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Generalized Monotonicity of a Separable Product of Operators: The Multivalued Case

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Abstract. This paper addresses the question of the generalized monotonicity of a separable product of operators. We extend the results of an earlier paper to the case where the operators are not continuous and multivalued. Necessary and sufficient conditions for the generalized monotonicity of the product are given in terms of the monotonicity indices of the factors.

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1. Introduction

Let E be a linear space paired with its dual E' by the bilinear form $\langle\ ,\ \rangle, C$ be a convex subset of E and F be a multivalued operator defined on C with values in $E'(\emptyset \neq F(x) \subset E'$ for all $x \in C)$. This operator is said to be

monotone on C if

$$\langle x_2^* - x_1^*, x_2 - x_1\rangle \geqslant 0, \quad \textit{whenever } x_i \in C,\ x_i^* \in F(x_i),\ i = 1,2;$$

pseudomonotone on C if

$$\langle x_2^*, x_2 - x_1\rangle > 0, \quad \textit{whenever } x_i \in C,$$
$$x_i^* \in F(x_i) \quad i = 1,2, \quad \textit{and} \quad \langle x_1^*, x_2 - x_1\rangle > 0;$$

quasimonotone on C if

$$\langle x_2^*, x_2 - x_1\rangle \geqslant 0, \quad \textit{whenever } x_i \in C,$$
$$x_i^* \in F(x_i), \quad i = 1,2, \quad \textit{and} \quad \langle x_1^*, x_2 - x_1\rangle > 0.$$

It is clear that F is pseudomonotone when it is monotone, quasimonotone when it is pseudomonotone.

A strong connection exists between convexity and monotonicity, for instance the subdifferential of a convex function is a monotone operator. Similar connections exist between generalized convexity and generalized monotonicity, the gradient of a pseudoconvex (quasiconvex) function is pseudomonotone (quasimonotone). For these connections and the interest of generalized monotonicity, the reader is referred to [5–9] and [11–17].

This paper addresses the question of the generalized monotonicity of a separable product of operators. The operator F is defined on $C = C_1 \times C_2 \times \cdots \times C_p$ by

$$F(x_1, x_2, \ldots, x_p) = (F_1(x_1), F_2(x_2), \ldots, F_p(x_p)),$$

where C_i is a convex set of a linear space E_i and F_i is a multivalued operator defined on C_i with values in E_i' the dual of E_i. We assume, of course, $p \geqslant 2, C_i$ nonempty and F_i nonnull on C_i. It is clear that all F_i are monotone (pseudomonotone, quasimonotone) when F is monotone (pseudomonotone, quasimonotone).

In an earlier paper [2], we analyzed this problem in the very particular case where for all i, E_i has a finite dimension, F_i is univalued and continuous. It was shown that if F is quasimonotone on C, then all factors F_i, except perhaps one, are monotone. A necessary and sufficient condition involves the monotonicity indices of F_i, a concept introduced in [2].

We deal now with a quite more general context: E_i has not necessarily a finite dimension, F_i is not continuous and multivalued. The topological structures, when needed, are minimal. The interior of a convex set is taken in a geometrical sense: $x \in \text{int}(C)$ if for all $d \in E$, there exists $s > 0$ so that $x + sd \in C$. The duality between E and E' needs only the condition: if x^* is a nonnull element of E', then there exists $x \in E$ such that $\langle x, x^* \rangle \neq 0$.

Despite this very general context, we obtain the same results as in [2], the proofs of the main results are essentially different.

The concept of the monotonicity index will be the main tool of this paper, as it was in our earlier paper. It is derived from the concept of convexity index introduced by Debreu and Koopmans [4], and revisited by Crouzeix and Lindberg [3]. These convexity indices appear in necessary and/or sufficient conditions for a separable sum of functions to be quasiconvex, a problem of a special interest in economics, in particular in consumer theory. The works of Crouzeix and Lindberg [3] and Debreu and Koopmans [4] are concerned with the quasiconcavity of a utility function which is a separable sum of functions. The present paper corresponds to the generalized monotonicity of a demand map which is a separable product of operators, a more general context since the approach of the behaviour of a consumer by demand maps is more general than the one by utility functions.

This paper is organized as follows. In Section 2, we list some results on the continuous and univalued case and the convexity indices of functions. In Section 3, we define what we call k-monotone operators, then we define the monotonicity index of an operator. Section 4 establishes the necessary and sufficient conditions for the generalized monotonicity of a separable product of two operators. These conditions are generalized to more than two factors in the last section.

2. The Continuous and Univalued Case

Throughout this section, we give a brief 'digest' of the notation and results contained in our earlier paper [2].

Let C be an open convex subset of $\mathbb{R}^n, F\colon C \to \mathbb{R}^n$ be univalued and continuous on C. For all a $\in C$, $d \in \mathbb{R}^n$, define

$$I_{a,d} = \{t \in \mathbb{R}\colon a + td \in C\},$$

$$F_{a,d}(t) = \langle F(a + td), d\rangle, \quad t \in I_{a,d},$$

and

$$f_{a,d}(t) = \int_0^t F_{a,d}(s)\,\mathrm{d}s.$$

The continuity of F ensures the existence of $f_{a,d}$. Then we define $m(F)$, the monotonicity index of F as

$$m(F) = \inf_{a,d}[m(F_{a,d})\colon a \in C, d \in \mathbb{R}^n], \tag{2.1}$$

where $m(F_{a,d}) = c(f_{a,d})$, the convexity index of $f_{a,d}$, such as that defined in [3] (see also [4] for an earlier and equivalent definition):

$$c(f_{a,d}) = \mathrm{Sup}[\mu\colon \mu \neq 0, \ \mu \exp(-\mu f_{a,d}) \text{ is concave}].$$

With by convention, the supremum is taken equal to $-\infty$ if no $\mu \neq 0$ exists satisfying the condition.

Let now D be an open convex set of $\mathbb{R}^p$, $G\colon D \to \mathbb{R}^p$ be a univalued and continuous operator. Define $H\colon C \times D \to \mathbb{R}^{n+p}$ by

$$H(x, y) = (F(x), G(y)).$$

Then ([2, Theorem 4.1])

$$H \text{ is quasimonotone if and only if } m(F) + m(G) \geqslant 0. \tag{2.2}$$

In the particular case where F is the gradient of a function f and G the gradient of another function g, H is the gradient of the function h:

$$h(x, y) = f(x) + g(y), \quad (x, y) \in C \times D.$$

Then (2.2) is related to a necessary and sufficient condition for the quasiconvexity of h on $C \times D$ [3, 4]

$$h \text{ is quasiconvex if and only if } c(f) + c(g) \geqslant 0. \tag{2.3}$$

In the next sections, (2.2) will serve as a basis for the extension of the monotonicity index to multivalued operators.

3. *K*-Monotone Operators

For any $k \neq 0$, define

$$G_k(t) = \frac{1}{kt}, \quad t \in I,\ I = (0, \infty). \tag{3.1}$$

Then by relations (2.2) to (2.4)

$$m(G_k) = m(G_k, a) = -k \quad \text{for any } a \in I.$$

Now, let F be a multivalued operator defined on a convex set C. Condition (2.2) suggests thinking for the monotonicity index of F of a formula like

$$m(F) = \mathrm{Sup}[k\colon\ H_k \text{ is quasimonotone on } C \times I], \tag{3.2}$$

where H_k is the multivalued operator defined by

$$H_k(x,t) = (F(x), G_k(t)), \quad x \in C,\ t \in I.$$

Indeed (3.2) is equivalent to (2.2) when F is continuous and univalued. Formula (3.2) leads to the following definition:

We say that F is k-monotone on C when H_k is quasimonotone on $C \times I$.

THEOREM 3.1 (Characterization of k-monotone operators). *F is k-monotone on C if and only if F is pseudomonotone on C and*

$$\frac{1}{\langle x_0^*, x_1 - x_0\rangle} - \frac{1}{\langle x_1^*, x_1 - x_0\rangle} \geqslant k \quad \textit{whenever } x_i \in C,$$

$$x_i^* \in F(x_i) i = 0,1 \quad \textit{and} \quad \langle x_0^*, x_1 - x_0\rangle > 0. \tag{3.3}$$

Proof. (i) F is k-monotone on C if and only if

$$\begin{aligned} &\langle x_1^*, x_1 - x_0\rangle \geqslant -\frac{1}{kt_1}(t_1 - t_0), \quad \text{whenever } x_i \in C,\ x_i^* \in F(x_i),\\ &t_i \in I \quad \text{for } i = 0,1 \quad \text{and} \quad \langle x_0^*, x_1 - x_0\rangle > -\frac{1}{kt_0}(t_1 - t_0). \end{aligned} \tag{3.4}$$

This condition is equivalent to

$$\langle x_1^*, x_1 - x_0\rangle \geqslant \frac{\mu}{1 - k\mu}, \quad \text{whenever } x_i \in C,\ x_i^* \in F(x_i), \quad \text{for } i = 0,1$$

$$\text{and} \quad \langle x_0^*, x_1 - x_0\rangle > \mu \quad \text{and} \quad 1 - k\mu > 0. \tag{3.5}$$

To prove (3.5), from (3.4), take $t_0 = 1$ and $t_1 = 1 - k\mu$. To prove (3.4) from (3.5) take $\mu = -1/k(t_1/t_0 - 1)$ (then $1 - k\mu = t_1/t_0 > 0$).

When k is positive, consider the function $\mu \mapsto \mu(1-k\mu)^{-1}$ on the interval $(-\infty, k^{-1})$, when k is negative, take the same function, but on the interval (k^{-1}, ∞). This function increases on its domain and condition (3.5) is equivalent to

$$\langle x_1^*, x_1 - x_0\rangle \geqslant \frac{\langle x_0^*, x_1 - x_0\rangle}{1 - k\langle x_0^*, x_1 - x_0\rangle}, \quad \text{whenever } x_i \in C,$$
$$x_i^* \in F(x_i), \quad i = 0, 1, \quad \text{and} \quad 1 > k\langle x_0^*, x_1 - x_0\rangle. \tag{3.6}$$

(ii) Assume that F is k-monotone and $\langle x_0^*, x_1 - x_0\rangle > 0$. To prove that $\langle x_1^*, x_1 - x_0\rangle$ is positive, it suffices to take μ positive small enough in order to have $\langle x_0^*, x_1 - x_0\rangle > \mu$ and $1 - k\mu > 0$, then apply (3.5). It remains to prove (3.3). For this, we consider two cases:

(a) k is negative. Then $1 > k\langle x_0^*, x_1 - x_0\rangle$ and by (3.6) we have:

$$\langle x_1^*, x_1 - x_0\rangle - \langle x_0^*, x_1 - x_0\rangle \geqslant k\langle x_0^*, x_1 - x_0\rangle\langle x_1^*, x_1 - x_0\rangle,$$

from which (3.3) holds.

(b) k is positive. Notice that $k\langle x_1^*, x_0 - x_1\rangle$ is negative. Hence, by (3.6), we have,

$$\langle x_0^*, x_0 - x_1\rangle \geqslant \frac{\langle x_1^*, x_0 - x_1\rangle}{1 - k\langle x_1^*, x_0 - x_1\rangle}$$

and (3.3) holds as well.

(iii) Assume that F is pseudomonotone, (3.3) holds and $1 > k\langle x_0^*, x_1 - x_0\rangle$.

(a) If $\langle x_0^*, x_1 - x_0\rangle \leqslant 0$ and $\langle x_1^*, x_1 - x_0\rangle \geqslant 0$, then (3.6) is obvious.

(b) If $\langle x_0^*, x_1 - x_0\rangle$ is positive, then (3.3) implies (3.6).

(c) If $\langle x_1^*, x_1 - x_0\rangle < 0$, then $\langle x_1^*, x_0 - x_1\rangle > 0$. Permute x_0 and x_1, then (3.6) holds again.

□

COROLLARY 3.2. (i) *If F is k-monotone on C and $k > r$, then F is r-monotone on C.*

(ii) *F is monotone on C if and only if F is k-monotone on C for all negative k.*

Proof. (i) A direct consequence of the theorem.

(ii) Assume that F is monotone. Then

$$\langle x_1^*, x_1 - x_0\rangle \geqslant \langle x_0^*, x_1 - x_0\rangle \quad \text{for all } x_1, x_0 \in C,$$

from what (3.3) holds for all negative k. Conversely, assume that F is pseudomonotone but not monotone. Then $x_i \in C$, $x_i^* \in F(x_i)$, $i = 0, 1$, are so that

$$\langle x_1^*, x_1 - x_0\rangle < \langle x_0^*, x_1 - x_0\rangle.$$

If $\langle x_0^*, x_1 - x_0\rangle$ is positive, $\langle x_1^*, x_1 - x_0\rangle$ is also positive by pseudomonotonicity and

$$\frac{1}{\langle x_0^*, x_1 - x_0\rangle} - \frac{1}{\langle x_1^*, x_1 - x_0\rangle} < 0.$$

Hence, F is not k-monotone for some negative k.

If not, $\langle x_1^*, x_1 - x_0\rangle$ is negative and

$$\langle x_1^*, x_0 - x_1\rangle < \langle x_0^*, x_0 - x_1\rangle.$$

It is the same case as just above. □

We notice that (3.3) gives an equivalent definition of the k-monotonicity of an operator which can be applied even when $k = 0$. We can now precise (3.2).

Let F be an operator defined on a convex set C. The monotonicity index of F on C is given by

$$m(F) = \begin{cases} -\infty, & \text{if } F \text{ is not pseudomonotone on C,} \\ \text{Sup}[k\colon\ F \text{ is } k\text{-monotone on } C], & \text{otherwise.} \end{cases}$$

We notice that F is $m(F)$ -monotone when $m(F)$ is finite, and $m(F) = +\infty$ when F is null on C. On the other hand, by Corollary 3.2, F is monotone if and only if $m(F) \geqslant 0$.

Expression (2.1) relates the monotonicity index of F to monotonicity indices of operators of one variable. We now seek a similar relation for multivalued operators.

As in Section 2, define

$$I_{a,d} = \{t \in \mathbb{R}\colon\ a + td \in C\}$$

and

$$F_{a,d}(t) = \{\langle x^*, d\rangle\colon\ x^* \in F(a + td)\}, \quad t \in I_{a,d}.$$

Clearly, F is monotone (pseudomonotone, quasimonotone, k-monotone) on C, if and only if for all $a \in C$, $d \in E$; $F_{a,d}$ is so on $I_{a,d}$.

Assume that F is pseudomonotone on C, then $F_{a,d}$ is also pseudomonotone on $I_{a,d}$. Hence, $t^-_{a,d}$ and $t^+_{a,d}$ exist so that

$$\begin{aligned}
&-\infty \leqslant t^-_{a,d} \leqslant t^+_{a,d} \leqslant +\infty,\\
&F_{a,d}(t) \subset (-\infty, 0) && \text{for all } t \in I_{a,d} \cap (-\infty, t^-_{a,d}),\\
&F_{a,d}(t) = 0 && \text{for all } t \in I_{a,d} \cap (t^-_{a,d}, t^+_{a,d}),\\
&F_{a,d}(t) \subset (0, \infty) && \text{for all } t \in I_{a,d} \cap (t^+_{a,d}, \infty),\\
&F_{a,d}(t) \subset [0, \infty) && \text{for all } t \in I_{a,d} \cap (t^-_{a,d}, \infty),\\
&F_{a,d}(t) \subset (-\infty, 0] && \text{for all } t \in I_{a,d} \cap (-\infty, t^+_{a,d}).
\end{aligned}$$

Before going further, we give an example:

EXAMPLES. $E = C = \mathbb{R}$

$$F_1(t) = \begin{cases} \frac{1}{t} & \text{if } t \neq 0, \\ [-1, 10] & \text{if } t = 0. \end{cases}$$

$$F_2(t) = \begin{cases} \frac{1}{t} & \text{if } t < -1, \\ (-1, 0) & \text{if } t = -1, \\ 0 & \text{if } -1 < t < 0, \\ [0, 1] & \text{if } t = 0, \\ 1 + \sqrt{t} & \text{if } t > 0. \end{cases}$$

Then F_1 and F_2 are pseudomonotone on $\mathbb{R}$.

Next, define

$$I^+_{a,d} = \begin{cases} I_{a,d} \cap [t^+_{a,d}, \infty) & \text{if } F(t^+_{a,d}) \cap (0, \infty) \neq \emptyset, \\ I_{a,d} \cap (t^+_{a,d}, \infty) & \text{otherwise,} \end{cases}$$

and $F^+_{a,d}$ defined on $I^+_{a,d}$ by

$$F^+_{a,d}(t) = F_{a,d}(t) \cap (0, \infty).$$

Then, by Theorem 3.1,

$$m(F) = \inf_{a,d}[m(F^+_{a,d}): \ I^+_{a,d} \neq \emptyset]. \tag{3.7}$$

This relation corresponds to relation (2.1). It suggests the analysis of the positive operators which are defined on intervals of $\mathbb{R}$.

Thus, we consider a multivalued operator ϕ: $I \to (0, \infty)$, where I is any interval of $\mathbb{R}$. If I is the empty set or a singleton, then $m(\phi) = +\infty$. Assume that the interior of I is not empty, then the definition of $m(\phi)$ and Theorem 3.1 imply

$$m(\phi) = \inf \left[\frac{1}{(t_1 - t_0)} \left(\frac{1}{t_0^*} - \frac{1}{t_1^*} \right) : \ t_0, \ t_1 \in I, \ t_i^* \in \phi(t_i), \ t_0 < t_1 \right]. \tag{3.8}$$

The introduction of the selections of ϕ simplifies the analysis of its monotonicity index. The map Σ: $I \to \mathbb{R}$ is said to be a *selection* of ϕ if it is a singlevalued operator such that:

$$\Sigma(t) \in \phi(t) \quad \text{for all } t \in I.$$

The operator ϕ is k-monotone (quasimonotone, pseudomonotone) if and only if all its selections are k-monotone (quasimonotone, pseudomonotone).

Let Σ be a selection of ϕ, t_0 be in the interior of I and σ be defined by

$$\sigma(t) = \int_{t_0}^{t} \Sigma(s)\,\mathrm{d}s, \quad t \in I \tag{3.9}$$

when this integral exists.

THEOREM 3.3. (i) *Assume that ϕ is k-monotone ($k \in \mathbb{R}$), then for any selection Σ of ϕ and any $t \in I$, we have*

$$\begin{aligned} \limsup_{s\uparrow t} \Sigma(s) &\leqslant \operatorname{Inf}[t^*\colon\ t^* \in \phi(t)] \leqslant \Sigma(t) \\ &\leqslant \operatorname{Sup}[t^*\colon\ t^* \in \phi(t)] \leqslant \liminf_{s\downarrow t} \Sigma(s). \end{aligned} \tag{3.10}$$

(*When t is a bound of I, only one inequality subsists.*)

(ii) *Assume in addition that ϕ is bounded on any closed interval contained in the interior of I, then σ is well defined and does not depend on the selection Σ, σ is finite on the interior of I but possibly infinite at the bounds of the interval.*

Proof. Let us consider the function

$$\theta(t) = -\frac{1}{\Sigma(t)} - kt, \quad t \in I.$$

Then θ is nondecreasing on I and

$$\limsup_{s\uparrow t} \theta(s) \leqslant \theta(t) \leqslant \liminf_{s\downarrow t} \theta(s).$$

Since Σ is any selection of ϕ, we have

$$\begin{aligned} \limsup_{s\uparrow t} \theta(s) &\leqslant \operatorname{Inf}\left[-\frac{1}{t^*} - kt\colon\ t^* \in \phi(t)\right] \leqslant \theta(t) \\ &\leqslant \operatorname{Sup}\left[-\frac{1}{t^*} - kt\colon\ t^* \in \phi(t)\right] \leqslant \liminf_{s\downarrow t} \theta(s), \end{aligned}$$

from which (3.10) follows.

The function θ is a function of finite variation because it is monotone, hence Σ is also a function of finite variation because it is bounded. Consequently, Σ can be expressed as the difference of two nondecreasing functions and is integrable. It is clear that σ does not depend on the selection Σ.

The above proof works for all real k. The results are obvious, when k is nonnegative, since Σ is then nondecreasing. □

The function σ can be not defined when ϕ is only k-monotone for $k < 0$. Consider for instance $I = (0, \infty)$ and cannot be ϕ be defined by

$$\phi(t) = \begin{cases} \frac{1}{t} & \text{if } 0 < t \leqslant 1, \\ \frac{1}{t-1} & \text{if } 1 < t. \end{cases}$$

It is easily seen that ϕ is (-1)-monotone, hence by definition of the k-monotonicity, the operator $H = (\phi, G_{-1})$ is quasimonotone. This shows the limits of the approach followed in our earlier paper where the proofs were based on the existence of the function σ and the conditions for the quasiconvexity for the separable sum of functions. In the example, σ does not exist, but H is quasimonotone.

When σ is well defined we have:

THEOREM 3.4. *If* (3.10) *holds and* σ *is well defined, then*

$$c(\sigma) = m(\phi).$$

Proof. (i) If $m(\phi) = c(\sigma) = -\infty$, there is nothing to prove.

(ii) Assume that $m(\phi) > -\infty$ and let us prove that $m(\phi) \leqslant c(\sigma)$. Let k be such that $-\infty < k \leqslant m(\phi)$ and let Σ be any selection of ϕ. Then Σ is k-monotone. Let $t \in I$ and s be small enough so that

$$t + s \in I \quad \text{and} \quad 1 - ks\Sigma(t) > 0.$$

Since Σ is k-monotone, we have

$$\Sigma(t+s) \geqslant \frac{\Sigma(t)}{1 - ks\Sigma(t)} \quad \text{if } s > 0,$$

$$\Sigma(t+s) \leqslant \frac{\Sigma(t)}{1 - ks\Sigma(t)} \quad \text{if } s < 0.$$

Then by (3.9), in both cases we get

$$\sigma(t+s) \geqslant -\frac{1}{k}\ln(1 - ks\Sigma(t)) + \sigma(t)$$

and

$$\exp(-k\sigma(t+s)) \geqslant \exp(-k\sigma(t)) - ks\Sigma(t)\exp(-k\sigma(t)) \quad \text{if } k < 0,$$
$$\exp(-k\sigma(t+s)) \leqslant \exp(-k\sigma(t)) - ks\Sigma(t)\exp(-k\sigma(t)) \quad \text{if } k > 0.$$

When k is negative, $-k\Sigma(t)\exp(-k\sigma(t))$ is a subgradient at t of the function $\exp(-k\sigma)$. This function is convex, since it is locally subdifferentiable at any point. Similarly, the function is concave when k is positive. Henceforth, $k \leqslant c(\sigma)$ for any $k \leqslant m(\phi)$.

(iii) Assume that $-\infty < c(\sigma)$ and let us prove that $m(\phi) \geqslant c(\sigma)$. Let k be such that $-\infty < k \leqslant c(\sigma)$. Condition (3.10) and relation (3.8) show that the monotonicity indices of ϕ on I and its interior are the same. Without loss of generality, we assume in the sequel that I is open. Then condition (2.3) implies that the function

$$\theta(x, u) = \sigma(x) + \frac{1}{k}\ln u$$

is quasiconvex on $I \times (0, \infty)$.

The function $\exp(-k\sigma)$ admits derivatives on the right and on the left at any point (since it is convex or concave). Hence, σ admits right and left derivatives too. On the other hand, by construction

$$\sigma'_-(x) \leqslant z \leqslant \sigma'_+(x) \quad \text{for any } x \in I \text{ and } \ z \in \phi(x). \tag{3.11}$$

Now let any $x, y \in I$ and $u, v \in (0, \infty)$. We consider the function

$$\mu(t) = \theta(x + t(y - x), u + t(v - u)).$$

Then μ is quasiconvex. Hence

$$\mu'_-(1) \geqslant 0, \quad \text{whenever } \mu'_+(0) > 0.$$

It follows that if $y \geqslant x$, one has

$$\sigma'_-(y)(y - x) + \frac{1}{kv}(v - u) \geqslant 0$$

$$\text{whenever } \sigma'_+(x)(y - x) + \frac{1}{ku}(v - u) > 0,$$

and if $y < x$, one has

$$\sigma'_+(y)(y - x) + \frac{1}{kv}(v - u) \geqslant 0$$

$$\text{whenever } \sigma'_-(x)(y - x) + \frac{1}{ku}(v - u) > 0.$$

Taking into account of (3.11), we deduce that ϕ is k-monotone. □

We close this section in relating the k-monotone operators to the so-called r-convex functions introduced by Avriel [1].

Let $r > 0$, f is said to be *r-convex* if $\exp(-rf)$ is convex.

We extend this definition to negative r by saying that f is r-convex $(r < 0)$ if $\exp(-rf)$ is concave.

Then it is easily seen that a differentiable function f is r-convex if and only if ∇f is $(-r)$-monotone.

4. The Case of Two Factors

In this section, $C(D)$ is a nonempty convex subset of a linear space $X(Y)$, $F(x) \subset X'$, $F(x) \neq \emptyset$ for all $x \in X(G(y) \subset Y'$, $G(y) \neq \emptyset$ for all $y \in Y)$. We consider the multivalued operator defined on $C \times D$ by

$$H(x, y) = (F(x), G(y)) \quad \text{for all } x \in C, \ y \in D.$$

THEOREM 4.1. *Assume that* $-\infty < m(F)$, $-\infty < m(G)$ *and* $m(F) + m(G) \geqslant 0$. *Then* H *is pseudomonotone on* $C \times D$.

Proof. The assumptions $-\infty < m(F)$ and $-\infty < m(G)$ imply that F and G are pseudomonotone. If both F and G are monotone, then H is monotone and therefore pseudomonotone. If not, $m(F)$ and $m(G)$ are nonnull. Take $k = m(F)$, then $m(G) \geqslant -k$.

Assume, for contradiction, that H is not pseudomonotone. Then $x_i \in C$, $y_i \in D$, $x_i^* \in F(x_i)$ and $y_i^* \in G(y_i)$, $i = 0, 1$, exist so that

$$\langle x_0^*, x_1 - x_0\rangle + \langle y_0^*, y_1 - y_0\rangle > 0 \tag{4.1}$$

and

$$\langle x_1^*, x_1 - x_0\rangle + \langle y_1^*, y_1 - y_0\rangle \leqslant 0. \tag{4.2}$$

At least one of the terms in (4.1) is positive, say for instance, $\langle x_0^*, x_1 - x_0\rangle$. Then $\langle x_1^*, x_1 - x_0\rangle$ is positive, by pseudomonotonicity of F. Then (4.2) implies that $\langle y_1^*, y_0 - y_1\rangle$ is positive and finally, by pseudomonotonicity of G, $\langle y_0^*, y_0 - y_1\rangle$ is also positive. (4.1) and (4.2) become

$$0 > \frac{1}{\langle x_0^*, x_1 - x_0\rangle} - \frac{1}{\langle y_0^*, y_0 - y_1\rangle}, \tag{4.3}$$

and

$$0 \geqslant \frac{1}{\langle y_1^*, y_0 - y_1\rangle} - \frac{1}{\langle x_1^*, x_1 - x_0\rangle}. \tag{4.4}$$

On the other hand, since $m(F) \leqslant k$ and $m(G) \leqslant -k$

$$\frac{1}{\langle x_1^*, x_0 - x_1\rangle} - \frac{1}{\langle x_0^*, x_0 - x_1\rangle} \geqslant k, \tag{4.5}$$

and

$$\frac{1}{\langle y_1^*, y_0 - y_1\rangle} - \frac{1}{\langle y_0^*, y_0 - y_1\rangle} \geqslant -k. \tag{4.6}$$

The contradiction is obtained by adding (4.3), (4.4), (4.5) and (4.6). □

Theorem 4.1 gives a sufficient condition for the pseudomonotonicity of H and therefore for the quasimonotonicity of H. The proof of a necessary condition is more complex and needs several steps.

PROPOSITION 4.2. *Assume that* H *is quasimonotone on* $C \times D$, F *and* G *nonnull and* C, D *have nonempty interiors. Then* F *and* G *are pseudomonotone.*

Proof. (i) Firstly, we prove that at least one of the operators F or G is pseudomonotone. If not, $x_i \in C,\ y_i \in D,\ x_i^* \in F(x_i)$ and $y_i^* \in G(y_i),\ i = 1, 2$, exist so that

$$\langle x_0^*, x_1 - x_0\rangle > 0 \quad \text{and} \quad \langle x_1^*, x_1 - x_0\rangle = 0,$$
$$\langle y_1^*, y_0 - y_1\rangle > 0 \quad \text{and} \quad \langle y_0^*, y_0 - y_1\rangle = 0.$$

Then

$$\langle x_0^*, x_1 - x_0\rangle + \langle y_0^*, y_1 - y_0\rangle > 0, \quad \text{and}$$
$$\langle x_1^*, x_1 - x_0\rangle + \langle y_1^*, y_1 - y_0\rangle < 0$$

which is in contradiction with H quasimonotone.

(ii) Assume now that F is pseudomonotone and G is not.

F is nonnull on $\mathrm{int}(C), x_1 \in \mathrm{int}(C)$ and $x_1^* \in F(x_1)$ exist so that $x_1^* \neq 0$. Then we take $x_0 \in C$ such that

$$\langle x_1^*, x_0 - x_1\rangle > 0,$$

and for any $t \in (0, 1),\ x_t = x_0 + t(x_1 - x_0)$.

Since F is pseudomonotone, we have

$$\langle x_0^*, x_t - x_0\rangle < 0 \quad \text{if } x_0^* \in F(x_0)$$

and

$$\langle x_t^*, x_t - x_0\rangle < 0 \quad \text{if } x_t^* \in F(x_t).$$

We now express that G is not pseudomonotone. There are $y_i \in D$ and $y_i^* \in G(y_i)$ such that

$$\langle y_0^*, y_1 - y_0\rangle > 0 \quad \text{and} \quad \langle y_1^*, y_1 - y_0\rangle = 0.$$

For t positive small enough, we have

$$\langle x_0^*, x_t - x_0\rangle + \langle y_0^*, y_1 - y_0\rangle > 0,$$

but

$$\langle x_t^*, x_t - x_0\rangle + \langle y_1^*, y_1 - y_0\rangle < 0.$$

This is in contradiction with the pseudomonotonicity of H. □

Remark. The assumptions are necessary. For a counter-example consider:

$$C = \mathbb{R} \times \{0\}, \qquad D = \mathbb{R},$$

$$F(x_1, 0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad G(y) = \begin{cases} \sqrt{-y} & \text{if } y < 0, \\ 0 & \text{if } y \geqslant 0. \end{cases}$$

Then H is quasimonotone but not pseudomonotone on $C \times D$, F and G are non-null, but the interior of C is empty.

In the next four propositions, we consider two real functions $\theta\colon I \to (0,\infty)$ and $\mu\colon J \to (0,\infty)$, where I and J are two nondegenerate intervals of the real line.

Let $\xi = (\theta, \mu)$ be defined

$$\xi(t,u) = (\theta(t),\ \mu(u)),$$

and for $t \in I, u \in J$

$$r(t) = \frac{1}{\theta(t)}, \qquad s(u) = \frac{1}{\mu(u)}.$$

PROPOSITION 4.3. *Assume that ξ is quasimonotone. Then for all u, t, Δ, δ and λ such that*

$$\delta > 0, \quad \Delta > 0, \quad u \text{ and } u - \Delta \in J; \quad t \text{ and } t + \delta \in I, \quad \lambda = \frac{\delta s(u)}{\Delta r(t)} > 1,$$

one has

$$\frac{s(u-\Delta)}{s(u)} \geqslant \frac{r(t+\delta)}{\lambda r(t)}. \tag{4.7}$$

Proof. Let u, t, Δ and δ be such that

$$\delta > 0, \quad \Delta > 0, \quad u \text{ and } u - \Delta \in J; \quad t \text{ and } t + \delta \in I.$$

Set:

$$A = \theta(t)\delta - \mu(u)\Delta \quad \text{and} \quad B = \theta(t+\delta)\delta - \mu(u-\Delta)\Delta.$$

Then ξ quasimonotone implies that

$$B \geqslant 0, \quad \text{whenever } A > 0.$$

Notice that $A > 0$ is equivalent to $\lambda > 1$, while $B \geqslant 0$ is equivalent to

$$\frac{s(u-\Delta)}{s(u)} \geqslant \frac{r(t+\delta)}{\lambda r(t)}.$$

□

PROPOSITION 4.4. *Assume that the operator ξ is quasimonotone. Then for all $t \in I$ and $u \in J$ (except the upper bounds of the intervals), one has*

$$\liminf_{\delta \downarrow 0} r(t+\delta) \leqslant r(t),$$
$$\liminf_{\Delta \downarrow 0} s(u+\Delta) \leqslant s(u).$$

Proof. We shall prove the statement for r. It is enough to prove that for any $k > 1$, there is a sequence $\{\delta_n\}$ of positive reals converging to 0 such that

$$\frac{r(t+\delta_n)}{r(t)} \leqslant k \quad \text{for all } n. \tag{4.8}$$

Let $a, b \in J$ such that $a < b$ and $\lambda \in (1, k)$. There exists $n_0 > 0$ such that

$$\left(\frac{k}{\lambda}\right)^{n_0} > \frac{s(a)}{s(b)}.$$

For $n > n_0$, set $\Delta_n = (b-a)/n$ and for $i = 0, 1, \ldots, n$; $u_i^n = a + i\Delta_n$. Then we have:

$$\prod_{i=1}^{n} \frac{s(u_{i-1}^n)}{s(u_i^n)} = \frac{s(a)}{s(b)} < \left(\frac{k}{\lambda}\right)^n. \tag{4.9}$$

We determine $\bar{u}_n$ by the following procedure:

(a) Start with $i = n$,

(b) If $s(u_i^n) \geqslant s(u_{i-1}^n)$, then take $\bar{u}_n = u_i^n$ (then $s(\bar{u}_n - \Delta_n)/s(\bar{u}_n) \leqslant 1$). Stop.

- If not, we have $s(u_{j-1}^n) > s(u_j^n) \geqslant s(b)$ for $j = i, i+1, \ldots, n$.
- If $i > 1$ set $i = i - 1$ and go to (b).
- If not, $i = 1$ and $s(u_0^n) > s(u_1^n) > \cdots > s(u_n^n) = s(b)$.

It results from (4.9) that $i \geqslant 1$ exists so that

$$\frac{s(u_{i-1}^n)}{s(u_i^n)} < \frac{k}{\lambda}.$$

Take $\bar{u}_n = u_i^n$.
End of the procedure.

In all cases, we have

$$\frac{s(\bar{u}_n - \Delta_n)}{s(\bar{u}_n)} < \frac{k}{\lambda} \quad \text{and} \quad s(\bar{u}_n) \geqslant s(b).$$

Take $\delta_n = \lambda r(t)\Delta_n / s(\bar{u}_n)$. It is clear that $\{\delta_n\}$ converges to 0. Expression (4.7) implies that:

$$\frac{k}{\lambda} > \frac{s(\bar{u}_n - \Delta_n)}{s(\bar{u}_n)} \geqslant \frac{r(t+\delta_n)}{\lambda r(t)}, \quad \text{for all } n \geqslant n_0$$

from what (4.8) follows. □

PROPOSITION 4.5. *Assume that the operator ξ is quasimonotone. Then for all $t \in I$ and $u \in J$ (except the lower bounds of the intervals), one has*

$$\liminf_{\delta \downarrow 0} r(t-\delta) \geqslant r(t)$$

and

$$\liminf_{\Delta \downarrow 0} s(u - \Delta) \geqslant s(u).$$

Proof. We shall prove the statement for s. Firstly, we prove the existence of some $\bar{t} \in \operatorname{int}(I)$ such that

$$\limsup_{t \to \bar{t}} \theta(t) < +\infty. \tag{4.10}$$

Let $t \in \operatorname{int}(I), u$ and Δ be such that $\Delta > 0$ and $u, u + \Delta \in J$. Take $\delta > 0$ small enough in order to have $\mu(u)\Delta > \theta(t)\delta$. Let $\{\delta_n\}$ any sequence of positive reals converging to δ. Since ξ is quasimonotone one has for n large enough $\mu(u + \Delta)\Delta \geqslant \theta(t - \delta_n)\delta_n$. Then

$$\limsup_{t' \to (t-\delta)} \theta(t') \leqslant \frac{\mu(u + \Delta)\Delta}{\delta}.$$

Take $\bar{t} = t - \delta$.

Assume now for contradiction that

$$s(u) > \liminf_{\Delta \downarrow 0} s(u - \Delta).$$

Then there exist $k_1 \in (0, 1)$ and a decreasing sequence $\{\Delta_n\}$ of positive reals converging to 0 such that

$$\frac{s(u - \Delta_n)}{s(u)} \leqslant k_1.$$

Take λ, k_2 and n_0 such that

$$\lambda > 1, \quad k_2 = \lambda k_1 < 1 \quad \text{and} \quad \bar{t} + \frac{\lambda \Delta_{n_0} r(\bar{t})}{s(u)(1 - k_2)} \in I.$$

Then for $n \geqslant n_0$ we construct a sequence $\{t_i^n\}_i$ by

$$t_0^n = \bar{t} \quad \text{and for } i \geqslant 0, \qquad t_{i+1}^n = t_i^n + \frac{\lambda \Delta_n r(t_i^n)}{s(u)}.$$

The sequence is well defined. Indeed, it can be proved by induction that for all i one has

$$\bar{t} \leqslant t_{i+1}^n \leqslant \bar{t} + \frac{\lambda \Delta_n r(\bar{t})}{s(u)} \left(1 + k_2 + \cdots + k_2^i\right) \leqslant \bar{t} + \frac{\lambda \Delta_{n_0} r(\bar{t})}{s(u)(1 - k_2)}$$

and

$$r(t_{i+1}^n) \leqslant k_2 r(t_i^n) \leqslant k_2^{i+1} r(\bar{t}).$$

To see this, notice that

$$\lambda_i^n = \frac{\delta_i^n s(u)}{\Delta_n r(t_i^n)} = \lambda.$$

Then by Proposition 4.3, one has

$$k_1 \geqslant \frac{s(u - \Delta_n)}{s(u)} \geqslant \frac{r(t_i^n + \delta_i^n)}{\lambda r(t_i^n)}.$$

From which the result follows.

Now we construct a sequence $\{t_n\}$ by taking $t_n = t_n^n$. Then $\{t_n\}$ converges to $\bar{t}$ and $\{r(t_n)\}$ to 0 in contradiction to (4.10). □

Recall that the upper and lower *Dini derivatives* of a function f at x_0 following a direction $h \in E$ are defined as

$$f'_+(x_0, h) = \limsup_{\lambda \downarrow 0} \frac{f(x_0 + \lambda h) - f(x_0)}{\lambda},$$

$$f'_-(x_0, h) = \liminf_{\lambda \downarrow 0} \frac{f(x_0 + \lambda h) - f(x_0)}{\lambda}.$$

The next proposition makes use of the famous Dini theorem.

DINI'S THEOREM (Theorem 7.2, Saks [10]). *If f is a finite function defined on I such that*

(i) $\limsup_{\delta\downarrow 0} f(t - \delta) \leqslant f(t) \leqslant \limsup_{\delta\downarrow 0} f(t + \delta)$ *at every $t \in I$.*
(ii) $f'_+(t, 1) \geqslant 0$ *at every point t except at most at those of an enumerable set.*

Then f is monotone nondecreasing.

PROPOSITION 4.6. *If ξ is quasimonotone on $I \times J$, then $m(\theta) + m(\mu) \geqslant 0$.*

Proof. Assume for contradiction that ξ is quasimonotone and $m(\theta) + m(\mu) < 0$. Without loss of generality, we assume that $m(\mu) < 0$. There exist k_1 and k_2 such that $0 < k_2 < k_1$, $m(\theta) < k_1 - k_2$ and $m(\mu) < -k_1$.

We consider the functions:

$$\nu(t) = -r(t) + (k_2 - k_1)t, \quad \text{for all } t \in I,$$

$$\chi(u) = -s(u) + k_1 u, \quad \text{for all } u \in J.$$

We know by Proposition 4.2 that θ and μ are pseudomonotone. Hence, Theorem 3.1 and the definition of monotonicity indices imply that ν and χ are not nondecreasing.

On the other hand, by Propositions 4.4 and 4.5, we have

$$\limsup_{\delta\downarrow 0} \nu(t - \delta) \leqslant \nu(t) \leqslant \limsup_{\delta\downarrow 0} \nu(t + \delta) \quad \text{at every } t \in \text{int}(I),$$

$$\limsup_{\delta\downarrow 0} \chi(u - \delta) \leqslant \chi(u) \leqslant \limsup_{\delta\downarrow 0} \chi(u + \delta) \quad \text{at every } u \in \text{int}(J).$$

Hence, according to Dini's theorem, $\bar{t} \in \text{int}(I)$ and $\bar{u} \in \text{int}(J)$ exist so that

$$\nu'_+(\bar{t}, 1) < 0 \quad \text{and} \quad \chi'_+(\bar{u}, 1) < 0.$$

Referring to the definitions of Dini derivatives, we deduce the existence of $\bar{\Delta}$ and $\bar{\delta}$ positive such that $\bar{t} + \bar{\delta} \in I, \bar{u} + \bar{\Delta} \in J$

$$\frac{r(\bar{t} + \delta) - r(\bar{t})}{\delta} > k_2 - k_1 \quad \text{for all } \delta \in (0, \bar{\delta})$$

and

$$\frac{s(\bar{u} + \Delta) - s(\bar{u})}{\Delta} > k_1 \quad \text{for all } \Delta \in (0, \bar{\Delta}).$$

Without loss of generality, we assume that

$$\bar{\Delta} \leqslant \frac{s(\bar{u})\bar{\delta}}{r(\bar{t})}.$$

Let $\Delta \in (0, \bar{\Delta})$, then

$$s(\bar{u} + \Delta) > s(\bar{u}) + k_1\Delta > s(\bar{u}) + k_2\Delta > s(\bar{u}).$$

Take

$$\lambda = \frac{s(\bar{u} + \Delta)}{s(\bar{u} + \Delta) - k_2\Delta}$$

and

$$\delta = \frac{\Delta r(\bar{t})\lambda}{s(\bar{u} + \Delta)} = \frac{\Delta r(\bar{t})}{s(\bar{u} + \Delta) - k_2\Delta},$$

then $\lambda > 1$ and $\delta \in (0, \bar{\delta})$. We have

$$\frac{s(\bar{u})}{s(\bar{u} + \Delta)} < 1 - \frac{k_1\Delta}{s(\bar{u} + \Delta)},$$
$$\frac{r(\bar{t} + \delta)}{\lambda r(\bar{t})} > \frac{1}{\lambda}(1 + \frac{(k_2 - k_1)\delta}{r(\bar{t})}) = 1 - \frac{k_1\Delta}{s(\bar{u} + \Delta)}.$$

On the other hand, by Proposition 4.3 and since $\lambda = \delta s(\bar{u} + \Delta)/\Delta r(\bar{t}) > 1$, we have

$$\frac{s(\bar{u})}{s(\bar{u} + \Delta)} \geqslant \frac{r(\bar{t} + \delta)}{\lambda r(\bar{t})}.$$

We have got a contradiction. □

Now, we can establish the necessary condition for the quasimonotonicity of the operator $H = (F, G)$.

THEOREM 4.7. *Assume that the interior of* $C(D)$ *is nonempty in* $X(Y)$, $F(G)$ *is nonnull on* $C(D)$ *and* $H = (F, G)$ *is quasimonotone on* $C \times D$. *Then* $m(F) + m(G) \geqslant 0$.

Proof. Assume for contradiction that $m(F) + m(G) < 0$. Since,

$$m(F) = \inf_{x_1, d_1} [m(F^+_{x_1,d_1}) \colon \; I^+_{x_1,d_1} \neq \emptyset],$$

$$m(G) = \inf_{x_2, d_2} [m(G^+_{x_2,d_2}) \colon \; I^+_{x_2,d_2} \neq \emptyset],$$

there exist $(x_1, d_1) \in C \times X$ and $(x_2, d_2) \in D \times Y$, such that

$$m(F^+_{x_1,d_1}) + m(G^+_{x_2,d_2}) < 0.$$

Then a selection θ of $F^+_{x_1,d_1}$ and a selection μ of $G^+_{x_2,d_2}$ exist so that

$$m(\theta) + m(\mu) < 0.$$

Consider $\xi = (\theta, \mu)$. Then, by Proposition 4.6, ξ is not quasimonotone, in contradiction with H quasimonotone. □

5. More than Two Factors

We consider now the general case

$$F(x_1, x_2, \dots, x_p) = (F_1(x_1), \; F_2(x_2), \dots, F_p(x_p))$$

defined on $C = C_1 \times C_2 \times \cdots \times C_p$ with $p \geqslant 2$ and for $i = 1, 2, \dots, p$, the set C_i is a convex set of E_i and F_i is a multivalued operator with values in E_i'. We assume that F_i is nonnull on C_i and C_i has a nonempty interior.

Assume that F is quasimonotone, Theorem 4.7 implies that when $p = 2$, at least one of the operators is monotone. Hence, it is easily obtained that for $p \geqslant 2$ all the operators except perhaps one are monotone. Necessary and sufficient conditions for the quasimonotonicity of H would be obtained from Theorems 4.1 and 4.7 by joining together the monotone factors. It remains to compute the monotonicity index of a product of monotone operators.

PROPOSITION 5.1. *Assume that for* $i = 1, 2, \dots, p$ $(p \geqslant 2), F_i$ *is monotone. Then* F *is monotone and*

$$\frac{1}{m(F)} = \frac{1}{m(F_1)} + \frac{1}{m(F_2)} + \cdots + \frac{1}{m(F_p)},$$

with the convention $1/0 = +\infty$.

Proof. It is clear that F is monotone. Hence, $m(F)$ is nonnegative. Let us prove the formula for two factors, then the general formula will be deduced by induction. Consider the operator $F = (F_1, F_2)$ when F_1 and F_2 are monotone. Then F is monotone and $m(F) \geqslant 0$. If $m(F_1) = 0$ ($m(F_2) = 0$), then $m(F) = 0$ and the inequality holds. On the other hand, F_1 and F_2 are nonnull, so that we assume henceforth that

$$0 < m(F_1) < +\infty \quad \text{and} \quad 0 < m(F_2) < +\infty.$$

Let any

$$(x, d) \in (C_1 \times C_2) \times (E_1 \times E_2), \quad x = (x_1, x_2), \quad d = (d_1, d_2),$$

and

$$F_{1(x_1,d_1)}(t) = \langle F_1(x_1 + td_1), d_1 \rangle, \quad t \in I_{x_1,d_1},$$
$$F_{2(x_2,d_2)}(t) = \langle F_2(x_2 + td_2), d_2 \rangle, \quad t \in I_{x_2,d_2}.$$

Then

$$\begin{aligned} F_{(x,d)}(t) &= \langle F(x + td), d \rangle \\ &= F_{1(x_1,d_1)}(t) + F_{2(x_2,d_2)}(t), \quad t \in I_{x,d} = I_{x_1,d_1} \cap I_{x_2,d_2}. \end{aligned}$$

Since $F_{(x,d)}, F_{1(x_1,d_1)}$ and $F_{2(x_2,d_2)}$ are monotone, we can define $f_{(x,d)}$, $f_{1(x_1,d_1)}$ and $f_{2(x_2,d_2)}$

$$f_{(x,d)}(t) = \int_0^t F_{(x,d)}(u)\,du, \quad t \in I_{x,d},$$
$$f_{1(x_1,d_1)}(t) = \int_0^t F_{1(x_1,d_1)}(u)\,du, \quad t \in I_{x_1,d_1},$$
$$f_{2(x_2,d_2)}(t) = \int_0^t F_{2(x_2,d_2)}(u)\,du, \quad t \in I_{x_2,d_2}.$$

It is clear that $f_{x,d}(t) = f_{1(x_1,d_1)}(t) + f_{2(x_2,d_2)}(t)$.

By Theorem 3.3, we have

$$c(f_{(x,d)}) = m(F_{(x,d)}), \qquad c(f_{1(x_1,d_1)}) = m(F_{1(x_1,d_1)})$$

and

$$c(f_{2(x_2,d_2)}) = m(F_{2(x_2,d_2)}).$$

The same argument as used for the proof of Theorem 5.1 [2] gives the formula:

$$\frac{1}{m(F)} = \frac{1}{m(F_1)} + \frac{1}{m(F_2)}.$$

An immediate induction generalizes the formula to more than two factors. □

Our main theorem which generalizes Theorem 6.1 of [2] is as follows.

THEOREM 5.2. (i) *If F is quasimonotone on C, then one of the following conditions holds:*

(a) *all F_i are monotone.*
(b) *all F_i except one are monotone and*

$$\frac{1}{m(F_1)}+\frac{1}{m(F_2)}+\cdots+\frac{1}{m(F_p)}\leqslant 0.$$

(ii) *If one of the conditions* (a) *or* (b) *holds, then F is pseudomonotone and*

$$\frac{1}{m(F)}=\frac{1}{m(F_1)}+\frac{1}{m(F_2)}+\cdots+\frac{1}{m(F_p)}. \tag{5.1}$$

Proof. Assertion (i) is a direct consequence of Proposition 5.1 and Theorem 4.7. Conversely, if condition (a) holds, then F is monotone and therefore pseudomonotone and formula (5.1) follows from Proposition 5.1. Assume that condition (b) holds. Then the pseudomonotonicity of F follows from Proposition 5.1 and Theorem 4.1. It remains to prove formula (5.1) when one of the factors is not monotone. For $k<0$, let the operator G_k be defined on $I=(0,+\infty)$ by

$$G_k(t)=\frac{1}{kt}\,.$$

Then G_k is monotone and $m(G_k)=-k$.

Define now on $C_1\times C_2\times\cdots\times C_p\times I$ the operator

$$H_k(x_1,x_2,\ldots,x_p,t)=(F_1(x_1),F_2(x_2),\ldots,F_p(x_p),G_k(t)).$$

Then by the first part of this theorem, H_k is quasimonotone (i.e. F is k-monotone) if and only if

$$\frac{1}{m(F_1)}+\frac{1}{m(F_2)}+\cdots+\frac{1}{m(F_p)}-\frac{1}{k}\leqslant 0.$$

Then (5.1) follows from the definition of $m(F)$. □

Theorem 5.2 suggests an improvement the results obtained by Debreu and Koopmans and Crouzeix and Lindberg for the generalized convexity of a separable sum of functions. But first, we recall the definition of pseudoconvexity for non differentiable functions. Let C be an open convex set of E, $f\colon C\to\mathbb{R}$ such that f admits directional derivatives at any $a\in C$. Then f is said to be *pseudoconvex* if

$$f'(a,x-a)<0 \quad \text{whenever } x\in C, a\in C \quad \text{and} \quad f(x)<f(a).$$

It is easy to see that f is pseudoconvex when $c(f)>-\infty$.

Assume that $p \geqslant 2$ and for $i = 1, 2, \ldots, p$ C_i is an open convex subset of a linear space E_i and f_i: $C_i \to \mathbb{R}$ is not constant. We consider s to be defined on $C = C_1 \times C_2 \times \cdots \times C_p$ by

$$s(x_1, x_2, \ldots, x_p) = f_1(x_1) + f_2(x_2) + \cdots + f_p(x_p).$$

We have

THEOREM 5.3. (i) *If s is quasiconvex on C, then one of the following conditions holds,*

(a) *all f_i are convex;*
(b) *all f_i except one are convex and*

$$\frac{1}{c(f_1)} + \frac{1}{c(f_2)} + \cdots + \frac{1}{c(f_p)} \leqslant 0.$$

(ii) *If one of the conditions* (a) *or* (b) *holds, then s is pseudoconvex and*

$$\frac{1}{c(s)} = \frac{1}{c(f_1)} + \frac{1}{c(f_2)} + \cdots + \frac{1}{c(f_p)}.$$

Proof. This theorem was proved by Crouzeix and Lindberg [3] when E_i has a finite dimension and with quasiconvex instead of the pseudoconvex in (ii). Actually, the proof does not involve the dimension of the spaces E_i so that it suffices to prove that s is pseudoconvex when (a) or (b) holds.

It suffices to consider two factors. An immediate induction would extend the result to the general case.

If both functions f_1 and f_2 are convex, then s is convex and therefore pseudoconvex. If one of them is not convex, say f_2, the other one, f_1, is convex and $c(f_1) + c(f_2) \geqslant 0$.

If $c(f_1) + c(f_2) > 0$, then $c(s) > -\infty$, s is $-c(s)$-convex and therefore pseudoconvex.

We are left with $c(f_1) + c(f_2) = 0$.

Let $(x, d) \in (C_1 \times C_2) \times (E_1 \times E_2)$ with $s(x) > s(x+d)$ and $x+d \in (C_1 \times C_2)$. We must prove that

$$s'(x, d) = f_1'(x_1, d_1) + f_2'(x_2, d_2) < 0.$$

The directional derivatives of f_1 and f_2 exist since $c(f_1)$ and $c(f_2)$ are finite. Since $c(f_1)$ is positive, the function $x_1 \longmapsto \exp(-c(f_1) f_1(x_1))$ is concave. Hence,

$$\exp(-c(f_1) f_1(x_1 + d_1)) \leqslant \exp(-c(f_1) f_1(x_1)) f_1'(x_1, d_1),$$

from what we deduce

$$\exp(-c(f_1)(f_1(x_1 + d_1) - f_1(x_1))) \leqslant 1 - c(f_1) f_1'(x_1, d_1),$$
$$-c(f_1)(f_1(x_1 + d_1) - f_1(x_1)) \leqslant \ln(1 - c(f_1) f_1'(x_1, d_1)), \tag{5.2}$$

and

$$f_1(x_1 + d_1) - f_1(x_1) \geqslant \frac{-1}{c(f_1)} \ln(1 - c(f_1) f_1'(x_1, d_1)).$$

Since $c(f_2)$ is negative, the function $x_2 \longmapsto \exp(-c(f_2) f_2(x_2))$ is convex. Then we have:

$$\begin{aligned} &\exp(-c(f_2)(f_2(x_2 + d_2) - f_2(x_2))) \geqslant (1 - c(f_2) f_2'(x_2, d_2)), \\ &-c(f_2)(f_2(x_2 + d_2) - f_2(x_2))) \geqslant \ln(1 - c(f_2) f_2'(x_2, d_2)), \end{aligned} \tag{5.3}$$

and

$$f_2(x_2 + d_2) - f_2(x_2) \geqslant \frac{-1}{c(f_2)} \ln(1 - c(f_2) f_2'(x_2, d_2)).$$

Recall that $c(f_1) = -c(f_2)$. Add (5.2) and (5.3) then

$$\begin{aligned} &f_1(x_1 + d_1) + f_2(x_2 + d_2) - f_1(x_1) - f_2(x_2) \\ &\geqslant \frac{1}{c(f_1)} \ln \frac{1 + c(f_1) f_2'(x_2, d_2)}{1 - c(f_1) f_1'(x_1, d_1)}. \end{aligned} \tag{5.4}$$

Recall that $s(x_1 + d_1, x_2 + d_2) < s(x_1, x_2)$. Hence,

$$\frac{1}{c(f_1)} \ln \frac{1 + c(f_1) f_2'(x_2, d_2)}{1 - c(f_1) f_1'(x_1, d_1)} < 0,$$

from which we deduce that

$$s'(x, d) = f_1'(x_1, d_1) + f_2'(x_2, d_2) < 0.$$

□

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