*Journal of Statistical Physics, Vol. 50, Nos. 3/4, 1988*

# Statistical Mechanics of the Nonlinear Schrödinger Equation

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*Received September 21, 1987*

We investigate the statistical mechanics of a complex field $\phi$ whose dynamics is governed by the nonlinear Schrödinger equation. Such fields describe, in suitable idealizations, Langmuir waves in a plasma, a propagating laser field in a nonlinear medium, and other phenomena. Their Hamiltonian

$$H(\phi)=\int_{\Omega}\left[\tfrac{1}{2}|\nabla\phi|^2-(1/p)\,|\phi|^p\right]dx$$

is unbounded below and the system will, under certain conditions, develop (self-focusing) singularities in a finite time. We show that, when $\Omega$ is the circle and the $L^2$ norm of the field (which is conserved by the dynamics) is bounded by $N$, the Gibbs measure $\nu$ obtained is absolutely continuous with respect to Wiener measure and normalizable if and only if $p$ and $N$ are such that classical solutions exist for all time—no collapse of the solitons. This measure is essentially the same as that of a one-dimensional version of the more realisitc Zakharov model of coupled Langmuir and ion acoustic waves in a plasma. We also obtain some properties of the Gibbs state, by both analytic and numerical methods, as $N$ and the temperature are varied.

**KEY WORDS:** Nonlinear Schrödinger equation; statistical mechanics; unbounded Hamiltonians; singularities; Gibbs measures.

## 1. INTRODUCTION

### 1.1. General Background

The essence of statistical mechanics is the replacement of the study of the microscopic dynamical trajectory of an individual macroscopic system by

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0022-4715/88/0200-0657$06.00/0 

the study of appropriate ensembles or probability measures on the phase space of the system. The success of this program, especially for systems in equilibrium, has made a virtue of necessity. Instead of trying to solve the initial value problem for a system containing a very large (say $10^{23}$) number of particles, which is clearly an impossible task even in principle, we obtain information about values of macroscopic observables by taking averages over Gibbs probability distributions containing only a few parameters (particle density, temperature, etc.).

While the rigorous justification of the theory is still not fully understood, its success leaves no doubt about its utility. In fact, the results obtained from a suitable probability measure, which includes information about both typical behavior and fluctuations, are generally more relevant than the solution of a specific initial value problem for understanding the behavior of real systems.(1) With this in mind, we present here an extension of the statistical mechanical formalism to a continuum field with a Hamiltonian that is unbounded below. The system models certain aspects of the behavior of a plasma excited by an electromagnetic field, say that of a laser(2–4) (this is our primary motivating example), as well as the propagation of a laser beam in a nonlinear medium and other phenomena (see Ref. 5 for references).

To put our work in context, we recall first the structure of Gibbs distributions of particle systems. In classical mechanics (the quantum case is similar) the Gibbs probability distribution for finding a system consisting of $N$ particles in a compact spatial region $\Omega$ is a set of microscopic states $dX_N$ is given by

$$\mu(dX_N) = Z_N^{-1} \exp[-\beta H(X_N)]\, dX_N$$

where $H$ is the Hamiltonian of the system, $\beta$ is the reciprocal temperature, and $Z_N$ is a normalization constant (also the partition function). Provided now that $H(X_N)$ is suitably bounded from below ($H$ stability) and satisfies other reasonable conditions,(6) one can take the thermodynamic limit $N \to \infty$, $|\Omega| \to \infty$, $N/|\Omega| \to \rho$ (where $|\Omega|$ is the volume of $\Omega$), the correct idealization of a macroscopic system, and obtain a well-defined measure on the resulting infinite-dimensional phase space.

The situation becomes more complicated when the appropriate microscopic description of the system is in terms of a continuum field, say $\phi(\mathbf{x})$, $\mathbf{x} \in \Omega$. Such a description arises either when $\phi$ represents a fundamental field of nature, e.g., the electromagnetic or Yang–Mills field, or when $\phi$ is itself a coarse-grained (reduced) description of the microscopic system, e.g., the density and/or velocity field of a fluid. While the statistical mechanical treatment of the latter "derivative" fields, which endows them with fluctuations, may appear at first sight artificial (first coarsening, then refining), there are often very good reasons for wanting to focus attention on the collective variables of the field rather than on the atomic degrees of freedom—e.g., in fluid turbulence it is in the hydrodynamic modes that the interesting action takes place.

The difficulties encountered in constructing statistical mechanical theories of fields are associated with the infinite number of degrees of freedom present already in a finite volume—the ultraviolet problem. This causes no real trouble in the linear case, i.e., when the Hamiltonian is quadratic in the fields, since the degrees of freedom essentially decouple—say, in a Fourier representation—to give the free field. (One of the theory's first applications, the statistical treatment of blackbody radiation, is an example of such a field.) In the nonlinear case, comonly treated in Euclidean quantum field theory, the difficulties are indeed serious when the spatial dimension $d$ is greater than three. For $d \leqslant 3$, on the other hand, all can be made well for the type of polynomial Hamiltonian usually considered in field theory. These Hamiltonians are bounded from below and permit extensions of the usual tricks.[7] The situation is quite different when the Hamiltonian appropriate to the continuum field is not bounded below, the case we consider there.

It is *a priori* clear that this situation can arise only when the system has, in the coarse-grained description provided by the collective modes, the potentiality for some kind of instability on the appropriate scale. This is indeed the case for many systems, including suitable plasmas irradiated with laser light. Here the collective modes represent Langmuir waves and the instability is the soliton or caviton collapse, which gives rise to plasma turbulence. The physics of the problem—including the reasons for expecting an equilibrium treatment of the collective modes to be of any relevance, even though the system is clearly not in thermodynamic equilibrium—are given in Section 3; there we also discuss the Zakharov model, which contains the ion acoustic waves in addition to Langmuir waves. Here we go on to describe our simpler model and some features of our approach to its statistical mechanics.

### 1.2. Formulation of the Problem

We construct and study equilibrium Gibbs measures of systems whose microscopic state is modeled by a complex continuum field $\phi(x)$ depending on one space dimension. We take the domain $\Omega$ to be the interval $[0, L]$, with periodic boundary conditions, and consider Hamiltonians of the form

$$H(\phi) = \frac{1}{2} \int_0^L |\phi'(x)|^2 \, dx - \frac{1}{p} \int_0^L |\phi(x)|^p \, dx \tag{1.1}$$

where the real and imaginary parts of $\phi$ are the conjugate canonical variables and $p$ will be assumed to satisfy $p \geqslant 3$. The corresponding equation of motion for a time-dependent field $u(x, t)$ $(= [\phi(x)](t))$ is

$$iu_t = -u_{xx} - |u|^{p-2} u \tag{1.2}$$

with $x \in [0, L]$, which is to be supplemented by boundary conditions

$$u(0, t) = u(L, t), \qquad t \in \mathbb{R} \tag{1.3}$$

and an initial condition

$$u(x, 0) = \phi_0(x), \qquad 0 \leqslant x \leqslant L \tag{1.4}$$

Equation (1.2) will be referred to as the nonlinear Schrödinger equation (NLSE)[8]; it is a generalization of the usual case, for which $p = 4$. A related vector NLSE in $d = 3$ space dimensions and with $p = 4$ describes in certain regimes the Langmuir waves in a plasma, i.e., the time evolution of the envelope of the propagating electric field. For the problem of a laser field propagating in a nonlinear medium, $t$ corresponds to the space coordinate in the direction of propagation and $d = 2$, $p = 4$ is the physically relevant case.

Starting from smooth initial data, solutions of the NLSE with $p = 4$ and $d \geqslant 2$ can develop singularities in a finite time. On the other hand, in one dimension (1.2) is well known to be integrable for $p = 4$.[9] To get a singular behavior in one dimension similar to that encountered in the physically interesting cases, one must consider (1.2) with $p \geqslant 6$. [Generally the critical value of $p$ in dimension $d$ is $p_c = (2d + 4)/d$.] This suggests, and there is in fact reason to believe,[10] that by studying (1.1) and (1.2) for $p > 4$ one may obtain insight into the behavior of the $p = 4$ system in high dimensions.

Our basic problems then will be to investigate the existence and properties of Gibbs measures with formal densities $Z^{-1} \exp[-\beta H(\phi)]$, with $H$ given in (1.1). A little thought shows that the unboundedness of $H(\phi)$ from below makes this totally impossible without some additional restrictions. Fortunately there is a natural way to restrict the (infinite-dimensional) phase space of this system: we make use of the fact that (1.2) conserves, in addition to the energy, the $L^2$ norm (analogous to the particle number) of the field:

$$N(\phi) = \int_0^L |\phi(x)|^2 \, dx \tag{1.5}$$

We can therefore try to construct Gibbs measures with specified values of $N$ or with a bound on $N$; this is exactly what we do in this paper.

We remark that these measures live on rather ragged field configurations $\phi(x)$ for which the classical energy is infinite; in fact, on Brownian paths. Physical and mathematical problems raised by these irregular configurations will be discussed in Sections 3 and 5.

We do not attempt to go to an infinite-volume (thermodynamic) limit, however, as we would in the traditional case with a Hamiltonian bounded from below. The reason is that, due to the unboundedness of $H$, our system has a tendency to concentrate its field into local packets, driving the kinetic energy up and the potential energy down; the ground state of the system is a soliton (or caviton in plasma language) whose strength grows with $N$. Now, if we tried to let $N \to \infty$ as $L \to \infty$, keeping $N/L$ fixed, the system would concentrate the available $N$ and make the field blow up locally. This is just what happens for $p > p_c$ even for finite $N$ and is in fact what makes this model interesting for the study of plasma turbulence.

The remainder of the paper is organized as follows. In Section 2 we describe some aspects of the behavior of the NLSE as a classical evolution equation (for the case $d = 1$ with periodic boundary conditions) and state our results on the existence of the statistical ensemble; we find that the conditions under which the ensemble is well defined correspond closely to conditions guaranteeing a well-defined classical evolution. In Section 3 we describe briefly some of the plasma physics underlying our model and related properties of the statistical ensemble. Section 4 is devoted to proofs of results given in preceding sections. Concluding in Section 5, we survey some open problems.

## 2. SUMMARY OF RESULTS

### 2.1. The Initial Value Problem

In this section we study the nonlinear Schrödinger equation on a finite interval as an initial value problem. The conclusions we reach are slight modifications of known results, and will not in fact be used subsequently. Nevertheless, they form an important motivational background for the construction of the statistical ensemble to be given in Section 2.2: in a technical sense, because both sets of results depend critically on a standard interpolation inequality, and in a physical sense, because the conditions for the existence of a solution to the initial value problem are essentially the same as those for the existence of the ensemble.

The problem we wish to study is given in (1.2)–(1.4). The two terms in the Hamiltonian (1.1) will be called, respectively, the kinetic and potential energy of the field, and we will assume that both these energies are finite for the initial field configuration $\phi_0(x)$.

Solutions of the initial value problem for (1.2) have been extensively studied in the case when the domain is the entire real line $\mathbb{R}$ (see in particular Refs. 8 and 11). Many of these results extend readily to the problem on an interval; the most relevant will be summarized in Theorem 2.1 below. In brief, solutions always exist for a time period which may in general depend on the initial condition. Under further conditions on the degree of nonlinearity and on the $L^2$ norm (1.5) of the initial data, solutions will exist for all time. As will be discussed in Section 2.2, these conditions also guarantee the existence of the statistical ensemble.

Before stating the theorem, we must establish some notation that will be used throughout this paper. We will in general be considering complex-valued functions defined either on the real line $\mathbb{R}$ or an interval $I \subset \mathbb{R}$ of length $L$, usually taken to be $[0, L]$. We use standard $L^q$ norms on $I$,

$$\|\phi\|_q \equiv \left[\int_{x \in I} |\phi(x)|^q \, dx\right]^{1/q}$$

and write $\|\phi\|_{0,q}$ for the corresponding norm on $\mathbb{R}$. Finally, we let $H^1 \equiv H^1(I)$ be the Sobolev space of functions with finite kinitic energy, that is, the space of functions $\phi$ on $I$ satisfying periodic boundary conditions and having finite norm

$$\|\phi\|_{H^1}^2 \equiv \|\phi'\|_2^2 + \|\phi\|_2^2$$

We can now state precisely the relevant results for the initial value problem. In doing so, it is convenient to reformulate (1.2) and (1.4) as an integral equation:

$$u(\cdot, t) = U(t)\phi_0 + i \int_0^t U(t-s)(|u(\cdot, s)|^{p-2} u(\cdot, s)) \, ds \tag{2.1}$$

where $U(t) = \exp(-itH_0)$ with $H_0 = -d^2/dx^2$. Our theorem will deal with functions in $H^1$, so that that boundary condition (1.3) will be satisfied automatically.

**Theorem 2.1.** Suppose that $p \geqslant 3$ and that $\phi_0(x) \in H^1$. Then:

1. Equation (2.1) has a solution $u(x, t)$, defined in some interval $|t| < T$, which lies in $H^1$ (as a function of $x$).
2. If $u(x, t)$ is a solution of (2.1) with $u(\cdot, t) \in H^1$, then $H(u(\cdot, t))$ and $\|u(\cdot, t)\|_2^2$ are constant in time.
3. There exists a constant $N_1$, with $0 < N_1 < \infty$, such that:
   (a) If $p < 6$ or if $p = 6$ and $\|\phi_0\|_2^2 < N_1$, we may take $T = \infty$.

(b) If $p \geqslant 6$, there exists an initial value $\phi_0$ and a finite $T$ such that $\lim_{t \to T} \|u_x(\cdot, t)\|_2^2 = \infty$. If $p > 6$, $\|\phi_0\|_2^2$ may be chosen arbitrarily small; if $p = 6$, $\|\phi_0\|_2^2$ may be chosen arbitrarily close to $N_1$.

4. The constant $N_1$, which occurs in the critical case $p = 6$, satisfies $N_0 \leqslant N_1 \leqslant 2N_0$, where $N_0$ is given by

$$N_0 = (3/C_6)^{1/2} = \|\hat{\phi}\|_2^2 \tag{2.2}$$

Here $C_6$ is the best possible constant in a certain interpolation inequality (4.1).

*Proof.* See Section 4.

We remark that in the corresponding theorem on $\mathbb{R}$, or on $I$ with Dirichlet of Neumann boundary conditions, one has $N_1 = N_0$; it seems likely that this is true for periodic boundary conditions also, although we do not have a proof. In the corresponding result for the existence of the statistical ensemble, Theorem 2.3, the critical value of the $L^2$ norm in the case $p = 6$ is also $N_0$.

## 2.2. The Statistical Ensemble

We now wish to define a statistical ensemble for the system considered in the previous section, with formal (unnormalized) Gibbs measure

$$\begin{aligned}\exp[-\beta H(\phi)] &\prod_{x \in [0,L]} d\phi(x) \\ &= \exp\left(\frac{\beta}{p}\int_0^L |\phi|^p \, dx\right)\left[\exp\left(-\frac{\beta}{2}\int_0^L |\phi'|^2 \, dx\right) \prod_{x \in [0,L]} d\phi(x)\right]\end{aligned} \tag{2.3}$$

where $d\phi$ denotes Lebesgue measure in the complex plane. Our goal in this section is to give (2.3) a precise meaning as a normalizable measure; we begin with a discussion of two preliminary difficulties.

First, the "Lebesgue measure" $\prod d\phi(x)$ is ill defined. Note, however, that the quantity in brackets on the right-hand side of (2.3) is a formal version of the well-known measure for a massless free field in the interval $[0, L]$; this is a Wiener measure supported on continuous but not differentiable fields. (More precisely, the measure is a product of measures for the real and imaginary parts of $\phi$; each of these is a superposition of Brownian bridge measures, since we impose periodic boundary conditions.)

Second, $H$ fails badly to be bounded below. The problem is acute for large fields: if we scale $\phi \to \alpha\phi$, the total energy $H(\phi)$ rapidly approaches negative infinity as $\alpha$ increases, and thus (2.3) cannot be normalizable.

Since the dynamics conserves the $L^2$ norm of $\phi$, however, it seems natural to consider either of two modified ensembles, in which we restrict to those $\phi$ that satisfy either

$$\|\phi\|_2^2 = N \tag{2.4}$$

or

$$\|\phi\|_2^2 \leqslant N \tag{2.5}$$

For technical simplicity, we wish to avoid the problem, inherent in (2.4), of restricting our Wiener measures to a sphere in $L^2$, and therefore adopt (2.5) in defining our ensemble.

*Remark 2.1.* More generally, one could consider measures of the form

$$e^{-\beta H(\phi)} G(\|\phi(x)\|_2^2) \prod_{x \in [0, L]} d\phi(x)$$

where $G$ falls off sufficiently fast at infinity to ensure normalizability; (2.5) is of this form with $G$ a characteristic function. The choice

$$G(X) = e^{-\beta\mu X}$$

yields the usual grand canonical ensemble, but a scaling argument as above shows that this will not be normalizable. For $p < 6$, a modified version, with

$$G(X) = e^{-\beta\mu X^q} \tag{2.6}$$

$q = (p+2)/(6-p)$ and sufficiently large $\mu$ avoids this difficulty, as discussed in Remark 4.1 below.

*Remark 2.2.* A lattice version of the ensemble (2.6) has been investigated numerically, in conjunction with J. O'Connell. In the spirit of the grand canonical ensemble, one would like to probe different values of $\|\phi\|_2^2$ by adjusting the "chemical potential" $\mu$. In the simulations, however, we find a sharp transition in typical behavior at a critical value $\mu_c$ corresponding roughly to that necessary for normalizability in the continuum. Above $\mu_c$, the measure is concentrated near uniform field configurations with rather low values of $\|\phi\|_2^2$; below $\mu_c$, on configurations with almost all the field at one lattice site and with large values of $\|\phi\|_2^2$. (These typical states are metastable over a range of values near $\mu_c$.) Thus, (2.6) does not appear suitable for investigating regions of moderate $\|\phi\|_2^2$. Numerical investigations of these regions using (2.4) are discussed in Section 3.3.

Thus, we will define our Gibbs measure by

$$\left[\exp\left(\frac{\beta}{p}\int_0^L |\phi|^p\,dx\right)\right]\chi_{\{\phi\mid\|\phi\|_2^2\leqslant N\}}\,d\mu_\beta(\phi) \tag{2.7}$$

where $\mu_\beta$ is the appropriate version of the Wiener measure. We will show that (2.7) is well defined and normalizable when $N$ and $p$ are such that the dynamical problem, with arbitrary initial data $\phi$ satisfying (2.5), has solitions (with finite kinetic energy) for all time, that is, when $p<6$ or when $p=6$ and $N<N_0$.

There is a simple formal argument that indicates that this should be true. Under these restrictions on $p$ and $N$, the interpolation inequalities of Section 4, (4.2) (for $p=6$) or (4.3) (for $p<6$), together with (2.5), immediately yield a lower bound for $H$ on the space $H^1$:

$$H(\phi)\geqslant \tfrac{1}{2}\varepsilon\int_0^L |\phi'|^2\,dx - K_\varepsilon$$

Formally, this implies the normalizability of (2.7) via

$$\exp[-\beta H(\phi)]\prod_{x\in[0,L]} d\phi(x)\leqslant \tilde{K}\left[\exp\left(-\frac{\varepsilon\beta}{2}\int_0^L|\phi'|^2\,dx\right)\right]\prod_{x\in[0,L]} d\phi(x) \tag{2.8}$$

since the right-hand side of (2.8) is again a well-defined Wiener measure. The goal is to transform this formal argument (which deals with smooth fields) into a rigorous argument for Wiener paths; the basic technique is always to integrate over a family of Wiener paths that are close to some smooth field to which the interpolation inequalities can be applied.

We can now state our main result. Let $C$ denote the set of continuous, complex-valued functions $\phi$ on $[0, L]$ that satisfy $\phi(0)=\phi(L)$, and let $\mu_\beta$ be the Wiener measure on $C$ corresponding to the bracketed terms in (2.3). (The measure $\mu_\beta$ is defined formally in Section 4.) Then we have the following result:

**Theorem 2.2.** The function

$$F(\phi)\equiv\left[\exp\left(\frac{\beta}{p}\|\phi\|_p^p\right)\right]\chi_{\{\phi\mid\|\phi\|_2^2\leqslant N\}}$$

(a) Is in $L^1(C;\mu_\beta)$ if $p<6$ and $N<N_0$.

(b) Is not in $L^1(C;\mu_\beta)$ if $p>6$ or if $p=6$ and $N>N_0$.

*Proof.* See Section 4.

Thus, when the parameters satisfy the hypotheses of case (a), we can define the partition function

$$Z(\beta, N) = \int_C F(\phi)\, d\mu_\beta(\phi) \tag{2.9}$$

and a probability measure $\nu_{\beta,N}$:

$$d\nu_{\beta,N}(\phi) = Z(\beta, N)^{-1} F(\phi)\, d\mu_\beta(\phi) \tag{2.10}$$

## 3. NATURE OF STATISTICAL STATES

Having constructed statistical states for the NLSE, we now discuss their significance for the study of plasmas and describe some of their properties. We begin by recalling a more sophisticated model for a plasma and its relation to the NLSE.

### 3.1. The Zakharov Model

The collective response of a nearly collisionless plasma when the electron temperature is much greater than the ion temperature is dominated by electron plasma waves (Langmuir waves) and ion acoustic waves. Under appropriate conditions (energy density of the waves small compared to particle thermal energy densities, and characteristic length scales large compared to a Debye length), Zakharov's model is a useful description of the nonlinear coupling of these collective modes. In one dimension, neglecting dissipation terms and using appropriate units, we can write the Zakharov equations (ZE) in the form

$$iu_t = -u_{xx} + nu \tag{3.1}$$

$$n_{tt} - c^2 n_{xx} = c^2(|u|^2)_{xx} \tag{3.2}$$

with $u(x, t)$ the electrostatic envelope field, $n(x, t)$ the ion density fluctuation field, and $c$ the speed of sound in the plasma. In discussing these equations, we will again consider the case of periodic boundary conditions on the interval $[0, L]$. We note that solutions of the ZE preserve the $L^2$ norm of $u$ and the mean ion density field $\int_0^L n\, dx$.

Equation (3.2) can also be written in terms of an auxiliary field $V(x, t)$ [a hydrodynamic flux for the transport of ions, constrained by $\int_0^L V(x, t)\, dx = 0$] as

$$n_t = -c^2 V_x \tag{3.3a}$$

and

$$V_t = -n_x - |u|_x^2 \tag{3.3b}$$

In the form (3.1), (3.3) these are Hamilton's equations for the Hamiltonian

$$H_Z = \frac{1}{2} \int_0^L \left[ |u_x|^2 + \frac{1}{2}(n^2 + c^2 V^2) + n|u|^2 \right] dx \tag{3.4}$$

with (Re $u$, Im $u$) and ($\tilde{n}$, $\tilde{V}$) as pairs of conjugate variables, where we write $\tilde{n} = 2^{-1/2} n$ and $\tilde{V} = 2^{-1/2} \int^x V\,dx$.

The nonlinear Schrödinger equation (with $p = 4$) is closely related to the Zakharov equations in several ways. First, in the adiabatic limit $c \to \infty$, (3.2) reduces to $n = -|u|^2 + K$ and (3.1) becomes equivalent to the NLSE (for rigorous results on this limit see Ref. 12). Second, any solution of the NLSE of the form $u(x, t) = e^{i\omega t} w(x)$ yields a solution of the ZE if $n$ is taken to be $n(x) = -|w(x)|^2$; thus, the ZE and the NLSE have corresponding plane wave and soliton solutions. Finally, consider a formal Gibbs measure for the Hamiltonian (3.4):

$$e^{-\beta H_Z} \chi_{\{\int |u|^2 dx \leqslant N\}} \prod_x \left[ d^2 u(x)\, d\tilde{n}(x)\, d\tilde{V}(x) \right] \tag{3.5}$$

Because the $n$ and $V$ fields appear quadratically in $H_Z$, we can integrate over $\tilde{n}$ and $\tilde{V}$ explicitly to compute the marginal distribution of the $u$ field; this yields precisely the formal NLSE Gibbs measure (2.3). In this light the results of Section 2.2 may be regarded as a construction of the probability distribution for the electrostatic envelope field and ion density field in the Zakharov model. On the other hand, while the NLSE is completely integrable in the case $p = 4$, the Zakharov equations (so far as we know) are not; as a result, much more is known about solutions of the former.

There have been other studies of the statistical mechanics of the NLSE and classical nonlinear field theories. In Ref. 13 (and references therein) attention is restricted to Hamiltonians that are bounded below. In Refs. 14 and 15 an attempt is made to study Langmuir wave turbulence as a critical phenomenon. Only Ref. 14 is directly related to our work in that the starting point is the grand canonical ensemble. Here an ultraviolet cutoff is assumed, but infrared questions for the existence of the measure (the system is treated in $\mathbb{R}^d$) are not discussed. No distinction is made between different dimensions, and the fact that the exponent $p = 4$ is respectively subcritical, critical, and supercritical in dimensions one through three does not influence the analysis. The significance of the cutoff measure in the supercritical case is not made clear.

In Ref. 16 the emphasis is different, in that the relationship between a statistical dynamical theory (the DIA of R. H. Kraichnan) and equilibrium statistical mechanics is explored. Also, attention was restricted to a Fourier truncation of the NLSE and only negative temperatures (so there is no tendency for soliton formation) are considered.

As regards other field theories, we have little to say, except to point out that there have been many studies of the equilibrium statistical mechanics of the Fourier-truncated Euler and incompressible ideal MHD equations (see Ref. 17). In these models there is the unusual circumstance that while the dynamics is nonlinear, the equilibrium measure can be Gaussian.

### 3.2. Physical Interpretation of the Ensembles $\nu_{\beta,N}$

We now discuss the extent to which the ensembles constructed in Section 2.2 are applicable to the physics of real systems and what light they can shed on that physics. Of course, the one-dimensional nature of our model implies that any application to two- or three-dimensional systems can be at best qualitative. On the positive side we would like to stress again that the conditions (on $p$ and $N$) for the existence of the measures $\nu_{\beta,N}$ are essentially the same as those for the existence of a global time evolution for the NLSE; by itself this suggests that the standard picture of a measure-preserving dynamics makes sense under these conditions. There are several difficulties with this simple picture, however.

We first remark that, with probability one, field configurations in our ensembles have infinite kinetic energy $\frac{1}{2}\int |u_x|^2\, dx$. This is closely connected with a problem in applying the standard picture of the relation between the ensemble and the dynamics: we have in fact no dynamics defined on these singular configurations. (The problem of defining such a dynamics is discussed further in Section 5.) This infinite energy also represents a problem for relating physical quantities calculated in our ensemble to those that might be measured in the real system; certainly we do not expect agreement for the kinetic energy or other quantities that depend on smoothness of the configurations. We argue, however, that microscopic processes in a physical system would provide an ultraviolet cutoff if included in a more realistic model and that, lacking this refined version, we expect that quantities with finite expectation in our model would accurately represent cutoff-independent quantities in a more realistic calculation. (In a plasma these processes can be dissipative, a point discussed briefly in Section 5.)

A second interpretational problem is that the usual justification for the introduction of equilibrium ensembles, ergodicity of the dynamics, certainly does not apply to the NLSE in the integrable case $p=4$. This comment does not hold for the NLSE with other values of $p$ or for the ZE. On the other hand, one of us (H.A.R., in conjunction with G. Doolen) has studied numerically the time evolution of solutions of the one-dimensional ZE, and the results suggest a lack of ergodicity even here: equipartition of kinetic energy into sound modes of different wavelengths seems to fail. In Section 5 we discuss briefly a Langevin model which has formally the same invariant measure and may be ergodic, but also point out that this is not a realisitic model of dissipation in an excited plasma.

A third and more serious difficulty is that the conditions on $p$ and $N$ under which our ensembles exist do not correspond, even qualitatively, to those of the most interesting physical systems. In the case of a plasma, for example, the space dimension $d$ is 3 and the exponent $p = 4$ is greater than the corresponding critical exponent $p_c = 10/3$. Thus, there are classical initial conditions for which solutions do not exist for all time; the typical behavior is a self-focusing instability or collapse. Dissipative processes in a true plasma will eventually halt this collapse of the Langmuir wave packet, and the Zakharov model is expected to be relevant only during time intervals when these processes are not acting. To gain insight into the collapse in our statistical model, we would have to go beyond what we have done here. Some possible approaches are discussed in Section 5. For dimension two, the exponent $p = 4$ for the laser propagation problem is the critical value, and the existence theory for smooth solutions of the NLSE, together with our results for $d = 1$, $p = 6$, would suggest the existence of an $N_0$ such that an ensemble exists for $N < N_0$.

Accepting for the moment some applicability of our ensembles to physical systems, or at the least to one-dimensional versions of them, we ask what information they can provide. In the numerical results mentioned above there appear to be qualtiatively different regimes of behavior for solutions of the ZE: at low energy the solution $u(x, t)$ is typically (i.e., at typical times) fairly uniform as a function of $x$, while at high energy the typical solution is a single, stable soliton. At intermediate conditions there is a regime in which several solitons may coexist, and solitons continually coalesce and are created. The question of whether the existence of these different regimes might be related to a phase transition in the usual statistical mechanical sense has been a prime motivating factor in this paper.

### 3.3. Phase Transitions

We now turn to a discussion of the properties of the measures $\nu_{\beta,N}$ defined in (2.10). In particular, we wish to address the question of whether such states exhibit any phase transition as $\beta$ and $N$ vary, corresponding to the qualitative differences noted in numerical studies. Such a transition could reveal itself in the context of a rigorous treatment of the ensemble through a lack of analyticity of the partition function or other physical quantities along certain curves in the $\beta$–$N$ plane.

We have done Monte Carlo studies on a lattice version of the ensemble studied in this paper—more specifically, of the ensemble with restriction (2.4), which we do not expect to differ significantly from the actually constructed ensemble with restriction (2.5). Here we see roughly two regimes in the $\beta$–$N$ plane: typical configurations show no spatial structure for low $N$ or for low $\beta$, and soliton-like structures at higher $N$ and $\beta$. One parameter used to distinguish these regimes is

$$S = \left\langle \int_0^L |\phi|^4 \, dx \right\rangle \bigg/ \left\langle \int_0^L |\phi|^2 \, dx \right\rangle^2$$

where now $\langle \cdot \rangle$ refers to a time average in the Monte Carlo stochastic process. When $\beta$ is varied at large $N$, $S$ shows a relatively sharp increase

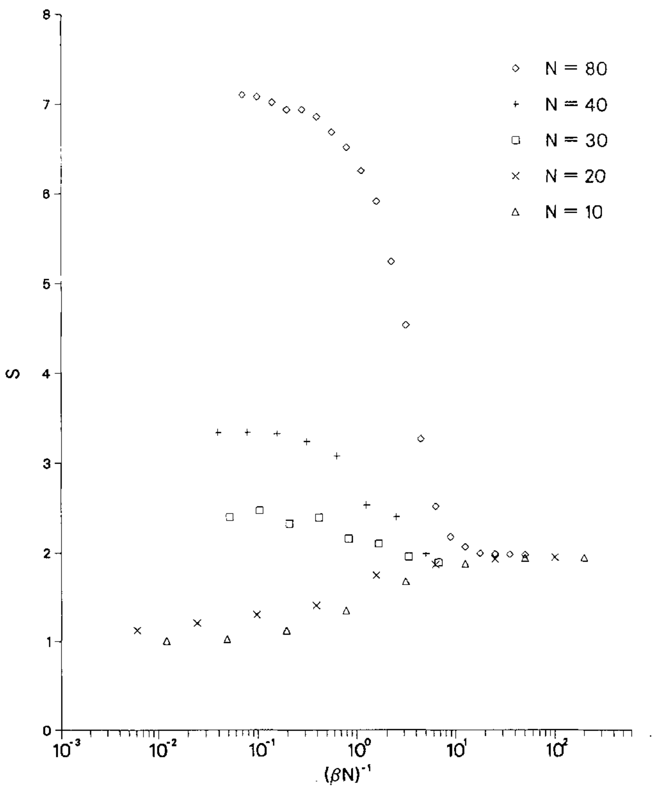


Fig. 1. Values of $S$ for $0.0005 \leqslant \beta \leqslant 4.096$ and $10 \leqslant N \leqslant 80$.

from its high-temperature value (2 in the continuum, slightly less on the lattice) to the value $S_N$ corresponding to the single-soliton solution of the time-independent NLSE with $L^2$ norm $N$. Figure 1 is a plot of values of $S$ versus a scaled temperature $(\beta N)^{-1}$, with $p=4$, for several values of $N$; this scaling tends to bring the transitions into coincidence and is the one suggested by Proposition 3.1 below. We also seek to determine the general character of a typical field configuration by computing the quantity

$$F(x) = \langle |\tilde{\phi}(x)|^2 \rangle$$

where $\tilde{\phi}$ is the translate of $\phi$ for which the maximum magnitude occurs at a fixed point, say $x_{\max}$. At high temperature, fluctuations produce an artificial spike in $F(x)$ at $x_{\max}$; to eliminate this effect, we look at the ratio $F(x)/F_0(x)$, where the subscript 0 identifies quantities computed in an ensemble with no potential energy term in the Hamiltonian. Figure 2 is a

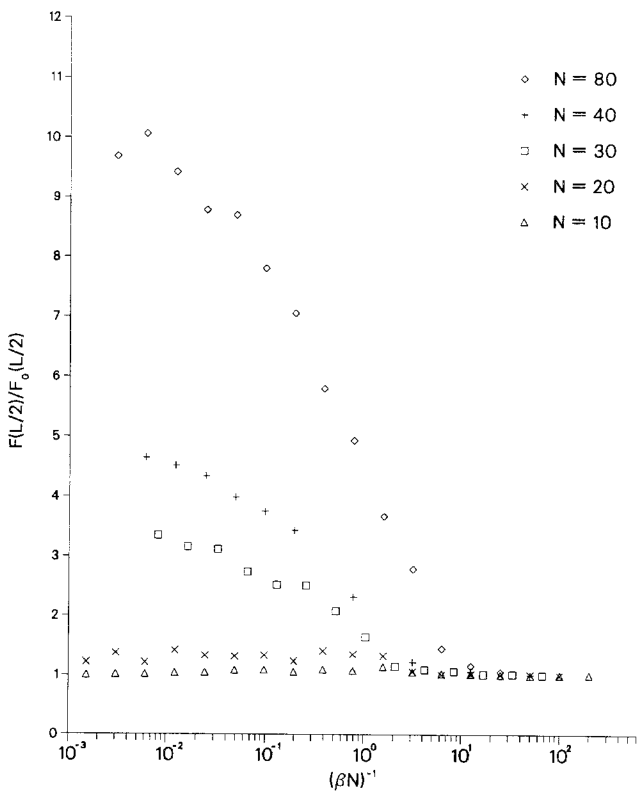


Fig. 2. Values of $F(L/2)/F_0(L/2)$ for $0.005 \leqslant \beta \leqslant 4.096$ and $10 \leqslant N \leqslant 80$.

plot of the quantity $F(x_{max})/F_0(x_{max})$, which represents a (normalized) height of the typical soliton, using the same horizontal scale as Fig. 1. Figures 3 and 4 present, for $N=80$ and $N=30$ respectively, the graphs of $F(x)/F_0(x)$ for 14 different values of $\beta$, ranging from 0.0005 to 4.096 and logarithmically equally spaced. For reference, we give in Fig. 5 the graphs of $F_0(x)$ for $N=80$ at the same values of $\beta$; since the model with no potential energy is invariant under the scaling $\phi \to \rho\phi$, $N \to \rho^2 N$, $\beta \to \rho^{-2}\beta$, the graphs for other values of $N$ are similar. (All data have been taken with $p=4$ and $L=1.0$ and with 40 lattice sites. In Figs. 3–5 we take $I$ to be the interval $[0, L]$ and choose $x_{max}=L/2$.) A phase transition does not appear to be inconsistent with these data, but the numerical studies do not settle the point.

Given the nature of the observed typical field configurations at low temperatures or high values of $N$, one might suspect the existence of measures not satisfying translation invariance, corresponding to concentration of the measure on fields near a particular soliton-like structure. The

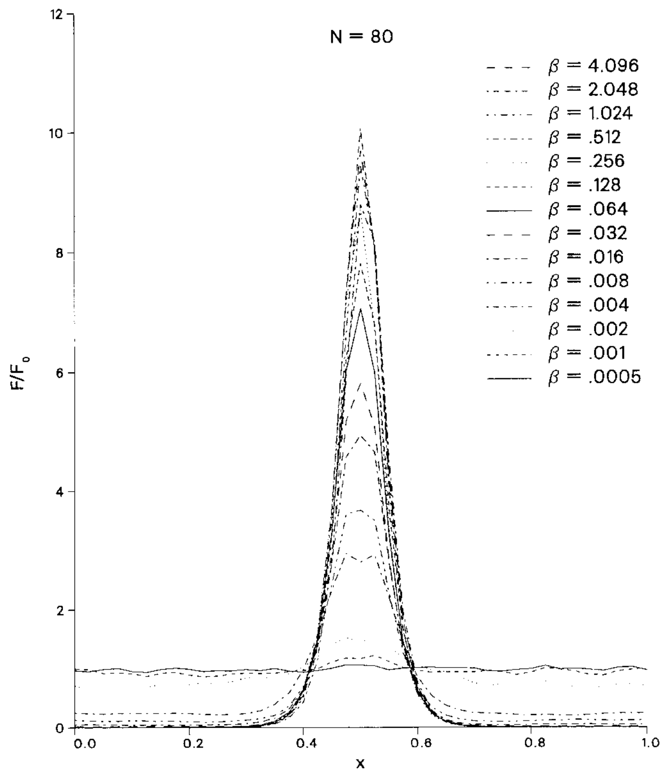


Fig. 3. Normalized field profiles at $N=80$ for $0.0005 \leqslant \beta \leqslant 4.096$.

translation-invariant measure $\nu_{\beta,N}$ constructed in Section 2.2 could then be a convex superposition of non-translation-invariant measures. We see no mechanism for the generation of such measures, however. One might try to localize typical solitions by introducing an external potential into the Hamiltonian (1.1) via a term $\gamma \int_0^L V(x)\, \phi(x)\, dx$, yielding a modified measure $\nu_{\beta,N,\gamma}$, and then consider the limit $\gamma \to 0$. The measure $\nu_{\beta,N,\gamma}$ will be absolutely continuous with respect to $\nu_{\beta,N}$, however, and we will have

$$\lim_{\gamma \to 0} \nu_{\beta,N,\gamma} = \lim_{\gamma \to 0} \left\{ \exp\left[ -\beta\gamma \int_0^L V(x)\, \phi(x)\, dx \right] \right\} \nu_{\beta,N} = \nu_{\beta,N}$$

(weakly) by the Lebesgue dominated convergence theorem.

We finally discuss the analyticity of the measure in the parameters $\beta$ and $N$. Although we are unable to settle completely the question of analyticity, the following result limits the possible curves of critical points

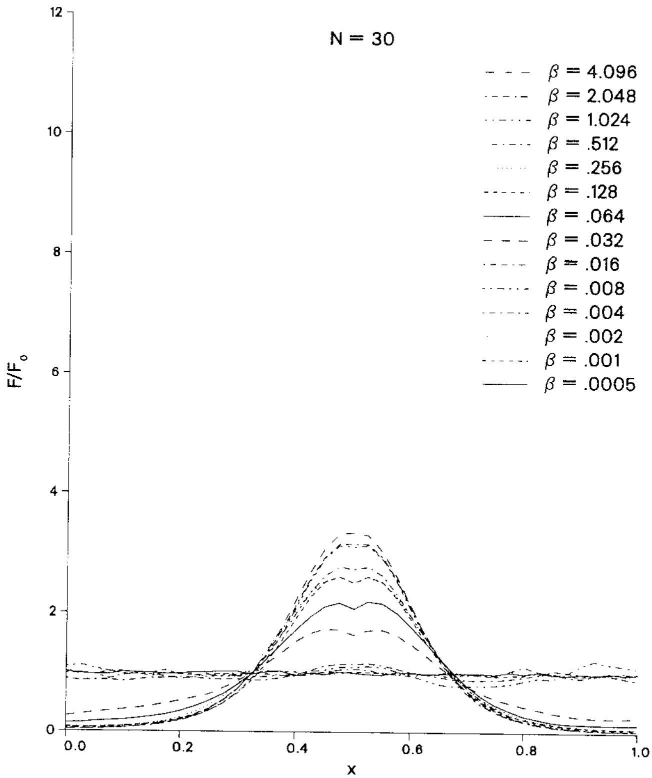


Fig. 4. Normalized field profiles at $N = 30$ for $0.0005 \leqslant \beta \leqslant 4.096$.

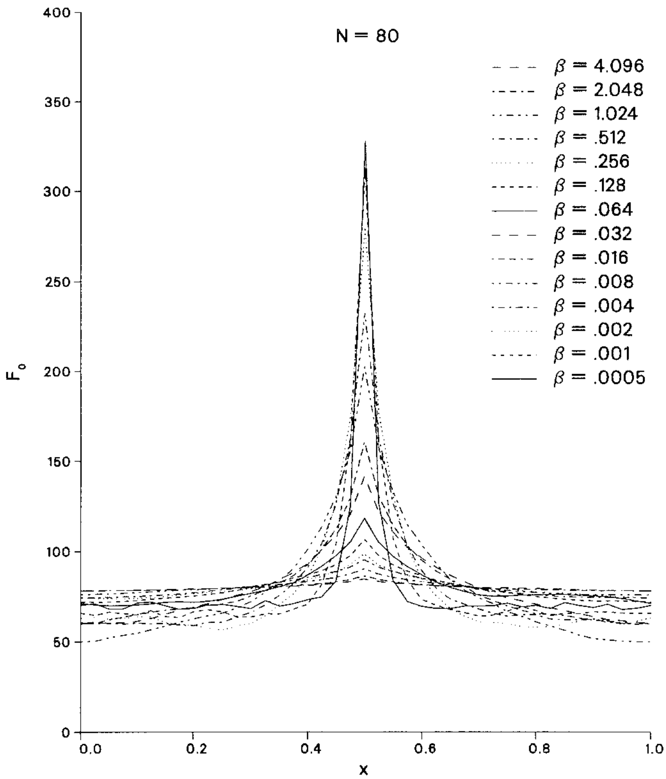


Fig. 5. Field profiles for Hamiltonian with no potential energy at $N=80$ for $0.0005 \leqslant \beta \leqslant 4.096$.

in the $\beta$–$N$ plane to those of the form $\beta N = K$, with $K$ a constant. (This is the scaling used in Figs. 1 and 2.) We will treat only the partition function $Z(\beta, N)$ defined in (2.9), although our conclusions would apply equally to the expectations of physical observables.

**Proposition 3.1.** Suppose that $p \leqslant 6$ and that $N > 0$. Then $Z(\alpha, \alpha^{-1}N)$ is (real) analytic in $\alpha$ for $\alpha > 0$ and, if $p = 6$, $\alpha^{-1}N < N_0$.

*Proof.* See Section 4.

## 4. PROOFS

In this section we give proofs of the theorems stated in Sections 2 and 3.

All the results depend criticaly on a standard interpolation inequality[18] relating various function norms. On $\mathbb{R}$ this has the form

$$\|\phi\|_{0,p}^{p} \leqslant C_p \|\phi'\|_{0,2}^{(p-2)/2} \|\phi\|_{0,2}^{(p+2)/2} \tag{4.1}$$

where $2 < p < \infty$. Weinstein[11] has shown that the best possible constant in (4.1) is

$$C_p = \frac{p}{2} \|\hat{\phi}\|_{0,2}^{2-p}$$

where $\hat{\phi}$ is a ground state for the equation

$$(p-2)\phi'' - (p+2)\phi + \phi^{p-1} = 0$$

that is, a nonzero solution with minimal $L^2$ norm. (4.1) implies a similar result on the interval $I$:

**Lemma 4.1.** (a) If $2 < p \leqslant 6$, then for any $\varepsilon > 0$ there is a constant $K_\varepsilon > 0$ such that, for $\phi \in H^1$,

$$\|\phi\|_p^p \leqslant (C_p + \varepsilon) \|\phi'\|_2^{(p-2)/2} \|\phi\|_2^{(p+2)/2} + K_\varepsilon \|\phi\|_2^p \tag{4.2}$$

(b) If $2 < p < 6$, then for any $\varepsilon > 0$ there is a constant $K'_\varepsilon > 0$ such that, for $\phi \in H^1$,

$$\|\phi\|_p^p \leqslant \varepsilon \|\phi'\|_2^2 \|\phi\|_2^{p-2} + K'_\varepsilon \|\phi\|_2^p \tag{4.3}$$

*Proof.* For $\phi \in H^1(I)$ we may be periodicity assume that $|\phi(0)|$ $[= |\phi(L)|] \leqslant L^{-1/2} \|\phi(x)\|_2$. Choose $\delta > 0$ and let $\hat{\phi}$ be the continuous functions on $\mathbb{R}$ which agrees with $\phi$ on $[0, L]$, vanishes on $(-\infty, -\delta]$ and $[L+\delta, \infty)$, and is affine on $[-\delta, 0]$ and $[L, L+\delta]$. Since

$$\|\phi\|_p^p \leqslant \|\hat{\phi}\|_{0,p}^p$$

$$\|\hat{\phi}\|_{0,2}^2 = \|\phi\|_2^2 + \frac{2\delta}{3} |\phi(0)|^2 \leqslant \left(1 + \frac{2\delta}{3L}\right) \|\phi\|_2^2$$

and

$$\|\hat{\phi}'\|_{0,2}^2 = \|\phi'\|_2^2 + \frac{2}{\delta} |\phi(0)|^2 \leqslant \|\phi'\|_2^2 + \frac{2}{\delta L} \|\phi\|_2^2$$

(4.2) follows from (4.1) and the inequality $(a+b)^q \leqslant a^q + b^q$, valid for $a, b \geqslant 0$ and $q = (p-2)/4$, by an appropriate choice of $\delta$. Similarly, (4.3) follows from the observation that for any $\eta > 0$ there is a $\tilde{K}_\eta > 0$ such that $(a+b)^q \leqslant \eta a b^{q-1} + \tilde{K}_\eta b^q$, and a choice of $\eta$. ∎

*Remark 4.1.* The constant $\tilde{K}_\eta$ varies (for small $\eta$) as $\eta^{(p-2)/(p-6)}$, so that, by taking $\eta \sim \|\phi\|_2^{-(p-2)}$, we may derive the inequality

$$\frac{1}{p}\|\phi\|_p^p \leqslant \tfrac{1}{2}\|\phi'\|_2^2 + C_1\|\phi\|_2^{(2p+4)/(6-p)} + C_2\|\phi\|_2^p$$

valid for $p < 6$. This is the point of departure for the proof of normalizability of the modified grand canonical ensemble discussed in Remark 2.1.

We now sketch the proof of Theorem 2.1, emphasizing only those elements that differ from corresponding proofs on the entire line $\mathbb{R}$.

*Proof of Theorem 2.1.* Parts 1 and 2. The fixed-point argument used to prove existence in Ref. 8, and the regularitzation argument used to prove the conservation laws, go through directly in the space $H^1 = H^1(I)$.

Part 3(a), $N_1 \geqslant N_0$. The elegant discussion of Ref. 11 applies here, with $N_0$ given by (2.2) and with the use of (4.2) or (4.3) instead of (4.1). Since the constant $N_0$ will reoccur in our discussion of the statistical ensemble, and since the discussion is based on the important interpolation inequality, it seems worthwhile to repeat the argument here. We discuss only the critical case $p = 6$. The fixed-point argument of Ref. 8 will in fact prove existence globally in time if supplemented by an *a priori* bound on the $H^1$ norm of the solution. But from (4.2) and the conservation laws,

$$\begin{aligned} H(\phi_0) &= H(u(\cdot, t)) \\ &= \tfrac{1}{2}\|u_x(\cdot, t)\|_2^2 - \tfrac{1}{6}\|u(\cdot, t)\|_6^6 \\ &\geqslant \tfrac{1}{2}[1 - (N_0^{-2} + \varepsilon)\|\phi_0\|_2^4]\,\|u_x(\cdot, t)\|_2^2 - K_{3\varepsilon}\|\phi_0\|_2^6 \end{aligned}$$

If $\|\phi\|_2^2 < N_0$, we may choose $\varepsilon$ so small that $(N_0^{-2} + \varepsilon)\|\phi_0\|_2^4 \equiv \kappa < 1$, yielding the bound

$$\|u_x(\cdot, t)\|_2^2 \leqslant (1-\kappa)^{-1}[H(\phi_0) + K_{3\varepsilon}\|\phi_0\|_2^6]$$

Part 3(b), $N_1 \leqslant 2N_0$. The argument of Glassey[19] shows that Part 3(b) holds, with $N_1 = N_0$, for the NLSE on $\mathbb{R}$, and has been extended by Kavian[20] to certain bounded domains. In particular, Kavian shows that 3(b) holds on the interval $I$ with Dirichlet, Neumann, or periodic boundary conditions; his example gives $N_1 \leqslant N_0$ in the first two cases, but $N_1 \leqslant 2N_0$ in the periodic case of interest to us. It seems likely that $N_1 = N_0$ here also, but we do not have a proof. ∎

We next turn to the discussion of Theorem 2.2, and begin by establishing some notation and defining the Wiener measures we will use. Recall that $C$ denotes the set of continuous, complex-valued functions $\phi$ on $[0, L]$ that satisfy $\phi(0) = \phi(L)$; for any $\psi \in C$ let $B_\varepsilon(\psi)$ denote the ball $\{\phi \in C \mid \|\phi - \psi\|_\infty < \varepsilon\}$. We also denote by $C^{a,b}$ the set of continuous functions on $[0, L]$ that satisfy $\phi(0) = a$, $\phi(L) = b$.

Now define the measure $\mu_\beta^{a,b;L}$ on $C^{a,b}$ as the natural restriction of the Wiener measure with diffusion constant $\beta^{-1}$, that is, as the measure with marginal distributions (see Ref. 21)

$$\operatorname{Prob}(\{\phi \mid \phi(x_i) \in A_i,\ 1 \leqslant i \leqslant n\})$$
$$= p_\beta^L(\alpha, b)^{-1} \int_{y_i \in A_i} \prod_{i=1}^{n+1} p_\beta^{x_i - x_{i-1}}(y_i, y_{i-1})\, dy_1 \cdots dy_n$$

Here $0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = L$, $A_1, \ldots, A_n$ are Borel subsets of $\mathbb{C}$, $dy$ is Lebesgue measure on $\mathbb{C}$, $y_0 = a$, $y_{n+1} = b$, and for $y, y' \in \mathbb{C}$,

$$p_\beta^x(y, y') \equiv (\beta/2\pi x) \exp(-\beta\,|y - y'|^2/2x)$$

Finally, define the measure $\mu_\beta$ on $C = \bigcup_{\alpha \in \mathbb{C}} C^{a,a} \simeq \mathbb{C} \times C^{0,0}$ as $da \times d\mu_\beta^{0,0;L}$, with $da$ Lebesgue measure on $\mathbb{C}$; equivalently, $\mu_\beta$ has marginal distributions

$$\operatorname{Prob}(\{\phi \mid \phi(x_i) \in A_i,\ 1 \leqslant i \leqslant n+1\})$$
$$= p_\beta^L(0, 0)^{-1} \int_{y_i \in A_i} \prod_{i=1}^{n+1} p_\beta^{x_i - x_{i-1}}(y_i, y_{i-1})|_{y_0 = y_{n+1}}\, dy_1 \cdots dy_{n+1} \qquad (4.4)$$

We will occasionally need the corresponding spaces and measures defined for real-valued paths; these will be denoted $C_{\mathbb{R}}^{a,b}$, $\mu_{\mathbb{R},\beta}^{a,b;L}$, etc. In particular, for $y, y' \in \mathbb{R}$,

$$p_{\mathbb{R},\beta}^x(y, y') \equiv (\beta/2\pi x)^{1/2} \exp(-\beta\,|y - y'|^2/2x)$$

Before proving the theorem we give a minor technical result.

**Lemma 4.2.** For any $\eta > 0$ there is an $\varepsilon > 0$ such that, if $\phi_1, \phi_2 \in C$ satisfy $\|\phi_1 - \phi_2\|_\infty < \varepsilon$, then

$$|\,\|\phi_1\|_q^q - \|\phi_2\|_q^q| < \eta(\|\phi_1\|_q^q + 1)$$

for $q = 2, p$.

*Proof.* We omit the straightforward proof.

As a preliminary to the proof of the first part of Theorem 2.2, we must define some quantities associated with ordinary (real-valued) Brownian motion in one dimension with diffusion constant $\beta^{-1}$, and with paths $\omega(x)$ satisfying $\omega(0)=0$. Fix $\theta>0$ and let $T_\beta(\omega)$ and $S_\beta(\omega)$ be the times the path first hits $y=\theta$ and $|y|=\theta$, respectively. Elementary calculations (see, e.g., Ref. 22) show that the random variables $T_\beta$ and $S_\beta$ have densities

$$f_\beta(x)=\theta(\beta/2\pi x^3)^{1/2}\exp(-\beta\theta^2/2x) \tag{4.5}$$

and

$$g_\beta(x)=2\theta\,(\beta/2\pi x^3)^{1/2}\sum_{k=0}^{\infty}(-1)^k\exp\{-\beta[(2k+1)\theta]^2/2x\}$$

respectively, so that

$$2c(\beta)^{-1}f_\beta(x)\leqslant g_\beta(x)\leqslant 2f_\beta(x) \tag{4.6}$$

for $c(\beta)=1-\exp(-8\beta\theta^2/2L)$. Finally, for $|y|<\theta$ and $|y'|<\theta$ we let $q_\beta^x(y,y')$ be the transition density from $y$ to $y'$ in time $x$ for Brownian motion with absorbing barriers at $\pm\theta$; that is, $q$ satisfies

$$\begin{aligned} 2\beta q_{\beta,x}&=q_{\beta,y'y'}\\ q_\beta^0(y,y')&=\delta(y-y')\\ q_\beta^x(y,\theta)&=q_\beta^x(y,-\theta)=0 \end{aligned}$$

Now suppose that $\omega(x)$ is a real-valued continuous path satisfying $\omega(0)=a$, $\omega(L)=a+l\theta$ for $l\in\mathbb{Z}$ (i.e., $\omega\in C_{\mathbb{R}}^{a,a+l\theta}$); we introduce variables that keep track of the way in which $\omega$ attains values of the form $a+j\theta$, $j\in\mathbb{Z}$. Specifically, if $D_{\alpha,\theta}$ is the set of numbers of this form, we define $U_0(\omega)$, $U_1(\omega)$,..., $U_{M(\omega)+1}(\omega)$ inductively by

$$U_0(\omega)=0$$

and, if $X_k\equiv\{x>U_k\,|\,\omega(x)\in D_{\alpha,\theta},\ \omega(x)\neq\omega(U_k)\}$ is nonempty,

$$U_{k+1}(\omega)=\inf X_k$$

We let $M(\omega)$ be the smallest value of $k$ for which $X_k$ is empty, and set $U_{M(\omega)+1}(\omega)=L$. Finally, we define $\Gamma_1(\omega)$,..., $\Gamma_{M(\omega)}(\omega)\in\{-1,1\}$ by the condition

$$\omega(U_k)=\omega(U_{k-1})+\Gamma_k\theta,\qquad k=1,\dots,M$$

As functions on the measure space $(C_{\mathbb{R}}^{a,\,a+l\theta}, \mu_{\mathbb{R},\beta}^{a,a+l\theta})$, $M$, $\Gamma$, and $U$ are random variables with joint distribution given by

$$\begin{aligned}\mu_{\mathbb{R},\beta}^{a,a+l\theta;L}(\{M=m,\ \Gamma_j=\gamma_j,\ j=0,...,m, \text{ and}\\ u_j \leqslant U_j \leqslant u_j+du_j,\ j=1,...,m\})\\ \equiv \delta_{\sum \gamma_i, l}\, p_{\mathbb{R},\beta}^{L}(a, a+l\theta)^{-1}\, h_\beta(m, u)\, du_1 \cdots du_m \end{aligned}\tag{4.7}$$

where

$$h_\beta(m, u) = \prod_{j=1}^{m} [\tfrac{1}{2} g_\beta(u_j - u_{j-1})]\, q_\beta^{(L-u_m)}(0, 0)\, du_1 \cdots du_m$$

Finally, we let $\xi(\omega) \in H_1([0, L])$ be the piecewise linear interpolant that agrees with $\omega$ at the points $x = U_j(\omega)$; note that $\|\omega - \xi(\omega)\|_\infty < 2\theta$.

*Proof of Theorem 2.2(a).* For a path $\phi \in C$ we let $A(\phi) \equiv \phi(0)$ and perform the construction of the preceding paragraph separately for $\phi^{\mathrm{R}} = \mathrm{Re}\,\phi$ and $\phi^{\mathrm{I}} = \mathrm{Im}\,\phi$, with $l=0$ in each case, producing random variables $M^{\mathrm{R}} \equiv M(\phi^{\mathrm{R}})$, etc. Let $\psi(\phi) = \xi(\phi^{\mathrm{R}}) + i\xi(\phi^{\mathrm{I}})$; $\psi$ is a piecewise linear interpolant of $\phi$ with $\|\psi(\phi) - \phi\|_\infty < 2\sqrt{2}\,\theta$. Lemma 4.3 guarantees that we can, first, choose $\theta$ small enough that for all $\phi$ with $\|\phi\|_2^2 \leqslant N$,

$$\|\psi(\phi)\|_2^2 < \begin{cases} N_1, & \text{if } p=6 \\ N+1, & \text{if } p<6 \end{cases}$$

where $N < N_1 < N_0$, and second, by another application of Lemma 4.3 and use of Eq. (2.2) and the inequality (4.2) for $p=6$, or (4.3) for $p<6$, choose $\eta$ and then $\theta$ so that

$$\frac{1}{p}\|\phi\|_p^p < \frac{1}{p}[(1+\eta)\,\|\psi(\phi)\|_p^p + \eta] < \frac{1}{2}(1-\lambda)\,\|\psi(\phi)'\|_2^2 + K$$

for constants $\lambda$, $K>0$ which are independent of $\phi$. Finally, observe that if $\|\phi\|_2^2 \leqslant N$, then $|\phi(x)|^2 \leqslant N/L$ for some $x \in [0, L]$, and since for all $x \in 0, L]$,

$$|\phi(0) - \phi(x)| \leqslant \theta[(M^{\mathrm{R}}+1)^2 + (M^{\mathrm{I}}+1)^2]^{1/2}$$

$M^{\mathrm{R}}$ and $M^{\mathrm{I}}$ must satisfy

$$(M^{\mathrm{R}})^2 + (M^{\mathrm{I}})^2 \geqslant c_1 A^2 - c_2 \tag{4.8}$$

for appropriate constants $c_1$ and $c_2$.

Because $\psi(\phi)$ depends on $\phi$ only through the random variables $A$, $M$, $U$, and $\Gamma$, we have

$$\begin{aligned}
&\int_C F(\phi)\, d\mu_\beta(\phi)\\
&\quad \leqslant \int_{\|\phi\|_2^2 \leqslant N} \exp[\beta(1-\lambda)\, \|\psi(\phi)'\|_2^2/2 + K]\, d\mu_\beta\\
&\quad \leqslant (\exp K) \int_{\mathbb{C}} da \sum_{m^{\mathrm{R}}, m^{\mathrm{I}}} \sum_{\gamma^{\mathrm{R}}, \gamma^{\mathrm{I}}} \int du^{\mathrm{R}}\, du^{\mathrm{I}}\, \delta_{\sum \gamma_i^{\mathrm{R}}, 0}\, \delta_{\sum \gamma_i^{\mathrm{I}}, 0}\, p_\beta^L(a, a)^{-1}\\
&\qquad \times \{\exp[\beta(1-\lambda)\, \|\psi(\phi)'\|_2^2/2]\}\, h_\beta(m^{\mathrm{R}}, u^{\mathrm{R}})\, h_\beta(m^{\mathrm{I}}, u^{\mathrm{I}}) \qquad (4.9)
\end{aligned}$$

Write

$$\|\psi(\phi)'\|_2^2 = \|\xi(\phi^{\mathrm{R}})'\|_2^2 + \|\xi(\phi^{\mathrm{I}})'\|_2^2$$

and note that

$$\|\xi(\phi^{\mathrm{R}})'\|_2^2 = \sum_{j=1}^{M^{\mathrm{R}}} \frac{\theta^2}{U_j^{\mathrm{R}} - U_{j-1}^{\mathrm{R}}}$$

Then (4.5), (4.6), and the easy estimate $q_\beta(y) \leqslant C q_{\lambda\beta}(y)$ for $|y| < \theta$ imply that

$$\begin{aligned}
&\{\exp[\beta(1-\lambda)\, \|\xi(\phi^{\mathrm{R}})'\|_2^2/2]\}\, h_\beta(m^{\mathrm{R}}, u^{\mathrm{R}})\\
&\quad \leqslant C[\lambda^{-1/2} c(\lambda\beta)]^{m^{\mathrm{R}}}\, h_{\lambda\beta}(m^{\mathrm{R}}, u^{\mathrm{R}})
\end{aligned}$$

with a similar inequality for the terms involving the imaginary part. Thus, with an obvious estimate of the sum over $\gamma$, (4.9) becomes

$$\begin{aligned}
&\int_C F(\phi)\, d\mu_\beta(\phi)\\
&\quad \leqslant C^2 e^K \int_{\mathbb{C}} da \sum_{m^{\mathrm{R}}, m^{\mathrm{I}}} [2\lambda^{-1/2} c(\lambda\beta)]^{m^{\mathrm{R}} + m^{\mathrm{I}}}\\
&\qquad \times \int du^{\mathrm{R}}\, du^{\mathrm{I}}\, p_\beta^L(a, a)^{-1}\, h_{\lambda\beta}(m^{\mathrm{R}}, u^{\mathrm{R}})\, h_{\lambda\beta}(m^{\mathrm{I}}, u^{\mathrm{I}}) \qquad (4.10)
\end{aligned}$$

Now observe that the last line of (4.10) is, except for an incorrect normalization factor, one contribution from (4.7) to $\mu_{\lambda\beta}^{a,b;L}(C^{a,b}) = 1$, where $b = a + m^{\mathrm{R}}\theta + i m^{\mathrm{I}}\theta$ [specifically, the contribution from terms for which $\gamma^{\mathrm{I}} \equiv \gamma^{\mathrm{R}} \equiv 0$ and $m^{\mathrm{I}} = m^{\mathrm{R}} = 0$ in (4.7)]. Hence

$$\begin{aligned}
&\int_C F(\phi)\, d\mu_\beta(\phi)\\
&\quad \leqslant C^2 e^K \int_{\mathbb{C}} da \sum_{m^{\mathrm{R}}, m^{\mathrm{I}}} \int du\, [2\lambda^{-1/2} c(\lambda\beta)]^{m^{\mathrm{R}} + m^{\mathrm{I}}}\, p_\beta^L(a, a)^{-1}\, p_{\lambda\beta}^L(a, b) \quad (4.11)
\end{aligned}$$

Since

$$p^L_{\lambda\beta}(a, b) = (\lambda\beta/2\pi) \exp\{-\lambda\beta\theta^2[(m^R)^2 + (m^I)^2]/2L\}$$

the sum over $m^R$ and $m^I$ in (4.11) is convergent; moreover, the lower bound (4.8) implies that the resulting integral over $a$ also converges. Thus

$$\int_C F(\phi)\, d\mu_\beta(\phi) < \infty \quad \blacksquare$$

For the proof of the second part of the theorem we will need an Onsager–Machlup formula for the measure $\mu_\beta$ (see Ref. 23 for further references).

**Lemma 4.3.** Suppose that $\psi \in C$ is absolutely continuous and that $\psi'$ is of bounded variation on $[0, L]$; without loss of generality we redefine $\psi'(L)$ so that $\psi'(L) = \psi'(0)$. Then

$$\begin{aligned} &\mu_\beta(B_\varepsilon(\psi)) \\ &\quad = \left\{\exp\left[-\frac{\beta}{2}\int_0^L |\psi'(x)|^2\, dx\right]\right\} \int_{B_\varepsilon(0)} \exp\left(-\beta \operatorname{Re} \int_0^L \phi\, d\psi'\right) d\mu_\beta(\phi) \\ &\quad \geqslant \left\{\exp\left[-\frac{\beta}{2}\int_0^L |\psi'(x)|^2\, dx\right] \exp[-\beta\varepsilon V(\psi')]\right\} \mu_\beta(B_\varepsilon(0)) \end{aligned}$$

where $V(\psi')$ is the total variation of $\psi'$ on $[0, L]$.

*Proof.* For $n \geqslant 1$ we introduce mesh points $x_k = kL/n$, $0 \leqslant k \leqslant n$, and set

$$B^{(n)}_\varepsilon(\psi) = \{\phi \in C \mid |\phi(x_k) - \psi(x_k)| < \varepsilon,\ 1 \leqslant k \leqslant n\}$$

Clearly $\mu_\beta(B_\varepsilon(\psi)) = \lim_{n\to\infty} \mu_\beta(B^{(n)}_\varepsilon(\psi))$. On the other hand, from (4.4)

$$\begin{aligned} &\mu_\beta(B^{(n)}_\varepsilon(\psi)) \\ &\quad = p^L_\beta(0, 0)^{-1} \int_{|z_i| < \varepsilon} \prod_{i=1}^n p^{L/n}_\beta(\psi(x_i) + z_i, \psi(x_{i-1}) + z_{i-1})\, dz_1 \cdots dz_n \\ &\quad = \exp\left[-(\beta n/2L) \sum_{i=1}^n |\psi(x_i) - \psi(x_{i-1})|^2\right] \\ &\qquad \times \int_{B^{(n)}_\varepsilon(0)} \exp\left\{-(\beta n/L) \operatorname{Re} \sum_{i=1}^n [\psi(x_i) - \psi(x_{i-1})]\right. \\ &\qquad \left. \times [\phi(x_i) - \phi(x_{i-1})]\right\} d\mu_\beta(\phi) \end{aligned}$$

where $z_0 = z_n$ by convention. But since $\psi' \in L^2$,

$$\lim_{n \to \infty} \frac{n}{L} \sum_{i=1}^{n} |\psi(x_i) - \psi(x_{i-1})|^2 = \int_0^L |\psi'(x)|^2 \, dx$$

Moreover, if for $\phi \in C$ we let $\phi^{(n)}$ be the piecewise linear interpolant of $\phi$ on the mesh points, we have

$$\frac{n}{L} \sum_{i=1}^{n} [\psi(x_i) - \psi(x_{i-1})][\phi(x_i) - \phi(x_{i-1})] = \int_0^L \psi'(x) \, \phi^{(n)\prime}(x) \, dx$$

$$= \int_0^L \phi^{(n)} d\psi'$$

Since $\phi^{(n)} \to \phi$ uniformly as $n \to \infty$, the equality in the statement of the lemma follows from the dominated convergence theorem; the inequality is then immediate. ∎

*Proof of Theorem 2.2(b).* For notational simplicity we replace the interval $[0, L]$ by the interval $[-L/2, L/2]$. We may choose the ground state solution $\hat{\phi}$ of the NLS on $\mathbb{R}$ discussed in Section 2 so that it is positive and symmetric under reflection through the origin, lies in $H^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$, and satisfies $\|\hat{\phi}\|_{0,p}^p = (p/2) \|\hat{\phi}'\|_{0,2}^0$, $\|\hat{\phi}\|_{0,2}^2 = N_0$. Let $\tilde{\phi} = \alpha\hat{\phi}$ for some $\alpha > 1$; in the case $p = 6$ we choose $\alpha$ such that $\|\tilde{\phi}\|_2^2 < N$, while for $p > 6$, $\alpha$ is arbitrary. Clearly, $H(\tilde{\phi}) < 0$.

Now for any $\phi$ defined on $\mathbb{R}$ and any $\rho > 0$ we define

$$(T_\rho \phi)(x) = \rho^{4/(p-2)} \tilde{\phi}(\rho^2 x)$$

For $\phi$ sufficiently regular that the corresponding norms are finite, both $\|\phi\|_{0,p}^p$ and $\|\phi'\|_{0,2}^2$ scale under $T_\rho$ as $\rho^{(2p+4)/(p-2)}$, while $\|\phi\|_{0,2}^2$ and $\|\phi''\|_{0,1}$ scale as $\rho^{2(6-p)/(p-2)}$ and $\rho^{2p/(p-2)}$, respectively. Finally, we define $\psi_\rho \in H^1$ by

$$\psi_\rho(x) = (T_\rho \tilde{\phi})(x), \qquad x \in [-L/2, L/2]$$

Thus, under the hypotheses of (b) we will have, for some positive constants $E$, $\eta_1$, $A$, $B$, and $\rho_0$, and for all $\rho > \rho_0$,

$$\|\psi_\rho\|_2^2 < N - \eta_1 \tag{4.12a}$$

$$\|\psi_\rho\|_p^p \leqslant \rho^{(2p+4)/(p-2)} B \tag{4.12b}$$

$$-H(\psi_\rho) \geqslant \rho^{(2p+4)/(p-2)} E \tag{4.12c}$$

$$V(\psi_\rho') = \|\psi_\rho''\|_1 + O(\rho) < \rho^{2p/(p-2)} A \tag{4.12d}$$

We now apply Lemma 4.2, with $\phi_1 = \psi_\rho$ and $\eta = \min\{\eta_1, E/(2B)\}$, to produce an $\varepsilon$ for which the conclusions of that lemma hold. Then, from Lemma 4.3, for $\rho > \rho_0$,

$$\begin{aligned}
&\int_C F(\phi)\, d\mu_\beta(\phi) \\
&\quad \geqslant \int_{B_\varepsilon(\psi_\rho)} \exp\left(\frac{\beta}{p} \|\phi\|_p^p\right) d\mu_\beta(\phi) \\
&\quad \geqslant \left\{\exp\left[\frac{\beta}{p}\left[(1-\eta)\|\psi_\rho\|_p^p - \eta\right]\right]\right\} \int_{B_\varepsilon(\psi_\rho)} d\mu_\beta(\phi) \\
&\quad \geqslant \exp\left\{\beta\left[-H(\psi_\rho) - \frac{\eta}{p}\|\psi_\rho\|_p^p - \varepsilon V(\psi_\rho') - \frac{\eta}{p}\right]\right\}
\end{aligned}$$

Then, as $\rho \to \infty$, we see from (4.12b)–(4.12d) that $F(\phi) \notin L^1(C; \mu_\beta)$. ∎

Finally, we give the following result:

*Proof of Proposition 3.1.* We modify the Hamiltonian of (1.1) by introducing a complex coupling constant $\lambda$:

$$H_\lambda(\phi) = \frac{1}{2}\int_0^L |\phi'(x)|_2^2\, dx - \frac{\lambda}{p} |\phi(x)|^p\, dx$$

Theorem 2.2 holds for this modified Hamiltonian, with the exception that if $p = 6$, then $F \in L^1$ requires

$$N^2 \operatorname{Re} \lambda \leqslant N_0^2 \tag{4.13}$$

Moreover, the Lebesgue dominated convergence theorem shows that the corresponding partition function

$$\tilde{Z}(\beta, N, \lambda) = \int_C \left[\exp\left(\frac{\lambda\beta}{p}\|\phi\|_p^p\right)\right] \chi_{\{\phi \mid \|\phi\|_2^2 \leqslant N\}}\, d\mu_\beta(\phi) \tag{4.14}$$

is analytic in $\lambda$ for fixed $\beta$, $N$, for all $\lambda \in \mathbb{C}$ if $p < 6$, and for all $\lambda$ satisfying (4.13) if $p = 6$.

Now a change of variables $\phi = \alpha^{1/2}\psi$ $(\alpha > 0)$ in (4.14) shows that

$$\tilde{Z}(\beta, N, \lambda) = \tilde{Z}(\alpha\beta, \alpha^{-1}N, \alpha^{(p-2)/2}\lambda)$$

In particular, taking $\lambda = \alpha^{-(p-2)/2}$ and $\beta = 1$ yields

$$\tilde{Z}(1, N, \alpha^{(2-p)/2}) = \tilde{Z}(\alpha, \alpha^{-1}N, 1) = Z(\alpha, \alpha^{-1}N)$$

The result is immediate. ∎

## 5. OPEN QUESTIONS

Many questions remain regarding the behavior of this and related models. Here are a few of them.

1. *Existence of equilibrium dynamics.* The Gibbs measure constructed in Section 2.2 lives on Wiener paths, which, although continuous, are typically not differentiable. On the other hand, the dynamical theory sketched in Section 2.1 works for differentiable (specifically, $H^1$) functions. It would be desirable to define a dynamics on the Wiener paths for which the measure would be invariant, but we have been unable to do so. Note that, at least for the definition of a local (in time) dynamics, the source of difficulty is the nonlinearity itself and not the "wrong" sign of the potential energy.

We remark that to apply a fixed-point method to (2.1) in a straightforward way (as in Section 2.1) one should replace the space $H^1(I)$ by a linear space $X$ of functions on $I$ such that: (i) the nonlinear map $\phi \mapsto |\phi|^{p-2}\phi$ is continuous on $X$; (ii) the free time evolution $U(t)$ is continuous on $X$; (iii) almost every Wiener path lies in $X$, i.e., $\mu_\beta(C \backslash X) = 0$. The obvious candidates fail. $H^\alpha(I)$ satisfies (ii) for all $\alpha$, but (i) only for $\alpha \geqslant 1/2$ and (iii) only for $\alpha < 1/2$; $L^\infty(I)$ satisfies (i) and (iii), but fails (ii) (as does $C$ itself), as shown by the following argument, for which we thank E. Stein. Choose $s \in \mathbb{R}$ so that $2\pi s/L^2$ is irrational, and let

$$\Phi_N = \sum_{|n| \leqslant N} \exp[i(2\pi n/L)^2 s] \exp(2\pi i n x/L)$$

so that

$$U(s)\,\Phi_N = \sum_{|n| \leqslant N} \exp(2\pi i n x/L)$$

Then $\|U(s)\,\Phi_N\|_\infty = 2N+1$, but Hardy and Littlewood[24] show that $\|\Phi_N\|_\infty = o(N)$. Thus, $U(s)$ is unbounded on $L^\infty$. [In fact, an application, given in Ref. 25, of the closed graph theorem shows that $U(s)$ cannot map $L^\infty$ to itself.] Incidentally, $U(t)$ is bounded on $L^\infty$ (or $C$) for $2\pi t/L^2$ rational, because it acts in Fourier space as multiplication by the Fourier coefficients of a complex measure of finite total variation.[25,26]

2. *Extension to higher dimensions.* The NLSE occurs naturally in various physical problems, including nonlinear optics as well as the plasma physics emphasized here, in which the position coordinate is two or three dimensional. The theory of smooth solutions extends readily to this case, at least on $\mathbb{R}^n$.[8] It would be interesting to extend the construction of the Gibbs measure to these cases for the appropriate values of $p$ and $N$; the

standard free field measures could play the role that Wiener measure has played here. We would of course have to deal with ultraviolet problems. Mathematically, these would lead to normal ordering of the fields in $d=2$, as is usual in quantum field theory,[27] and to more complicated renormalizations in $d=3$, but we do not have a physical interpretation of such manipulations in, for example, the plasma physics case. Nor do we know whether the standard methods of constructive quantum field theory would suffice for Hamiltonians unbounded below.

3. *Properties of the measure* $\nu_{\beta,N}$. Various additional questions remain open for the measures constructed in this paper. Are physical quantities in fact analytic in $\beta$ and $N$? What are the properties of the correlation functions? Is the measure normalizable in the critical case $p=6$, $N=N_0$? Can an ensemble be defined via (2.4), that is, by restricting the fields to lie on the surface of a sphere in $L^2$, and do the resulting measures differ significantly from those when the restriction is to the ball?

4. *Parameter ranges where collapse can occur.* The ensembles as constructed here do not exist for $p>6$ or for $p=6$ and $N>N_0$. Can a modified approach throw any light on the behavior of physical systems with parameters in this range? One approach would be the construction of a set $X$ of nonzero Wiener measure, invariant under the dynamics, such that

$$\int_X \exp\left(\frac{\beta}{p}\int |\phi(x)|^p\, dx\right) d\mu_\beta < \infty$$

$X$ should in some sense correspond to the set of field configurations giving rise to smooth solutions globally in time. (In the case $p=6$ one would require that the set $X\backslash\{\phi \mid \|\phi\|_2^2 \leqslant N_0\}$ have nonzero Wiener measure.) The Gibbs ensemble could then be defined on $X$. One could also study modified versions of the interaction energy, introducing a cutoff at high field amplitudes to prevent collapse. For example, the replacement of $\int |\phi|^4\, d^2x$ by

$$\int \frac{|\phi|^4}{1+\lambda|\phi|^2}\, d^2x$$

has been used in the study of laser propagation in a nonlinear medium[28] and a similar modification could be studied in one dimension.

5. *Stochastic dynamics.* It is possible to introduce a stochastic dynamics for the Zakharov model which preserves the $L^2$ norm of the $u$ field and has the Gibbs state (3.5) as an invariant measure. The simplest such model, which must also include a dissipative term, is achieved by modifying only (3.3a), retaining (3.1) and (3.3b) in their original form. If

$\psi(k)$ denotes the $k$th Fourier coefficient of the field $\psi$, the modified (3.3a) is

$$V(k)_t = (-n_x + |u|_x^2)(k) - 2\rho(k)\, V(k) + \left(\frac{4\rho(k)}{\beta c}\right)^{1/2} \xi(k)$$

for $k = \pm 1, \pm 2,...$. Here $\rho(k) > 0$ is the coefficient of dissipation in the $k$th mode and $\{\xi(k)\}$ are independently distributed white noise random variables with covariance

$$E[\xi(k)(t)\, \xi(k')(t')] = \delta_{k,k'}\, \delta(t - t')$$

In this model it seems possible that the Gibbs ensemble is the unique invariant measure. As already mentioned, however, this introduction of steady Gaussian "microscopic noise" into the ZE is not physically complete, since in a real plasma there are other, intermittent, dissipative processes associated with collapse that do not satisfy a fluctuation-dissipation relation.

## ACKNOWLEDGMENTS

We thank Michael Aizenman, Josef Fritz, Sheldon Goldstein, Eli Stein, and Michael Weinstein for very helpful suggestions, and Joseph O'Connell for his investigation of the ensemble discussed in Remark 2.2. One of us (H. A. R.) acknowledges the support of the INLS at UCSD, where part of this research was performed. This work was supported in part by AFSOR grant 87-0010, by the U.S. DOE, and in part by ONR contract DP-F85-28.

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