studied the case where E is a finite dimentional space. Lucchetti [9], Penot and Sterna-Karwat [12], Luc [6], Dolecki and Malivert [5], and Attouch and Riahi [1] investigated more general cases, including a perturbation of the ordering cone.

The purpose of our paper is to further study the stability of efficient sets Min(A, C) and WMin(A, C) when both A and C are under perturbations, by using several convergence concepts of set-valued analysis. In the context of multicriteria optimization these sets represent efficient values in the evaluation space. For the behavior of efficient solutions in the decision space, we refer the reader to [6, section 4, chapter 4] (see also [8]).

Recall that given a sequence of subsets $\{A_n\}$ in E , the Kuratowski–Painlevé lower and upper limits are defined as

$$\begin{aligned} \text{Li}\, A_n &= \{x \in E : x = \lim_{n\to\infty} x_n, x_n \in A_n \text{ for all large } n\} \\ \text{Ls}\, A_n &= \{x \in E : x = \lim_{k\to\infty} x_k, x_{n_k} \in A_{n_k}, \\ &\qquad n_k \text{ a selection of the integers}\} \end{aligned}$$

The condition $\text{Ls}\, A_n \subset A \subset \text{Li}\, A_n$ is referred to in the literature as the convergence of A_n to A in the sense of *Kuratowski–Painlevé.*

Considering the limits in the weak topology rather than the norm topology, we denote the subsets described above by w-Ls A_n and w-Li A_n. If w-Ls $A_n \subset A \subset$ Li A_n we say that A is the limit of A_n in the sense of *Mosco.*

Let us mention a classical stability result [12, 5], which is close to what we want to develop in this paper: If A_n converges to A in the sense of Kuratowski–Painlevé then

$$\text{Ls WMin}(A_n, C) \subset \text{ WMin } (A, C).$$

When the space E is finite-dimensional and A is closed this result is equivalent to say that if

$$\lim_{n\to\infty} d(A_n, x) = d(A, x) \quad \text{for every } x \in E \tag{W}$$

then one has also the inequality

$$\text{liminf } d(\text{WMin}(A_n, C), x) \geqslant d(\text{WMin}(A, C), x) \qquad \text{for all } x \text{ in } E.$$

Note that a sequence $\{A_n\}$ satisfying (W) is said converging to A in the sense of *Wijsman.* In this paper we study the infinite-dimensional case and further, we consider the case where the cone C is also under perturbations.

The next section provides sufficient criteria for relations of the form

$$\text{liminf } d(\text{WMin}(A_n, C_n), x) \geqslant d(\text{WMin}(A, C),\ x) \text{ for every } x \text{ in } E \tag{1}$$

or

$$\text{liminf } d(\text{Min}(A_n, C_n), x) \geqslant d(\text{Min}(A, C),\ x) \text{ for every } x \text{ in } E. \tag{2}$$

The other part of the convergence, namely

$$\text{limsup } d(\text{Min}(A_n, C_n), x) \leq d(\text{Min}(A, C),\ x) \text{ for every } x \text{ in } E \tag{3}$$

in general, requires more severe assumptions and will be obtained for efficient sets only. Note that (3) is equivalent, even in the infinite-dimensional setting, to

$$\text{Min}\,(A, C) \subset \text{Li}\,\text{Min}(A_n, C_n). \tag{4}$$

An assumption that usually plays a crucial role in this latter type of results is to suppose that $\bigcup_{n=1}^{\infty} \text{Min}\,(A_n, C_n)$ is nonempty and relatively compact. A simple but useful observation is that such an assumption provides in fact a much stronger convergence than (3) or (4). To illustrate this point let us recall the Attouch–Wets topology. B_ρ denotes the ball centered at 0 and with radius ρ. Given $x \in E$ and two nonempty sets A and B, define

$$d(x, A) = \inf_{a \in A} d(x, a) \quad (d(x, \emptyset) = \infty),$$
$$e(A, B) = \sup_{a \in A} d(a, B) \quad e(\emptyset, A) = 0 \quad (e(\emptyset, \emptyset) = 0) \quad e(A, \emptyset) = \infty,$$
$$e_\rho(A, B) = e(A \cap B_\rho, B) \text{ and } h_\rho(A, B) = \max\{e_\rho(A, B), e_\rho(B, A)\}.$$

We shall say that the sequence A_n converges to A in the Attouch–Wets sense if

$$\lim_{n \to \infty} h_\rho(A, A_n) = 0 \text{ for all } \rho > 0.$$

There is another useful way to describe the upper part of the Attouch–Wets convergence. Indeed it can be shown [3] that a sequence A_n converges to A in this sense if and only if, for each nonempty bounded set B

$$\text{liminf}\; d(A_n, B) \geqslant d(A, B).$$

Here $d(A, B)$ stands for $\inf_{a \in A} \inf_{b \in B} d(a, b)$, where we agree that $d(A, B) = +\infty$ iff at least one of the two sets is empty.

2. First Type of Convergence

C_n, $n = 1, 2, \ldots$ denote a sequence of convex cones in E, and $C_n^c = E \backslash C_n$ the complementary sets of C_n . Unless otherwise specified, Min A and Min A_n will always stand for Min (A, C) and Min (A_n, C_n), respectively.

THEOREM 2.1. *Assume that every bounded subset of* $\bigcup_{n=1}^{\infty} WMinA_n$, *is relatively compact. If*

(i) Ls $A_n \subset A \subset$ Li A_n,
(ii) Ls $C_n^c \subset$ cl (C^c)

then liminf $d(\text{WMin}A_n, B) \geqslant d(\text{WMin}\ A,\ B)$ *for each bounded part* B *in* E.

If, in addition, infinitely many A_n *possess weakly efficient points, which are located in a bounded set, then* WMin A *is nonempty.*

Proof. The conclusion of the Theorem is trivial if liminf $d(\text{WMin } A_n, B) = \infty$. Therefore, it suffices to consider the case where this lower limit is finite. Suppose on the contrary that there is some bounded subset $B \subset E$ with $\alpha = d(\text{WMin } A,\ B)$ (α possibly infinite) and some positive number γ such that

$$\liminf d(\text{WMin } A_n, B) < \gamma < \alpha.$$

By taking a subsequence of $\{A_n\}$ if necessary we may assume that $d(\text{WMin } A_n, B) < \gamma$ for all n and, hence, there exists $y_n \in \text{WMin } A_n$ such that $d(y_n, B) < \gamma$ for every n. By the compactness assumption, it can be assumed that a subsequence of $\{y_n\}$ converges to some $y_o \in E$. From (i), one gets $y_o \in A$. This point y_o cannot be a weakly efficient point of A because $d(y_o, B) \le \gamma < \alpha$ while $d(\text{WMin } A,\ B) = \alpha$. One can find some $a \in A$ such that $y_o - a \in intC$. Therefore $y_o - a \notin \text{cl}(C^c)$ and from (ii), using $\text{Ls } C_n^c = \text{Ls cl}(C_n^c)$, we have

$$y_o - a \notin \text{Ls cl}(C_n^c). \tag{5}$$

On the other hand (i) implies that there is a sequence $\{a_n\}$, $a_n \in A_n$ such that $\lim_{n\to\infty} a_n = a$. Consequently $\lim_{n\to\infty} y_n - a_n = y_o - a$ and by (5) there exists $n \in N$ such that $y_n - a_n \notin \text{cl}(C_n^c)$. Thus $y_n - a_n \in intC_n$, contradicting the weak-efficiency of y_n.

Under the additional hypothesis of the Theorem, liminf $d(\text{WMin } A_n, B)$ is a finite number, hence $d(\text{WMin } A,\ B)$ is finite too, and WMin A must be nonvoid.

THEOREM 2.2. *Assume that E is a reflexive space, C_n and C are pointed, convex cones. If*

(i) *w-Ls* $A_n \subset A \subset$ *w-Li* A_n,

(ii) *w-Ls* $C_n^c \subset C^c \cup \{0\}$,

then liminf $d(\text{Min } A_n, x) \geqslant d(\text{Min } A,\ x)$ *for every* $x \in E$.

Proof. As in the previous theorem, it is sufficient to consider x such that liminf $d(\text{Min } A_n, x)$ is finite. Suppose that the result does not hold. By taking a subsequence of $\{A_n\}$ if necessary we can find $y_n \in \text{Min } A_n$ and a positive number γ satisfying for all n,

$$d(y_n, x) < \gamma < d(\text{Min } A, x).$$

By the reflexivity of E, it can be assumed that a subsequence of $\{y_n\}$ weakly converges to some $y_o \in E$ and by (i), $y_o \in A$. Since such y_o belongs to the closed ball centered at x with radius γ we have for the weak limit y_o, $d(y_o, x) \le \gamma < d(\text{Min } A, x)$, and then $y_o \notin \text{Min } A$. Thus, there exists $a \in A$ such that

$$y_o - a \in C\backslash\{0\}.$$

Using $A \subset$ w-Li A_n, one may suppose the existence of $a_n \in A_n$ with w-$\lim_{n\to\infty} a_n = a$. We claim that there exists $n_o > 0$ with $y_n - a_n \in C_n\backslash\{0\}$ for $n \geqslant n_o$, contradicting the efficiency of y_n. Otherwise there is a subsequence of

$\{y_n - a_n\}$, $y_{n_k} - a_{n_k} \in C^c_{n_k} \cup \{0\}$ and from the assumption (ii), $y_o - a$ belongs to $C^c \cup \{0\}$, a contradiction. □

From this theorem we easily deduce another result for weak efficiency, when $C_n = C$ for all n. Recall that C is said to be polyhedral if it is the intersection of a finite number of half spaces.

COROLLARY 2.1. *Assume that E is a reflexive space and C is a polyhedral cone with a nonempty interior. If*

(i) *w-Ls $A_n \subset A \subset$ w-Li A_n,*

then $\liminf d(\text{WMin}(A_n, C), x) \geqslant d(\text{WMin } A,\ x)$ *for every* $x \in E$.

Proof. Setting $D = \text{int } C$, one applies Theorem 2.2 with $C_n = C = D$. Note that weak efficiency with respect to C, coincides with efficiency with respect to D. Further C being polyhedral, D^c is closed for the weak topology and the assumption (ii) of Theorem 2.2, w-Ls $D^c \subset D^c \cup \{0\}$, is satisfied. □

Condition (i) in Theorem 2.2 and Corollary 2.1 is Kuratowski–Painlevé convergence with respect to the weak topology, which is clearly weaker than Mosco convergence.

Recall that a Banach space E is said to be dual Kadec if for every sequence x^*_n in the dual of E, weakly convergent to x^*, with $\| x^*_n \| = \| x^* \| = 1$ one has $\lim_{n \to \infty} \| x^*_n - x^* \| = 0$. Borwein and Fitzpatrick [2] proved that if E is a reflexive, dual Kadec–Banach space, Mosco and Wijsman convergences coincide for sequences of closed nonempty convex subsets. Then we can deduce from 2.1 the following result.

COROLLARY 2.2. *Assume that E is a reflexive dual Kadec–Banach space and C a polyhedral cone with a nonempty interior. If A_n n = 1, 2, ... and A are closed convex sets such that* $\lim_{n \to \infty} d(A_n, x) = d(A, x)$ *for every* $x \in E$, *then* $\liminf d(\text{WMin}(A_n, C), x) \geqslant d(\text{WMin } A, x)$ *for every* $x \in E$.

If E is a Hilbert space the assumption of convexity on A_n can be relaxed as shown in the following result.

THEOREM 2.3. *Assume that E is a Hilbert space, C a polyhedral cone with a nonempty interior and A a nonempty closed convex set. If* $\lim_{n \to \infty} d(A_n, x) = d(A, x)$ *for every* $x \in E$, *then* $\liminf d(\text{WMin}(A_n, C), x) \geqslant d(\text{WMin } A,\ x)$, *for every* $x \in E$.

Proof. As previously we prove the result at those points where $\liminf d(\text{WMin}(A_n, C), x)$ is finite. By supposing that the result is not true, one can find a positive number γ such that

$$\forall n_o \in N\ \exists n \geqslant n_o \exists y_n \in \text{WMin}(A_n, C) \qquad d(y_n, x) < \gamma < d(\text{WMin } A, x).$$

Since E is a reflexive space, it can be assumed that the sequence $\{y_n\}$ weakly converges to some $y_o \in E$. Clearly, $d(y_o, x) \leq \gamma$, and also

$$d(y_n, y_o) \leq d(y_n, x) + d(y_o, x) \leq 2\gamma. \tag{6}$$

We claim that $y_o \in A$. If not, using a separation theorem one can find a vector $\xi \in E$, with $\| \xi \| = 1$, and a positive number η such that

$$\langle \xi, y \rangle \leq \langle \xi, y_o \rangle - 2\eta, \quad \text{for every } y \in A. \tag{7}$$

Set $y_t = y_o + t\xi$ with $t > 0$. Observe that the ball $B(y_t, t + \eta)$ does not meet A, because every $y \in B(y_t, t + \eta)$ satisfies the relation

$$\langle \xi, y_t - y \rangle \leq \| y - y_t \| \leq t + \eta,$$

which implies

$$\langle \xi, y \rangle \geqslant \langle \xi, y_t \rangle - (t + \eta) \geqslant \langle \xi, y_o \rangle - \eta$$

and by (7), $y \notin A$. Furthermore, if t is sufficiently large, there exists $n_o > 0$ such that $y_n \in B(y_t, t + \eta)$ for all $n \geqslant n_o$. To see this, let us calculate $d(y_t, y_n)$:

$$\| y_t - y_n \|^2 = \| y_o - y_n \|^2 + t^2 + 2t\langle \xi, y_o - y_n \rangle. \tag{8}$$

From (6), one takes t such that $\| y_n - y_o \|^2 \leq (2\gamma)^2 \leq \eta t$ and then, using $\lim_{n \to \infty} \langle \xi, y_o - y_n \rangle = 0$, choose n_o (depending on t), such that

$$\langle \xi, y_o - y_n \rangle \leq \eta^2/(2t), \text{ for } n \geqslant n_o.$$

The value of (8) can now be estimated as

$$\| y_t - y_n \|^2 \leq \eta t + t^2 + \eta^2 \leq (t + \eta)^2, \qquad \text{for } n \geqslant n_o.$$

In this way, $y_n \in B(y_t, t + \eta)$ for $n \geqslant n_o$ and consequently $d(A_n, y_t) \leq t + \eta$. This and the fact that $d(A, y_t) > t + \eta$ contradict the assumption of the theorem at the point y_t saying that $\lim_{n \to \infty} d(A_n, y_t) = d(A, y_t)$. Thus, we have shown that $y_o \in A$.

Since $d(y_o, x) \leq \gamma < d(\text{WMin } A, x)$ the point y_o cannot be a weakly efficient point of A, i.e.

$$y_o \in a + intC \text{ for some } a \in A. \tag{9}$$

Let $a_n \in A_n$ with $\lim_{n \to \infty} a_n = a$. Since C is polyhedral and $\{y_n\}$ weakly converges to y_o, it follows from (9) that $y_n \in a_n + \text{int } C$ whenever n is large enough. This contradicts the fact that $y_n \in \text{WMin}(A_n, C)$ and completes the proof. □

3. Second Type of Convergence

Another kind of convergence often considered for efficient sets, is the lower part of the Kuratowski–Painlevé convergence, namely

$$\text{Min } A \subset \text{Li Min } A_n. \tag{10}$$

Before mentioning two important results in that vein, we recall [6], that a set $A \subset E$ is said to satisfy the domination property if, for every $x \in A$, there is $a \in \text{Min } A$ such that $x \in a + C$.

THEOREM 3.1. *[12, 5] Assume that the following conditions hold*

(i) $\text{Ls } A_n \subset A \subset \text{Li } A_n$,
(ii) A_n *satisfy the domination property for all large n,*
(iii) *if* $a_n \in A_n$ *is such that* $\lim_{n\to\infty} a_n$ *exists and* $e_n \in \text{Min } A_n \cap (a_n - C)$, *then* $\{e_n\}$ *admits a convergent subsequence,*
(iv) $\text{Ls } C_n \subset C$ *with C a closed pointed convex cone,*
then $\text{Min } A \subset \text{Li Min } A_n$.

A similar result has been proved by Attouch and Riahi when E is a Banach space and the cone C is not under perturbations. In that latter result it is supposed that C is a closed convex cone satisfying the condition

$$C \subset \{x \in E : l(x) \geqslant \epsilon \parallel x \parallel\} \tag{11}$$

where $\epsilon > 0$ and $l \in E*$, the topological dual of E.

THEOREM 3.2. *[1] Assume that the following conditions hold*

(i) $\text{Ls } A_n \subset A \subset \text{Li } A_n$,
(a) $\inf_{n\in N} \inf_{x\in A_n} l(x) > -\infty$,
(b) *for every* $\rho > 0$, $(\cup_{n\in N}\text{Min}(A_n, C)) \cap B\rho$ *is relatively compact, then* $\text{Min } A \subset \text{ Li Min}(A_n, C)$.

It can be observed that in a Banach space the conditions (11) and (a) imply the domination property (ii). Moreover, conditions (a) and (b) entail the compactness assumption (iii).

Now we present sufficient criteria in order to obtain a convergence stronger than (4).

THEOREM 3.3. *Assume that the following conditions hold*

(i) $\text{Ls } A_n \subset A \subset \text{Li } A_n$,
(ii) A_n *satisfy the domination property for all large n,*
(iii) *if* $a_n \in A_n$ *is such that* $\lim_{n\to\infty} a_n$ *exists and* $e_n \in \text{Min } A_n \cap (a_n - C)$, *then* $\{e_n\}$ *admits a convergent subsequence,*
(iv) $\text{Ls } C_n \subset C$ *with C a closed pointed convex cone,*

(v) *for every* $\rho > 0$, Min $A \cap B\rho$ *is relatively compact.*
Then for each $\rho > 0$, $\lim_{n\to\infty} e_\rho(\text{Min } A, \text{Min } A_n) = 0$.

Proof. Suppose that the conclusion of Theorem does not hold, then there exist $\rho > 0$, $\epsilon > 0$ and a subsequence of $\{A_n\}$, denoted by $\{A'_n\}$, such that

$$\forall k \qquad e_\rho(\text{Min } A, \text{Min } A'_k) > \epsilon \tag{12}$$

Then for each k, there exists $e_k \in \text{Min } A \cap B_\rho$ satisfying

$$d(e_k, \text{Min } A'_k) > \epsilon \tag{13}$$

By (v), $\{e_k\}$ admits a subsequence converging to some $e \in E$ and then from (13) there exists $K > 0$ such that

$$d(e, \text{Min } A'_k) > \epsilon/2 \qquad \text{for all } k > K. \tag{14}$$

Using Theorem 3.1 we know that Min $A \subset$ Li Min A_n. Thus, for each $k, e_k = \lim_{k\to\infty} e_i^k$ with $e_i^k \in A'_i$. It follows that for each k we can choose $e_{i(k)}^k \in A'_{i(k)}$ such that

$$| e_k - e_{i(k)}^k |< 1/k$$

and then $\lim_{k\to\infty} e_{i(k)}^k = e$ which contradicts (14). □

The conclusion of Theorem 3.3 corresponds to the lower part of the Attouch-Wets convergence and it is known that it implies the lower part of the Kuratowski–Painlevé convergence. The opposite implication, as proved here under assumption (v), can also be derived using a result of [11].

If we replace assumptions (iii) and (v) in 3.3, by

(iii′) $\bigcup_{n=1}^\infty$ Min A_n is relatively compact,

we get in addition to the conclusion of Theorem 3.3, that Min A is nonempty. More precisely we have

THEOREM 3.4. *Assume that the following conditions hold*

(i) Ls $A_n \subset A \subset$ Li A_n,
(ii) A_n *satisfy the domination property for all large n,*
(iii′) $\bigcup_{n=1}^\infty$ Min A_n *is relatively compact,*
(iv) Ls $C_n \subset C$ *with C a closed pointed convex cone.*
Then Min A *is nonempty, compact and* $\lim_{n\to\infty} e(\text{Min } A, \text{Min } A_n) = 0$.

Proof. To show that Min $A \neq \emptyset$, one sets

$$A_o = \text{Ls Min } A_n. \tag{15}$$

Since $A \neq \emptyset$ and $A \subset \text{Li } A_n$, all A_n are nonempty for large n. Then from (ii) and (iii′), A_o is a nonempty compact set. In view of an existence theorem [7], the set Min A_o is nonempty. We claim that

$$\text{Min}(A_o, C) \subset \text{Min } A. \tag{16}$$

In fact, let $e \in \text{Min } A_o$. Then $e \in A_o \subset A$. If $e \notin \text{Min } A$, there is $a \in A$ such that $e \in a + C\backslash\{0\}$. By (i), there exist $a_n \in A_n$ with $\lim_{n\to\infty} a_n = a$ and by (ii), there exist $e_n \in \text{Min } A_n$ with $a_n \in e_n + C$, $n = 1, 2, \ldots$ In view of (iii′) we may assume that $\{e_n\}$ converges to some $e_o \in E$. It is clear that $e_o \in A_o$. Moreover, from (iv), $a \in e_o + C$. Consequently,

$$e \in e_o + C + C\backslash\{0\} \subset e_o + C\backslash\{0\}.$$

which contradicts the efficiency of e. Thus, Min A is nonempty.

Now we prove that Min A is closed and, consequently, compact. Consider a sequence $e_n \in \text{Min } A$ such that $\lim_{n\to\infty} e_n = e$. Since A is closed (from (i)), $e \in A$. Suppose that $e \notin \text{Min } A$, then there exists $x \in A$ such that $e - x \in C\backslash\{0\}$. By (i), $x = \lim_{n\to\infty} a_n$, $a_n \in A_n$ and from (ii) and (iii′) there exists a sequence $y_{n_i} \in \text{Min } A_{n_i} \cap (a_{n_i} - C)$, with $\lim_{i\to\infty} y_{n_i} = y$, n_i being a selection of integers. By (i), $y \in A$ and by (iv), $x - y \in C$. Therefore $e - y = (e - x) + (x - y) \in C\backslash\{0\}$ which contradicts the optimality of e.

For the last conclusion of the Theorem, suppose on the contrary that there exist $\epsilon > 0$ and a subsequence $\{A'_n\}$ of $\{A_n\}$ such that

$$e(\text{Min } A, \text{Min } A'_n) > \epsilon \text{ for all } n > 0. \tag{17}$$

Then for each n there is $e_n \in \text{Min } A$ satisfying

$$d(e_n, \text{Min } A'_n) > \epsilon.$$

From the compactness of Min A there exist $e \in \text{Min } A$ and $n_o > 0$ such that

$$d(e, \text{Min } A'_n) > \epsilon \qquad \text{for } n > n_o. \tag{18}$$

As (iii′) implies (iii), we have from Theorem 3.1

$$\text{Min } A \subset \text{Li Min } A_n$$

which contradicts (18).

Remark 3.1. The proof of Theorem 3.4 can be carried out in the same way by replacing the assumption (iii′) by the weaker one: (iii″) the sequence $\{\text{Min } A_n\}_{n\in N}$ is compactoid. Under (iii″), Ls Min A_n is a nonempty compact set [4, Prop 3.1, Cor 4.13].

The conclusions of Theorem 3.4 can also be obtained under assumptions involving the weak topology on E.

THEOREM 3.5. *Suppose that the following conditions hold*

(i) $\text{Ls } A_n \subset A \subset \text{w} - \text{Li } A_n$ *and* A *closed, nonempty,*
(ii) A_n *satisfy the domination property for all large n,*
(iii′) $\bigcup_{n=1}^{\infty} \text{Min } A_n$ *is relatively compact,*
(iv) $\text{w} - \text{Ls } C_n \subset C$ *with* C *a closed pointed convex cone.*
Then $\text{Min } A$ *is a nonempty compact subset and* $\lim_{n\to\infty} e(\text{Min } A, \text{Min } A_n) = 0$.

Proof. Setting $A_o = \text{Ls Min } A_n$ we prove in a very similar way as in Theorem 3.4 that $\text{Min } A$ is a nonempty compact subset such that $\emptyset \neq \text{Min } A_o \subset \text{Min } A$.

Now suppose that there exist $\epsilon > 0$ and a subsequence $\{A'_n\}$ of $\{A_n\}$ such that

$$e(\text{Min } A, \text{Min } A'_n) > \epsilon \text{ for } n > 0 \tag{19}$$

As previouly we get from the compactness of Min A that there exist $e \in \text{Min } A$ and $n_o > 0$ such that

$$d(e, \text{Min } A'_n) > \epsilon \qquad \text{for } n > n_o. \tag{20}$$

On the other hand by (i), $e = \text{w} - \lim a_k$, $a_k \in A'_{n_k}$. By (ii), for each $k \in N$, there exist $e_k \in \text{Min } A'_{n_k} \cap (a_k - C)$ and (iii′) entails that a subsequence of $\{e_k\}$ converges to some $x \in \text{Ls Min } A'_{n_k}$. It follows from (i) that $x \in A$ and from (iv) that $e - x \in C$. The efficiency of e implies $e = x$ and then $e \in \text{Ls Min } A'_{n_k}$ a contradiction with (17). □

In the last section, we propose some examples to clarify the role of the assumptions.

4. Examples

The first example shows that the condition (ii) in Theorem 2.2 cannot be relaxed even when $C_n = C$ for all $n \in N$.

Let l^2 be the space of sequences $x = \{x_k\}$ with $\sum_{k=1}^{\infty} x_k^2 < \infty$. The norm of l^2 is given by $||x|| = (\sum_{k=1}^{\infty} x_k^2)^{1/2}$. Let C be the cone of nonnegative vectors i.e.

$$C = \{x = \{x_k\} : x_k \geqslant 0, k = 1, 2, ...\}.$$

Note that in this example int $C = \emptyset$ so that we cannot consider weakly efficient points. Let a^n, $n = 1, 2...$ be a sequence of vectors in l^2 with $a^n = \{a_k^n\}$,

$$a_k^n = \begin{cases} -1/(2n)^k & \text{if } k \neq n \\ -1/4 & \text{if } k = n \end{cases}$$

and $a^o = -(1/\sqrt{2}, 1/\sqrt{4}, 1/\sqrt{8}, 1/\sqrt{16}, ...)$. Set

$$A = \{ta^o : 0 \leq t \leq 1\} \quad \text{and} \quad A_n = A \cup \{a^n\}$$

As the sequence $\{a^n\}$ weakly converges to 0, condition (i) in Theorem 2.2 is satisfied. However we prove that liminf $d(\text{Min } A_n, -a^o) < d(\text{Min } A, -a^o)$.

We have Min $A = \{a^o\}$ while for $n \geqslant 5$, Min $A_n = \{a^o\} \cup \{a^n\}$. Thus

$$d(\text{Min } A, -a^o) = d(a^o, -a^o) = 2||a^o|| = 2.$$

Let us calculate

$$d(a^n, -a^o)^2 = ||-a^o||^2 + ||a^n||^2 - 2 < a^n, -a^o >$$

Observe that $\lim_{n\to\infty}\langle a^n, -a^o\rangle = 0$ because $\{a_n\}$ weakly converges to 0. Therefore

$$\begin{aligned}
&\lim_{n\to\infty} d(a^n, -a^o)^2 \\
&= ||-a^o||^2 + \lim_{n\to\infty} ||a^n||^2 \\
&= 1 + \lim_{n\to\infty}\left(1/4^2 + \sum_{k=1,k\neq n}^{\infty} 1/(2n)^{2k}\right) \leq 1 + 1/4^2.
\end{aligned}$$

Finally for $n \geqslant 5$ one has $d(\text{Min } A_n, -a^o) = \text{Min}(d(a^o, -a^o), d(a^n, -a^o))$ and

$$\text{liminf } d(\text{Min } A_n, -a^o) \leq (1 + 1/16)^{1/2} < 2.$$

In this example $C^c \cup \{0\}$ is not weakly closed so that Theorem 2.2 does not apply.

The second example shows that the compactness assumption (iii′) in Theorem 3.4 or Theorem 3.5 cannot be omitted.

Let a^n, $n = 1, 2, \ldots$ be elements of l^2 which are given by $a^n = \{a_k^n\}$ with

$$a_k^n = \begin{cases} -1/(2n)^k & \text{if } k \neq n \\ -1 & \text{if } k = n \end{cases}$$

Set $A = \{0\}$ and $A_n = \{ta^n : 0 \leq t \leq 1\}$ for $n = 1, 2, \ldots$. We have Min $A_n = \{a^n\}$ and Min $A = \{0\}$. We show that

$$\lim_{n\to\infty} d(A_n, x) = d(A, x) \quad \text{for every } x \in l^2,$$

and

$$\lim_{n\to\infty} d(0, \text{Min } A_n) > 0.$$

The first limit means that A_n converges to A in the sense of Wijsman, which implies conditions (i) of Theorem 3.4 and Theorem 3.5 (since A is closed).

In fact, for any $t \in [0, 1]$ and any $x = \{x_k\} \in l^2$,

$$||x - ta^n||^2 = ||x||^2 + t^2||a^n||^2 - 2t \sum_{k=1,k\neq n}^{\infty} x_k/(2n)^k - 2tx_n.$$

As in the first example, one sees that $\lim_{n\to\infty} ||x - ta^n||^2 = ||x||^2 + t^2$ because $\lim_{n\to\infty} ||a^n||^2 = 1$. This means that for every $x \in l^2$,

$$\lim_{n\to\infty} d(A_n, x) = ||x|| = d(A, x). \tag{21}$$

We have then

$$\lim_{n\to\infty} d(0, \text{Min } A_n) = \lim_{n\to\infty} d(0, a^n) = \lim_{n\to\infty} ||a^n|| = 1.$$

In this example the sequence $\{a_n\} = \{\text{Min } A_n\}$ does not admit any convergent subsequence.

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