Z. Wahrscheinlichkeitstheorie verw. Gebiete
57, 509–534 (1981)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Central Limit Theorems for Dependent Variables. I.

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Summary. This paper gives a flexible approach to proving the Central Limit Theorem (C.L.T.) for triangular arrays of dependent random variables (r.v.s) which satisfy a weak 'mixing' condition called *ℓ-mixing*. Roughly speaking, an array of real r.v.s is said to be ℓ-mixing if linear combinations of its 'past' and 'future' are asymptotically independent. All the usual mixing conditions (such as strong mixing, absolute regularity, uniform mixing, ρ-mixing and ψ-mixing) are special cases of ℓ-mixing. Linear processes are shown to be ℓ-mixing under weak conditions. The main result makes no assumption of stationarity. A secondary result generalises a C.L.T. that Rosenblatt gave for strong mixing samples which are 'nearly second order stationary'.

§ 1. Summary

In this paper we consider conditions for the C.L.T. to hold for the sum

$$S_N = \sum_{j=1}^{n_N} X_{jN}$$

where $\{X_{jN}\colon j=1, \ldots, n_N,\ N=1, 2, \ldots\}$ is a triangular array of dependent real r.v.s. That is, we consider conditions under which

$$(S_N - ES_N)/(\operatorname{var}(S_N))^{\frac{1}{2}} \xrightarrow{\mathscr{L}} \mathscr{N}(0,1) \qquad \text{as } N \to \infty.$$

Our basic condition, *ℓ-mixing*, is defined at the beginning of § 2. There are many mixing approaches in use in the literature, such as α-mixing (strong mixing), β-mixing (absolute regularity), ϕ-mixing (uniform strong mixing), ρ-mixing and ψ-mixing. (See for example Theorem 8.3 of Billingsley (1971), Theorems 18.5.1–18.5.4 of Ibragimov and Linnik (1971), Theorems 2.1 and 2.2 of Ibragimov (1975) and Corollary 1 of Yoshihara (1978).) Like these conditions, ℓ-mixing requires the asymptotic decoupling of the 'past' and the

0044-3719/81/0057/0509/$05.20

'future'. However, unlike these conditions, ℓ-mixing does not require invariance under transformations of the form

$$\{X_{iN}\} \to \{f_{iN}(X_{iN})\} \quad \text{where } \{f_{iN}\} \text{ are one to one functions,}$$

so that ℓ-mixing is more widely applicable.

Our main C.L.T. is Theorem 2.1 and the special case of Corollary 2.1. Variants of these results are given in Theorems 2.2 and 2.3 for processes which are 'nearly second order stationary'. Theorem 2.1 requires a moment inequality, conditions for which are given in Propositions 2.1, 2.2 and 2.3. In particular Theorem 2.1 requires that for some $\varepsilon > 0$ the $2+\varepsilon$ absolute moment of S_N be finite, whereas Theorems 2.2 and 2.3 do not.

The theorems also require that $\operatorname{var}(S_N)/n_N$ be bounded above or below, or be 'slowly varying' or 'very slowly varying' in n_N. These conditions are shown to hold in §4 under weak conditions called *r-mixing*.

Conditions for a linear process to be ℓ-mixing are considered in §5 and compared with those required for α-mixing (strong mixing). In particular a class of first-order autoregressive processes is exhibited which is ℓ-mixing but not α-mixing nor β-mixing (absolutely regular).

Finally in §6 we briefly consider the relationship of ℓ-mixing, r-mixing and its variants, and α-mixing – in particular for Gaussian processes.

The notions of ℓ-mixing and the results given in this paper may be extended to 'spatial arrays' $\{X_{jN}\}$ where j is now an integer vector – as has been done for ϕ-mixing, for example, by Deo (1976).

We now introduce some notation that we shall require. In particular we extend the standard mixing concepts to non-stationary triangular arrays; c.f. Withers (1975).

Consider a series of random processes

$$(1.1) \qquad \mathscr{X} = \{\mathscr{X}_N,\ N \geqq 1\} \quad \text{where } \mathscr{X}_N = \{X_{jN},\ m_N \leqq j \leqq n_N\},$$

not necessarily real, defined on some probability space $(\Omega, \mathscr{A}, P)$, with m_N, n_N integers such that

$$-\infty \leqq m_N < n_N \leqq \infty \quad \text{and} \quad n_N - m_N \to \infty \quad \text{as } N \to \infty.$$

For any (not necessarily real) r.v.s Y, Z we set

$$\mathscr{M}(Y) = \text{the } \sigma\text{-algebra generated by } Y$$

(1.2) $\alpha(Y,Z) = \sup |P(A \cap B) - P(A)P(B)|$, where sup is over all A in $\mathscr{M}(Y)$ and all B in $\mathscr{M}(Z)$,

(1.3) $\beta(Y,Z) = E \sup |P(B \mid Y) - P(B)|$, where sup is over all B in $\mathscr{M}(Z)$,

(1.4) $\phi(Y,Z) = \sup |P(B \mid A) - P(B)|$, where sup is over A, B as in (1.2),

(1.5) $\rho(Y,Z) = \sup |\mathit{correlation}\,(y,z)|$, where sup is over $\mathscr{M}(Y)$-measurable real r.v.s, y, and $\mathscr{M}(Z)$-measurable real r.v.s, z, with finite variance.

Let $'$ denote the transpose of a vector.

For $Y=(Y_1, \ldots, Y_p)' \in R^p$, $Z=(Z_1, \ldots, Z_q)' \in R^q$ define

(1.6) $r(Y, Z)=\sup$ *correlation* $(a' Y, b' Z)$, where sup is over all $\alpha \in R^p$, $b \in R^q$ such that the correlation is well-defined,

(1.7) $r^*(Y, Z)=\sup |$*correlation* $(a' Y, b' Z)|$, where sup is over $a \in R^p$, $b \in R^q$ with components 0 or 1,

and

(1.8) $r^{**}(Y, Z)=\left|\text{correlation}\left(\sum_1^p Y_j, \sum_1^q Z_j\right)\right|.$

Thus r, r^* and $r^{**}(Y, Z)$ depend only on covar $\begin{pmatrix} Y \\ Z \end{pmatrix}$.

Now define

$$(1.9) \qquad \alpha_N(k)=\max_{m_N \leqq j \leqq n_N-k} \alpha(\{X_{m_N N}, \ldots, X_{jN}\}, \{X_{j+k, N}, \ldots, X_{n_N N}\}), \qquad 0 \leqq k \leqq n_N-m_N,$$

$$(1.10) \qquad \alpha(k)=\max_{\{N: k \leqq n_N-m_N\}} \alpha_N(k), \quad 0 \leqq k<\infty$$

and analogously define

$$(1.11) \qquad \beta_N(k),\ \beta(k),\ \phi_N(k),\ \phi(k),\ \rho_N(k),\ \rho(k),\ r_N(k),\ r(k),\ r_N^*(k),\ r^*(k),\ r_N^{**}(k), \text{ and } r^{**}(k).$$

The coefficients $\alpha(k)$, $\beta(k)$, $\phi(k)$, $\rho(k)$ and $r(k)$ are called *the k-th strong-mixing, absolutely regular, uniform mixing maximal correlation*, and *maximal linear correlation coefficients* respectively, of $\mathscr{X}$.

Similarly, $\alpha_N(k), \ldots, r_N(k)$ are called *the k-th strong-mixing*, ..., and *maximal linear correlation coefficients of* $\mathscr{X}_N$. When $\{X_{jN} \equiv X_j\}$ is second-order stationary, define

$$r^{***}(k)=\limsup_{a, b \to \infty}\left|\text{correlation}\left(\sum_1^a X_j, \sum_{k+a}^{k+a+b} X_j\right)\right|, \quad 0 \leqq k<\infty,$$

so that $r^{***}(k) \leqq r^{**}(k)$.

When $\{X_{jN} \equiv X_j\}$ is stationary, the definitions of $\alpha(k)$, $\beta(k)$, $\phi(k)$, $\rho(k)$, and $r(k)$ do not depend on the choice of m_N or n_N: $-m_N=n_N=\infty$ yields the definitions of α, ϕ, ρ in Ibragimov and Linnik (1971) and Ibragimov (1975), while $m_N=1$, $n_N=\infty$ yield the definition of ϕ on p. 26 of Billingsley (1971).

It is well known that

$$(1.12) \qquad 4 \alpha(Y, Z) \leqq \rho(Y, Z) \leqq 2 \phi(Y, Z)^{\frac{1}{2}}$$

and

$$(1.13) \qquad 4 \alpha_N(k) \leqq \rho_N(k) \leqq 2 \phi_N(k)^{\frac{1}{2}} \quad \text{and} \quad 4 \alpha(k) \leqq \rho(k) \leqq 2 \phi(k)^{\frac{1}{2}}.$$

Also

$$\beta(Y,Z)\leqq\phi(Y,Z),\quad \beta_N(k)\leqq\phi_N(k),\quad \beta(k)\leqq\phi(k). \tag{1.14}$$

Clearly, for Y,Z real r.v.s

$$r^{**}(Y,Z)\leqq r^*(Y,Z)\leqq r(Y,Z)\leqq\rho(Y,Z)\leqq 1 \tag{1.15}$$

so that $0\leqq r^{**}(k)\leqq r^*(k)\leqq r(k)\leqq\rho(k)\leqq 1$.

For (Y,Z) Gaussian, by Theorems 1, 2 of Kolmogorov and Rozanov (1960), $\rho(Y,Z)=r(Y,Z)$ and $4\alpha(Y,Z)\leqq r(Y,Z)\leqq\sin\{2\pi\alpha(Y,Z)\}\leqq 2\pi\alpha(Y,Z)$.

Hence if $\mathscr{X}_N$ is Gaussian, the analogous statements hold for $\{\rho_N(k),\ r_N(k),\ \alpha_N(k)\}$ and for $\{\rho(k),\ r(k),\ \alpha(k)\}$. (Their proofs do not require stationarity.)

Unlike $\{\alpha(k),\ \beta(k),\ \phi(k),\ \rho(k)\}$, the coefficients $\{r(k),\ r^*(k),\ r^{**}(k)\}$ are *not* invariant to transformations of type $\{X_{jN}\}\to\{f_{jN}(X_{jN})\}$ where $\{f_{jN}\}$ are one to one measurable functions. Thus a condition on $\{r(k)\}$ will generally be much weaker than the same condition on $\{\rho(k)\}$. Processes, such as Gaussian ones, for which $r(k)$, $\rho(k)$ and $\alpha(k)$ are equivalent, are exceptional.

The set of processes $\mathscr{X}$ is said to be *α-mixing* (or strong mixing) if $\alpha(k)\to 0$ as $k\to\infty$, *β-mixing* (or absolutely regular) if $\beta(k)\to 0$ as $k\to\infty$, and *ϕ-mixing* (or uniformly strong mixing) if $\phi(k)\to 0$ as $k\to\infty$. Likewise *ρ-mixing, r-mixing, r^*-mixing, r^{**}-mixing* and *r^{***}-mixing* may be defined.

One may also define 'complex' versions of ρ and r. In particular, set

$$\rho_c(Y,Z)=\sup|E\,y\bar{z}| \quad \text{where sup is over all } \mathscr{M}(Y)\text{-measurable}$$

complex r.v.s. y and $\mathscr{M}(Z)$-measurable complex r.v.s. z satisfying

$$E\,y=E\,z=0,\qquad E\,|y|^2=E\,|z|^2=1.$$

In §6 we shall need

Theorem 1.1. $\rho_c(Y,Z)=\rho(Y,Z)$.

The proof follows from

Lemma 1.1. *Let $\mathscr{H}$ be a real Hilbert space with inner product $\langle .,.\rangle$ and norm $\|.\|$. Let $A\colon\mathscr{H}\to\mathscr{H}$ be a bounded linear operator. Let $\mathscr{H}_c,\langle .,.\rangle_c, A_c$ be their complexifications. Then the norm of A satisfies $\|A\|=\sup|\langle f,g\rangle|$ where* sup *is over f,g in $\mathscr{H}$ satisfying $\|f\|=\|g\|=1$. The analogous result holds for $\|A_c\|_c$, the norm of A_c in $\mathscr{H}_c$. Moreover, $\|A\|=\|A_c\|_c$.*

Proof. The proof is straightforward but tedious. □

As usual, $[x]$ will denote the integral part of x. For X a real r.v. with finite mean, set

$$\|X\|_p=\begin{cases}(E\,|X-E\,X|^p)^{1/p}, & 1\leqq p<\infty,\\ \text{ess sup}\,|X-E\,X|, & p=\infty.\end{cases}$$

§ 2. C.L.T.s for ℓ-Mixing Arrays

Let $\mathscr{X}$ be a triangular array, as in (1.1), whose elements are real with finite means. Set

$$S_N = \sum_{j=m_N}^{n_N} X_{jN}, \qquad \sigma_N^2 = \operatorname{var}(S_N). \tag{2.1}$$

To introduce the notion of ℓ-mixing, set

$$\ell_N(k, u) = \max_{m_N \leqq j \leqq n_N - k} \sup |\operatorname{covar}(e^{iuP}, e^{-iuF})|$$

for u real, $0 \leqq k \leqq n_N - m_N$, $N \geqq 1$, where

$$P = \sigma_N^{-1} \sum_{\ell=m_N}^{j} \delta_\ell X_{\ell N}, \quad F = \sigma_N^{-1} \sum_{\ell=j+k}^{n_N} \delta_\ell X_{\ell N}, \quad \text{sup is over } \{\delta_j = 0 \text{ or } 1\}$$

and by the covariance of complex r.v.s is meant

$$\operatorname{covar}(Y, Z) = E Y\bar{Z} - E Y \overline{E Z}.$$

Now set

$$\ell(k, u) = \sup_{\{N : k \leqq n_N - m_N\}} \ell_N(k, u), \qquad 0 \leqq k < \infty, \ u \text{ real}. \tag{2.2}$$

Definition 2.1. The triangular array $\mathscr{X}$ is said to be *ℓ-mixing* if for all real u, $\ell(k, u) \to 0$ as $k \to \infty$.

Roughly speaking, this means that all zero-one linear combinations of the 'past' and the 'future' observations are asymptotically independent.

Definition 2.2. $\mathscr{X}$ is said to be *strongly ℓ-mixing* if for all u there exists $K(u) < \infty$ such that $\ell(k, u) \leqq \ell(k) K(u)$ where $\ell(k) \to 0$ as $k \to \infty$.

For example, if there exists $\theta(k) \downarrow 0$ as $k \uparrow \infty$ such that for all u, $\ell(k, u) = O(\theta(k))$ as $k \to \infty$, then $\mathscr{X}$ is strongly ℓ-mixing with $\ell(k) = \theta(k)$.

One possible choice is

$$\ell(k) = \alpha(k).$$

This is because $\ell_N(k, u) \leqq 16 \alpha_N(k)$ and $\ell(k, u) \leqq 16 \alpha(k)$ as follows from p. 307 of Ibragimov and Linnik (1971).

Another possible choice is

$$\ell(k) = \beta(k).$$

This is because by Lemma 1 of Yoshihara (1978),

$$\ell_N(k, u) \leqq 4 \beta_N(k) \quad \text{and} \quad \ell(k, u) \leqq 4 \beta(k).$$

Thus if a process is α-mixing or β-mixing or ρ-mixing or ϕ-mixing or ψ-mixing, then it is strongly ℓ-mixing.

Under certain regularity conditions

$$\ell_N(k, u) u^{-2} \to r_N^*(k) \qquad \text{as } u \to 0.$$

If in fact as $u \to 0$ $\ell(k, u) u^{-2}/r^*(k) \to 1$ uniformly in k, then one may choose

$$\ell(k) = r^*(k).$$

In §6 it will be shown that this choice is possible for Gaussian processes. These considerations suggest that this choice is possible for a variety of processes.

Henceforth we assume $m_N \equiv 1$ and suppress the dependence of n_N on N. Define

$$S_N(a, b) = \sum_{j=a+1}^{a+b} (X_{jN} - E X_{jN}), \qquad 0 \leqq a, \ 1 \leqq b \leqq n-a,$$

$$\tilde{c}_N(k) = \sup |\operatorname{covar}(X_{\ell N}, X_{mN})|, \qquad 0 \leqq k < n,$$

where sup is over $\{\ell, m: |\ell - m| \geqq k, \ 1 \leqq \ell \leqq n, \ 1 \leqq m \leqq n\}$,

$$C_N(k) = \sum_{j=k}^{n-1} \tilde{c}_N(j),$$

and

$$\tilde{c}(k) = \max_{\{N: k<n\}} \tilde{c}_N(k).$$

Thus

$$\sigma_N^2/n \leqq 2\, C_N(0) \leqq 2 \sum_0^\infty \tilde{c}(k).$$

Theorem 2.1. *The following conditions are sufficient for the C.L.T. to hold.*

For some $\varepsilon > 0$ and $\gamma \geqq 0$, $\mathscr{X}$ satisfies the moment inequality

$$\sup_{a,N} E\,|S_N(a, b)|^{2+\varepsilon} = O(b^{1+\varepsilon/2+\gamma}) \qquad \text{as } b \to \infty; \tag{2.3}$$

$\mathscr{X}$ is ℓ-mixing and for all real u

$$\ell(k, u) = o(k^{-\theta}) \qquad \text{as } k \to \infty, \quad \textit{where } \theta = 2\gamma/\varepsilon;$$

either (A): $\sigma_N^2 \to \infty$ as $N \to \infty$ and $\sum_0^\infty \tilde{c}(j) < \infty$

or (B): $1 + C_N(0) = O(\sigma_N^2/n)$ as $N \to \infty$.

Note 2.1. Under (A) σ_N^2/n may approach 0, but not ∞. Under (B) σ_N^2/n may approach ∞ (if $\gamma > 0$) but not 0, and must behave like $C_N(0)$.

The moment inequality (2.3) has been used by several authors; see for example Theorem 4.1 of Serfling (1968) and Theorem 5.3 of Dvoretzky (1972). Conditions for it to hold are given after Theorem 2.3. Putting $\gamma = 0$ in Theorem 2.1 yields the following result.

Corollary 2.1. *Suppose that $\mathscr{X}$ is ℓ-mixing and for some $\varepsilon > 0$,*

$$\sup_{a,N} \left\| \sum_{j=a+1}^{a+b} X_{jN} \right\|_{2+\varepsilon} = O(b^{\frac{1}{2}}) \qquad \text{as } b \to \infty,$$

and that (A) *or* (B) *of Theorem* 2.1 *hold.*

Then the C.L.T. holds.

The remaining C.L.T.s require $\mathscr{X}$ to be 'nearly second-order stationary'

Definition 2.3. $\mathscr{X}$ is said to be *nearly second-order stationary* (with respect to H) if for some function $H(.)$ of integral argument

(2.4) $$\sup_a |E S_N(a,b)^2/H(b)-1| \to 0 \quad \text{as } b\to\infty,\ N\to\infty.$$

For example, if $\{X_{jN}\equiv X_j\}$ is second-order stationary, then it is nearly second-order stationary w.r.t.

$$H(b)=\operatorname{var}\left(\sum_1^b X_j\right),$$

provided $H(b)>0$ for all b.

For the case $X_{jN}\equiv X_j$, this concept has been used by Rosenblatt (1956), Serfling (1968) and other authors.

The next theorem weakens the moment inequality of Corollary 2.1.

Theorem 2.2. *Suppose that $\mathscr{X}$ is ℓ-mixing, and nearly second-order stationary w.r.t. H. Suppose also that*

(2.5) $$\limsup_{N\to\infty,\, b\to\infty} \max_a \int_{|x|>\zeta} x^2\, dF_{abN}(x)\to 0 \quad \text{as } \zeta\to\infty$$

where $F_{abN}(x)=P(S_N(a,b)/\|S_N(a,b)\|_2\leqq x)$, and that (A) *or* (B) *of Theorem* 2.1 *holds.*

Then the C.L.T. is satisfied.

Note 2.2. For $\mathscr{X}$ nearly second-order stationary, (2.5) is equivalent to (2.5) with $\|S_N(a,b)\|_2$ replaced by $H(b)^{\frac{1}{2}}$, and is satisfied if for some $\varepsilon>0$

(2.6) $$\limsup_{N\to\infty,\, b\to\infty} \max_a E|S_N(a,b)|^{2+\varepsilon}/H(b)^{1+\varepsilon/2}<\infty,$$

in which $H(b)$ may be replaced by $ES_N(a,b)^2$.

The first part of the next result shows that if $\mathscr{X}$ is strongly ℓ-mixing, then σ_N^2/n must be *slowly varying* in n. Recall that $h(.)$ is said to be slowly varying if

$$\text{for}\quad \lambda>0,\quad h(\lambda x)/h(x)\to 1 \quad \text{as } x\to\infty.$$

If $h(x)$ is only defined for $x=1,2,\ldots$ – as is the case in this section – then x and λ in this definition are restricted to the values $1,2,\ldots$.

If the dependence among the observations is strong enough, the behaviour of S_N is generally non-Gaussian – as illustrated by Taqqu (1975). If the dependence is strong but 'not too strong', then an intermediate situation exists where the C.L.T. does not hold but *the modified C.L.T.* holds, that is there exists $\tau_N>0$ such that

$$(S_N-ES_N)/\tau_N \xrightarrow{\mathscr{L}} \mathscr{N}(0,1) \quad \text{as } N\to\infty.$$

In this case τ_N^2 is called a *normaliser*. If τ_N^2 and λ_N^2 are both normalisers, then

$$\tau_N^2/\lambda_N^2\to 1 \quad \text{as } N\to\infty.$$

Theorem 2.3. *Suppose that $\mathscr{X}$ is ℓ-mixing and nearly second-order stationary w.r.t. H, and that* (2.5) *holds, and*

either (I) $H(b)\to\infty$ as $b\to\infty$ and $\sup_{a\leqq b} H(a)/H(b)<\infty$,

or (II) *H is regularly varying of exponent $\theta>0$.*

Let $h(b)=H(b)/b, \quad b=1,2,\ldots.$

(a) *If $\mathscr{X}$ is strongly ℓ-mixing, then $h(.)$ is slowly varying and the modified C.L.T. holds.*

(b) *If $h(.)$ has an extension to* $(0,\infty)$ *which is slowly varying, then the C.L.T. holds.*

(c) *If $r^*(j)\to 0$ as $j\to\infty$ and it is possible to choose $\ell(j)\equiv r^*(j)$, then the C.L.T. holds.*

(d) *If for some $w>0$ $\ell(j,u)+r^*(j)=O(j^{-w})$ for u real, and* (II) *holds, then the C.L.T. holds.*

If $h(b)\equiv\sigma^2$ and $0<\sigma^2<\infty$ - as is true under the conditions of Theorems 4.1(b), 4.2(b) - then the extension to $(0,\infty)$ given by $h(x)\equiv\sigma^2$ is obviously slowly varying. Similarly the usually quoted examples of slowly varying functions of integral argument have slowly varying extensions on $(0,\infty)$.

Theorem 4.1(a) shows that for $\{X_{jN}\equiv X_j\}$ second-order stationary such an extension exists if $r^{***}(j)\to 0$ as $j\to\infty$.

Under (d), $h(.)$ is actually 'very slowly varying' in the sense given in Theorem 4.2(a).

The second part of Theorem 2.3(a) was proved by Rosenblatt (1956) for the case where $X_{jN}\equiv X_j$, $\mathscr{X}$ is α-mixing, $H(.)$ is non-decreasing, and (2.6) holds. (A misprint in the statement of his result is corrected by Blum and Rosenblatt (1956).)

Special cases of Theorem 2.3(b) with $h(b)\equiv\sigma^2$ are given by Theorem 5.3 of Longnecker and Serfling (1978) and Theorem 2.1 of Gastwirth and Rubin (1975) - both assuming $\{X_{jN}\equiv X_j\}$ are bounded, α-mixing and stationary, and by Corollary 1 of Yoshihara (1978) - assuming $\{X_{jN}\equiv X_j\}$ is β-mixing.

Various variations of these theorems are possible. For example, one may give conditions under which the modified C.L.T. holds but *not* the C.L.T. - or, in Theorem 2.1(B), one may allow $n/\sigma_N^2=O(n^\lambda)$ by modifying the other conditions.

We end this section by giving conditions for the moment inequality (2.3) to hold.

A slight modification of the proofs of Theorems 2.1, 3.1, 3.2, 3.3 of Serfling (1968) gives the following results.

Proposition 2.1. *Suppose that*

$$\max_{jN}\|X_{jN}\|_{2+\delta}<\infty, \qquad \text{where } 0<\delta\leqq\infty,$$

and

$$\max_{a,b,N} ES_N(a,b)^2/b<\infty.$$

(a) *For* $0 \leqq \varepsilon \leqq \delta < \infty$ *the moment inequality holds with* $\gamma = \varepsilon(\frac{1}{2} + \delta^{-1})$.

(b) *For* $0 < \varepsilon$, $\delta = \infty$ *the moment inequality holds with* $\gamma = \varepsilon/2$.

(c) *Let* $\mathscr{P}_N(a)$ *be any vector r.v. determining* $(X_{1N}, \ldots, X_{aN})$.

A necessary and sufficient condition for the moment inequality to hold with $\gamma = 0$ *is that for some* $\beta > 0$

$$\max_{a,N} E|E\{S_N(a,b)^2 | \mathscr{P}_N(a)\} - ES_N(a,b)^2|^{1+\beta} = O(b^{1+\beta}) \quad \text{as } b \to \infty. \tag{2.7}$$

In particular this condition holds if for some $\theta > 0$

$$\max_{a,N} E|E\{S_N(a,b)^2 | \mathscr{P}_N(a)\} - ES_N(a,b)^2| = O(b^{1-\theta}) \quad \text{as } b \to \infty. \tag{2.8}$$

Note 2.3. In fact (2.3) with $\gamma = 0$ implies (2.7) with $\beta = \varepsilon/2$; conversely (2.7) implies (2.3) with $\gamma = 0$ and $\varepsilon = \min(1, \delta, 2\beta)$. Also (2.8) $\Rightarrow$ (2.7) if $0 < \beta < \theta\delta/(2 + \theta + \delta)$.

Now consider the case where the sample fourth moments are bounded. For $0 \leqq k < n$ set

$$c_N^{(13)}(k) = \max_{1 \leqq a,\, b = a+k \leqq c \leqq d \leqq n} EX'_a X'_b X'_c X'_d$$

and

$$c_N^{(31)}(k) = \max_{1 \leqq a \leqq b \leqq c,\, d = c+k \leqq n} EX'_a X'_b X'_c X'_d$$

where $X'_j = X_{jN} - EX_{jN}$.

For $k = 0, 1, 2, \ldots$ set $c(k; 1, 3) = \sup_{\{N: k < n\}} (c_N^{(13)}(k) + c_N^{(31)}(k))$. Thus

$$\max_k c(k; 1, 3) \leqq 2 \max_{j,N} \|X_{jN}\|_4^4.$$

Proposition 2.2. *Suppose that* $\max_{jN} \|X_{jN}\|_4 < \infty$ *and*

$$\sum_0^b (k+1)\, c(k; 1, 3) = O(b^\gamma) \quad \text{as } b \to \infty \text{ where } \gamma \geqq 0. \tag{2.9}$$

Then the moment inequality (2.3) holds with $\varepsilon = 2$.

For example, (2.9) holds if $c(k; 1, 3) = O(k^{\gamma-2})$ and $\gamma > 0$.

Longnecker and Serfling (1978) have given a number of bounds for $E|S_N(a,b)|^{2+\varepsilon}$ for ε an even integer. These usually yield a weaker result than Proposition 2.2, but can be strengthened considerably by the following.

Lemma 2.1. *In Lemma* 4.1, *Corollary* 4.2 *and* (4.15) *of Longnecker and Serfling* (1978), $h = 0$. *Consequently in the righthand side of their Eqs.* (4.10), (4.18), (4.20), (4.22), (4.27), (4.35), (4.39), (4.44) *and* (4.47) *the exponent* $v/2$ *may be replaced by* 1.

It is also worth noting that their results remain valid with their b_i redefined as any number greater than or equal to EX_i^v. For example, with these changes their Corollary 4.6 now yields

Lemma 2.2. *Suppose for some positive even integer* v,

$$K_v = \max_{jN} \|X_{jN}\|_v < \infty.$$

Then for $q \geqq 1$, $ES_N(a, b)^v \leqq (v!\, \beta(b, q) + D_v) K_v^v b^{v/\theta}$, *where* D_v *is a finite constant depending only on* v,

$$\theta = \begin{cases} 2, & q = 1 \\ q, & 1 < q \leqq 3/2 \\ q(2q-2)^{-1}, & 3/2 \leqq q, \end{cases}$$

$$\beta(b, q) = \left\{ \sum_{k=1}^{b} k^{v/2-1} f_v(k)^q / (v/2-1)! \right\}^{1/q}$$

and $f_v(.)$ *is any function such that for* $1 \leqq i_1 < \ldots < i_v \leqq n$.

(2.10) $|EX'_{i_1} X'_{i_2} \ldots X'_{i_v}| \leqq \min\{f_v(i_2 - i_1), f_v(i_4 - i_3), \ldots, f_v(i_v - i_{v-1})\} K_v^v$

where $X'_i = X_{iN} - EX_{iN}$.

Applying this result with $v = 4$, $f_4(k) = c(k; 1, 3) K_4^{-4}$ and $q = 1$ yields an alternative proof of Proposition 2.2. Another immediate application is

Proposition 2.3. *Suppose that for some even integer* $v \geqq 4$ $f_v(.)$ *satisfies the inequality* (2.10), *and*

$$K_v = \max_{jN} \|X_{jN}\|_v < \infty.$$

(a) *If* $\sum_{k=1}^{\infty} k^{v/2} f_v(k) < \infty$ *then the moment inequality is satisfied with* $\gamma = 0$ *and* $\varepsilon = v - 2$.

(b) *If* $f_v(k) = O(k^{-w})$ *where* $0 < w < v/2$, *then the moment inequality is satisfied with* $\gamma = v/2 - w$ *and* $\varepsilon = v - 2$.

§ 3. Proofs of the C.L.T.s

We shall use Bernstein's decomposition for S_N:

$$S_N = S'_N + S''_N$$

where

$$S'_N = \sum_{j=0}^{k-1} \xi_j, \qquad S''_N = \sum_{j=0}^{k} \zeta_j,$$

$$\xi_j = \sum_{\ell = jp+jq+1}^{(j+1)p+jq} X_{\ell N}, \qquad 0 \leqq j \leqq k-1,$$

$$\zeta_j = \sum_{\ell = (j+1)p+jq+1}^{(j+1)(p+q)} X_{\ell N}, \qquad 0 \leqq j \leqq k-1,$$

$$\zeta_k = \sum_{kp+kq+1}^{n} X_{\ell N}$$

and p, q, k are non-negative integers depending on N *whose dependence on N is suppressed* such that $k(p+q)\leqq n$.

The key to the C.L.T.s of this paper is the following result (c.f. the proof of Theorem 18.4.1 of Ibragimov and Linnik (1971)).

Lemma 3.1. *Suppose that $EX_{jN}\equiv 0$. For u real and p, q, k as above, define*

$$f_N(j,u)=\operatorname{covar}\left(\prod_{s=0}^{j} Y_s, \overline{Y}_{j+1}\right) \quad \textit{where } Y_j=\exp(iu\sigma_N^{-1}\xi_j).$$

(a) *When*

$$\sigma_N^{-2} E S_N''^2 \to 0 \quad \textit{as } N\to\infty, \tag{3.1}$$

then the condition

$$\sigma_N^{-2}\sum_{j=0}^{k-1} E\xi_j^2 \to 1 \quad \textit{as } N\to\infty \tag{3.2}$$

is equivalent to

$$\sigma_N^{-2}\sum_{0\leqq s<j\leqq k-2} \operatorname{covar}(\xi_s,\xi_j)\to 0 \quad \textit{as } \to\infty. \tag{3.3}$$

(b) *Suppose that the conditions* (3.1) *and* (3.2) *are satisfied and that*

$$k\to\infty \quad \textit{as } N\to\infty \tag{3.4}$$

and

$$\sum_{j=0}^{k-2} |f_N(j,u)|\to 0 \quad \textit{as } N\to\infty \textit{ for all real } u. \tag{3.5}$$

Then the condition

$$\textit{for } \varepsilon>0, \sum_{j=0}^{k-1}\int_{|z|>\varepsilon} z^2\, dP(\xi_j/\sigma_N\leqq z)\to 0 \quad \textit{as } N\to\infty \tag{3.6}$$

holds if and only if both the C.L.T. holds and $\{\sigma_N^{-1}\xi_j\}$ are asymptotically negligible in the sense that

$$\textit{for } \varepsilon>0, \max_{j=0}^{k-1} P(|\xi_j|\geqq\varepsilon\sigma_N)\to 0 \quad \textit{as } N\to\infty.$$

(c) *If $\{p, q, k\}$ are allowed to depend on u as well as N, and* (3.1), (3.2), (3.4), (3.5), (3.6) *hold for all real u, then the C.L.T. holds.*

(d) *Condition* (3.2) *in (b) and (c) can be removed if σ_N is replaced throughout – including in the statement of the C.L.T. – by any τ_N independent of u, such that*

$$\textit{for all } u \quad \tau_N^2/\lambda_N^2\to 1 \quad \textit{as } N\to\infty, \textit{ where}$$

$$\lambda_N^2=\sum_{j=0}^{k-1} E\xi_j^2. \tag{3.7}$$

Proof. For $0 \leqq j \leqq k-1$ let P_j denote the probability measure generated by P treating $\{\xi_0, \ldots, \xi_j\}$, $\xi_{j+1}, \ldots, \xi_{k-1}$ as independent, and let E_j denote E under P_j.

(a) Condition (3.1) is equivalent to

$$\sigma_N^{-2} E S_N'^2 \to 1$$

under which condition (3.2) is equivalent to

$$(E - E_0) S_N'^2/(2\sigma_N^2) = \text{L.H.S. } (3.3) \to 0.$$

(b) By (3.1), $\sigma_N^{-1} S_N'' = o_p(1)$ so that the C.L.T. is equivalent to

$$\sigma_N^{-1} S_N' \xrightarrow{\mathscr{L}} \mathcal{N}(0, 1) \quad \text{as } N \to \infty. \tag{3.8}$$

For $W = \exp(iu S_N'/\sigma_N)$

$$\begin{aligned}(E - E_0) W &= \sum_0^{k-2} (E_{j+1} - E_j) W = \sum_0^{k-2} \operatorname{covar}_{P_{j+1}} \left(\prod_0^j Y_s, \prod_{j+1}^{k-1} \overline{Y}_s\right) \\ &= \sum_0^{k-2} \left(\prod_{j+2}^{k-1} E Y_s\right) \operatorname{covar} \left(\prod_0^j Y_s, \overline{Y}_{j+1}\right) \to 0 \quad \text{as } N \to \infty, \quad \text{by (3.5).}\end{aligned}$$

Thus (3.8) is equivalent to (3.8) under P_0, which with the asymptotic negligibility condition is equivalent to (3.6) by p. 103 of Gnedenko and Kolmogorov (1954).

(c) By (3.2) and (3.6), $\sigma_N^{-1} S_N' \xrightarrow{\mathscr{L}} \mathcal{N}(0, 1)$ as $N \to \infty$ under P_0, so that $E_0 W \to \exp(-u^2/2)$, and $E W \to \exp(-u^2/2)$.

But $|E \exp(iu S_N/\sigma_N) - E W| \leqq E |u S_N''/\sigma_N| \to 0$ by (3.1), so that $\mathrm{E} \exp(iu S_N/\sigma_N) \to \exp(-u^2/2)$.

(d) This is proved similarly. □

Several other lemmas are required. When (2.4) holds, set $h(b) = H(b)/b$.

Lemma 3.2. *If $kp/n \to 1$ as $N \to \infty$ and* (2.4) *holds, then* (3.2) *is equivalent to*

$$h(p)/h(n) \to 1 \quad \textit{as } N \to \infty. \tag{3.9}$$

Proof. Use $\sigma_N^2 = n h(n)(1 + o(1))$ and $\sum_0^{k-1} E \xi_j^2 = k p h(p)(1 + o(1))$. □

Lemma 3.3 *L.H.S.* (3.3) *is bounded by $2\sigma_N^{-2} kp\, C_N(q+1)$.*

Proof. $\sigma_N^2 |\text{L.H.S.} (3.3)| \leqq \sum_{j=0}^{k-2} \sum_{\ell=0}^{j} |\operatorname{covar}(\xi_\ell, \xi_{j+1})| \leqq \sum_{j=0}^{k-2} \sum_{s=1}^{j+1} A(s)$, where $A(s) = p F_{Np}(sm + 1 - p)$, $m = p + q$ and $F_{Nb}(a) = \sum_{i=a}^{a+b-1} \tilde{c}_N(i)$.

Also $A(s) \leqq p^2 \tilde{c}_N(sm + 1 - p)$.

Hence

$$m \sum_1^{j+1} A(s) \leqq p^2 \sum_{m+1}^{(j+1)m} \tilde{c}_N(i + 1 - p) \leqq p^2 C_N(q+1) \quad \text{and} \quad A(1) \leqq p\, C_N(q+1). \qquad \square$$

Lemma 3.4. (a) *If* $q \leqq p$,

$$ES_N''^2 = O((p+kq)\, C_N(o)). \tag{3.10}$$

(b) *Under* (2.4),

$$\tau_N^{-2} ES_N''^2 = O(k^{-1} + kH(q)\, H(p)^{-1}).$$

Proof. (a) $E(X+Y)^2 \leqq 2EX^2 + 2EY^2$. Set $X = \sum_0^{k-1} \zeta_j$, $Y = \zeta_k$. Also, $ES_N(a,b)^2 \leqq 2b\, C_N(o)$.

Hence for $j<k$, $E\zeta_j^2 \leqq 2q\, C_N(o)$ and $E\zeta_k^2 \leqq 2(p+q)\, C_N(o)$.

$$EX^2 = \sum_0^{k-1} E\zeta_i^2 + 2T,$$

where for $m=p+q$ and

$$\tilde{F}_N(a,b) = \sum_{j=a}^{a+b-1} \tilde{c}_N(j), \qquad T = \sum_{0 \leqq i<j<k} E\zeta_i \zeta_j$$

is bounded absolutely by

$$q \sum_{s=1}^{k-1} (k-s)\, \tilde{F}_{Nq}(ps),$$

so that

$$ES_N''^2 \leqq (4(p+q)+kq)\, C_N(o) + 2qk \sum_{s=1}^{k-1} \tilde{F}_{Nq}(ps).$$

The second term $\leqq 2q^2 kp^{-1}\, C_N(o)$.

(b) Observe that $\tau_N^2 = kH(p)(1+o(1))$ and

$$ES_N''^2 \leqq 2k \sum_0^{k-1} E\zeta_i^2 + 2E\zeta_k^2 \leqq 2k^2 H(q) + 4H(p) + 4H(q). \qquad \square$$

Lemma 3.5. *Let* A, B *be positive functions.*

(a) *There exists* p *such that* $A(N)\, p^{-1} + B(N)^{-1}\, p \to 0$ *as* $N \to \infty$ *if and only if*

(b) $A(N)\, B(N)^{-1} \to 0$ *as* $N \to \infty$.

If $A(N)$, $B(N)$ *increase to* ∞ *as* $N \to \infty$, *then conditions* (a) *and* (b) *imply*

(c) *there exists* $p \to \infty$ *such that* $A(p)\, n^{-1} + B(p)^{-1}\, n \to 0$ *as* $N \to \infty$.

Proof. Take $p = A(N)^{\frac{1}{2}} B(N)^{\frac{1}{2}}$ in (a). In (c) take p such that $n^2 \approx A(p)\, B(p)$. $\square$

Lemma 3.6. *If* $h(.)$ *is slowly varying on* $(0, \infty)$ *then there exists* $L(.)$ *on* $(0, \infty)$ *such that*

$$L(x) \downarrow 0 \quad \text{as } x \uparrow \infty$$

and

$$h(xM(x))/h(x) \to 1 \quad \text{as } x \to \infty,$$

whenever

$$L(x)^{-1} \geqq M(x) \geqq L(x).$$

Proof. By Karamata's Representation Theorem - see for example Theorem 1.2 of Seneta (1970) - the proof for $M=L$ amounts to proving that if $\varepsilon(.)$ is a continuous function such that $\varepsilon(x)\to 0$ as $x\to\infty$, then there exists $L(.)$ such that $L(x)\downarrow 0$ and $\int_{xL(x)}^{x} \varepsilon(t)t^{-1}\,dt\to 0$ as $x\uparrow\infty$. The integral is bounded absolutely by

$$I_x=A(xL(x))\,|\log L(x)|$$

where $A(t)=\sup_{s\geqq t}|\varepsilon(s)|+t^{-1}$ is continuous and strictly decreasing. Choose m_x such that $m_x/A(m_x)=x$ and set $L(x)=m_x/x$. Then

$$I_x=A(m_x)\,|\log A(m_x)|\downarrow 0.$$

The proof for general M follows similarly. □

Lemma 3.7. *Suppose that $\mathscr{X}$ is nearly second order stationary and satisfies* (2.5), *and $k\to\infty$ as $N\to\infty$.*

Then (3.6) *holds with σ_N replaced by λ_N given by* (3.7).
Hence (3.2) *implies* (3.6).

Proof. The first condition implies $kE\xi_j^2\approx\tau_N^2$ uniformly in j, so that it suffices to prove

$$k^{-1}\sum_{j=0}^{k}\int_{|x|>\varepsilon k^{\frac{1}{2}}} x^2\,dF_{a_j pN}(x)\to 0 \qquad \text{as } N\to\infty$$

where $a_j=j(p+q)+1$. This is implied by

$$\max_a \int_{|x|>\varepsilon k^{\frac{1}{2}}} x^2\,dF_{apN}(x)\to 0 \qquad \text{as } N\to\infty,$$

which holds since for $\varepsilon>0$, $\delta>0$ we may choose $N(\delta)$, $\zeta(\delta)$, $M(\varepsilon,\delta)$, such that for $N>N(\delta)$, $N>M(\varepsilon,\delta)$ $\varepsilon k^{\frac{1}{2}}<\zeta(\delta)$ and $\max_a \int_{|x|>\zeta(\delta)} x^2\,dF_{apN}(x)<\delta$. □

Proof of Theorem 2.1. (A) Suppose $2C_N(0)\leqq K$. Apply Lemma 3.1 with $k=[K^{-1}\sigma_n^2/(p+q)]$. This satisfies $k(p+q)\leqq n$. By Lemma 3.4, (3.1) holds if $q/p+p/\sigma_N^2\to 0$. By Lemma 3.3, (3.3) holds if also $q\to\infty$ since $C_N(q+1)\leqq\sum_{q+1}^{\infty}\tilde{c}(j)$. L.H.S. (3.5) $\leqq kl(q,u)$. Condition (3.6) holds if $kp^{1+\varepsilon/2+\gamma}/\sigma_N^{2+\varepsilon}\to 0$. Hence it suffices to show that there exist sequences $p=p_N(u)$ and $q=q_N(u)$ such that as $N\to\infty$

$$p^{-1}\sigma_N^2\ell(q,u)+p^{\delta}\sigma_N^{-2}+qp^{-1}+q^{-1}\to 0$$

where $\delta=1+2\gamma/\varepsilon$. By Lemma 3.5 it suffices to find $q\to\infty$, such that $n^{-1}q^{\delta}+n^{\delta-1}\ell(q,u)^{\delta}\to 0$. This holds by Lemma 3.5 since $q^{\delta-1}\ell(q,u)\to 0$ as $q\to\infty$. (B) is proved similarly using $k=[n/(p+q)]$. □

Proof of Theorem 2.2. This is as for Theorem 2.1, except that Lemma 3.7 is used to prove (3.6). □

Proof of Theorem 2.3. We first prove the modified C.L.T. with $\tau_N^2 = nh(p)$ for a suitable sequence p. By Lemmas 3.1, 3.7 with $k=[n/(p+q)]$ this will be so if

$$p^{-1}q+k^{-1}+k\,l(q,u)+\tau_N^{-2}ES_N''^2 \to 0.$$

By Lemma 3.4(b) the last term is $O(kH(q)H(p)^{-1}+k^{-1})$. Hence it suffices to show there exist sequences p, q such that

$$pn^{-1}+p^{-1}\{q+n\,l(q,u)+nH(q)H(p)^{-1}\} \to 0 \quad \text{as } N\to\infty. \tag{3.11}$$

Set

$$R(j)=\begin{cases} \ell(j)+(\log j)^{-1} & \text{if } \mathscr{X} \text{ is strongly } \ell\text{-mixing;}\\ \ell(j,u)+(\log j)^{-1} & \text{if } h(.) \text{ is slowly varying on } (0,\infty).\end{cases}$$

Choose $q\to\infty$ as $n\to\infty$ slowly varying and (if (I) holds) such that for n large

$$H(q)(\log q)^{\frac{1}{2}} \leqq H(n^{\frac{1}{2}})^{\frac{1}{2}}$$

and (if h is slowly varying on $(0,\infty)$) $L(n)\leqq R(q)^{\frac{1}{2}}\leqq 1$ for $L(.)$ in Lemma 3.6. (The first condition is possible since $H(b)\to\infty$ as $b\to\infty$.)

Let $p=Kp_1$ where $K>0$ is an arbitrary integer and $p_1=[nR(q)^{\frac{1}{2}}]+1$. Then (3.11) holds. Hence if $\mathscr{X}$ is strongly ℓ-mixing the modified C.L.T. holds with normaliser $nh(Kp_1)$ and hence $h(p_1)/h(Kp_1)\to 1$ as $n\to\infty$. Also there exists $n_0<\infty$ such that $\{p_1\}$ takes all the values $n_0, n_0+1, n_0+2, \ldots$, so that $h(.)$ is slowly varying. This proves (a). For (b), by Lemma 3.7, $\tau_N^2/\sigma_N^2\to 1$ so that the C.L.T. holds.

(c) Let $\mathscr{Y}=\{Y_{jN}\}$ be a Gaussian array with covariances as for $\mathscr{X}$. By Theorem 6.1 we may choose

$$\ell(j,\mathscr{Y})=r^*(j), \qquad \text{where } \ell(j,\mathscr{X})=\ell(j).$$

For p_1 above set $p_1=p_1(\ell)$. By (a), $\mathscr{X}$ and $\mathscr{Y}$ both satisfy the modified C.L.T. with normaliser $nh(p)$ where $p=p_1(r^*)$. Hence $nh(p)/\sigma_N^2\to 1$.

(d) Condition (3.11) is satisfied by $q=[n^b]$, $p=[n^c]$ if $b=(1+2w)^{-1}$ and $(1+w)b<c<1$. Hence, by Theorem 6.1, for $\mathscr{Y}$ as in (c), $\mathscr{X}$ and $\mathscr{Y}$ both satisfy the modified C.L.T. with normaliser $nh(p)$. Hence $nh(p)/\sigma_N^2\to 1$. □

Note 3.1. Theorems 2.1–2.3 remain true if $\ell(j,u)$ is redefined as any function such that for u real and $0\leqq j<n$, $\ell_N(j,u)\leqq l(j,u)+\varepsilon_N(u)$ and $\varepsilon_N(u)$ is a function such that the sequence $k=k_N(u)$ used in Lemma 3.1 satisfies $\varepsilon_N(u)\,k_N(u)\to 0$ as $N\to\infty$ for u real.

Proof of Proposition 2.2. This follows from

$$ES_N(a,b)^4 \leqq 4!\sum EX_c'X_d'X_e'X_f'$$

summed over

$$\begin{aligned} &a+1\leqq b\leqq c\leqq d\leqq e\leqq f\leqq a+b\\ &\leqq 4!\sum_{c=a+1}^{a+b}\ \sum_{0\leqq i,j,k<b} \min\left(c_N^{(13)}(i),\, c_N^{(31)}(k)\right)\\ &\leqq 4!\,b^2\sum_{k=0}^{b-1}(k+1)\left(c_N^{(13)}(k)+c_N^{(31)}(k)\right)=A_N(b), \quad \text{say.}\end{aligned}$$

Here X_j' denotes $X_{jN} - EX_{jN}$. □

Proof of Lemma 2.1. By the line above their (4.3), $T_1^\nu \leqq T_2^{\nu/2}$, so that $A \leqq B$ where $A = ET_1^\nu$, $B = ET_2^{\nu/2}$. Set $\lambda = h/\nu$. By the definition of h, λ maximises $A^\lambda B^{1-\lambda}$ subject to $0 \leqq \lambda \leqq 1 - 2/\gamma$. Hence $\lambda = 0$. Substituting $h = 0$ into their (4.15) yields Lemma 2.1. □

§ 4. The Variance of a Sum

The C.L.T.s of § 2 require the ratio σ_N^2/n to be bounded away from 0 or ∞, or $\sigma_N^2/n \approx h(n)$ where $h(.)$ is slowly varying, or has an extension to $(0, \infty)$ which is slowly varying, or is in a certain sense very slowly varying. In this section it is shown that these requirements are satisfied under fairly weak conditions when $\mathscr{X}$ is approximately second-order stationary. Recall that

$$0 \leqq r^*(j) \leqq r(j) \leqq \rho(j) \leqq \min(1, 2\phi(j)^{\frac{1}{2}})$$

and that if

$$\{X_{jN} \equiv X_j\} \quad \text{is second-order stationary,} \tag{4.1}$$

then $0 \leqq r^{***}(j) \leqq r^*(j)$.

Theorem 4.1. *Suppose that* (4.1) *holds. Let f be the spectral density of $\{X_j\}$.*

(a) *Suppose that $\sigma_1^2 < \infty$, and $\sigma_N^2 \to \infty$ as $N \to \infty$, and $r^{***}(j) \to 0$ as $j \to \infty$. Then $h(n) = \sigma_N^2/n$ is slowly varying and has an extension to the real line which is slowly varying.*

(b) *The condition*

$$\sigma_N^2/n \to \sigma^2 \quad \text{as } N \to \infty \tag{4.2}$$

holds if and only if

$$\sigma^2 = \sum_{-\infty}^{\infty} \operatorname{covar}(X_0, X_j) \quad \text{exists.} \tag{4.3}$$

(c) *Suppose that $f(.)$ is right-continuous at 0 and left-continuous at 0. Then* (4.2) *holds with*

$$\sigma^2 = \pi f(0+) + \pi f(0-).$$

(d) *Suppose that $\sum_0^\infty r(2^j) < \infty$. Then $f(.)$ is continuous.*

Note 4.1. Under (4.1), if $\operatorname{covar}(X_0, X_j) \to 0$ as $j \to \infty$ then by Theorem 18.2.2 of Ibragimov and Linnik (1971) – referred to hence as IL – either $\sigma_N^2 \to \infty$ as $N \to \infty$ or σ_N^2 is bounded.

Note 4.2. In § 6 it is shown that the condition in (a)

$$r^{***}(j) \to 0 \quad \text{as } j \to \infty$$

is satisfied if $\mathscr{X}$ is α-mixing and satisfies (2.5).

Proof. (a) follows from Lemma 4.1 below.

(b) $\sigma^2-\sigma_N^2/n=2n^{-1}\sum_1^n s_j$ if and only if $s_j=\sum_{i=j}^{\infty}\operatorname{covar}(X_0,X_i)$ exists.

(c) Apply the argument on pp. 322, 323 of IL to $[-\pi,0)$ and $(0,\pi]$ separately.

(d) Let $\{Y_j\}$ be a stationary Gaussian process with $\operatorname{covar}(Y_0,Y_j)\equiv\operatorname{covar}(X_0,X_j)$. Apply Theorem 2.2 of Ibragimov (1975) to $\{Y_j\}$ and use the fact that for $\{Y_j\}$, $\rho=r$. □

Theorem 4.2. (a) *Suppose that* $\{X_{jN}\}$ *is nearly second-order stationary, that is, for some* $h(.)$

$$\sup_a|ES_N(a,b)^2b^{-1}h(b)^{-1}-1|\to 0\quad \text{as } b\to\infty,\ N\to\infty.$$

Suppose also that H *satisfies* (II) *of Theorem* 2.3 *and* $r^*(j)\to 0$ *as* $j\to\infty$. *Then* $h(.)$ *is slowly varying.*

If also

$$r^*(j)=O(j^{-w})\quad \text{where } w>0 \tag{4.4}$$

and c *lies in* $((1+\omega)/(1+2w),1)$ *then*

$$h([n^c])/h(n)\to 1\quad \text{as } n\to\infty. \tag{*}$$

(b) *Suppose that*

$$\delta_N=\max_{ij}|\operatorname{covar}(X_{iN},X_{jN})-c_0(i-j)|\to 0\quad \text{as } N\to\infty$$

and

$$\sum_0^{\infty}\tilde{c}_0(j)<\infty\quad \text{where } \tilde{c}_0(j)=\max_{i\geq j}c_0(i).$$

Then $\sigma_N^2/n\to\sum_{-\infty}^{\infty}c_0(j)$ *as* $N\to\infty$.

Proof. (a) This follows by applying the proof of Theorem 2.3(d) to $\mathscr{Y}$ defined in its proof. Condition (2.6) holds for $\mathscr{Y}$ with $\varepsilon=2$.

(b) follows from $\sigma_N^2/n=\sum_{|j|<n}(1-j/n)\Delta_{jN}$ where $|c_0(j)-\Delta_{jN}|\leq\delta_N$. □

Note 4.3. (a) above gives conditions for (*) to hold for all c in $[\varepsilon,1]$ for some ε in $(0,1)$. If in fact for some ε in $(0,1)$(*) holds *uniformly* for c in $[\varepsilon,1]$ then $h(.)$ is *very slowly varying* in the sense that (*) holds for *all* c in $(0,1)$. For example, by Dini's Theorem, (*) holds uniformly for c in $[\varepsilon,1]$ if $h(.)$ is non-decreasing and for c in $[\varepsilon,1]$, $h([n^c])/h(n)\downarrow 1$ as $n\uparrow\infty$.

How weak are the mixing conditions of these two theorems?

Example 4.1. Suppose that (4.1) holds and $\mathscr{X}$ is α-mixing. By Theorem 18.1.1 of IL, if

$$(S_N-ES_N)/\sigma_N\xrightarrow{\mathscr{L}}Z,\quad \text{some r.v.,}$$

then Z has a stable distribution, with exponent a, say, and $\sigma_N^2 = n^{2/a} h(n)$ where h is slowly varying. Hence if $a \neq 2$, $r^*(j) \nrightarrow 0$ as $j \to \infty$.

Example 4.2. Suppose that (4.1) holds and $\operatorname{covar}(X_0, X_j) = j^{-\varepsilon} L(j)$ where L is slowly varying and $0 < \varepsilon < 1$. Then $\sigma_N^2 = n^{2-\varepsilon} h(n)$ where h is slowly varying, and hence $r^*(j) \nrightarrow 0$ as $j \to \infty$.

This case has been studied by Taqqu (1975) when $X_j \equiv G(Y_j)$ and $\{Y_j\}$ is Gaussian: Z in Example 4.1 is generally not normally distributed, so that in general $a \neq 2$.

Example 4.3. If (4.1) holds and f is continuous and bounded away from 0 then $r(j) \to 0$ as $j \to \infty$; if also $f^{(k)}$ is bounded then $r(j) = O(j^{-k})$. This follows by Theorem 4 of Kolmogorov and Rozanov (1960). More generally, by Bedzanjan (1975) we have

Example 4.4. If (4.1) holds and f is bounded away from 0 and $f^{(k)}$ satisfies a Lipschitz condition of order β, $0 \leqq \beta < 1$, then $r(j) = O(j^{-k-\beta})$. (He also gives conditions for $r(j) = O(e^{-j\delta})$.)

It would be preferable to give sufficient conditions for the mixing requirements of these theorems simply in terms of the 'maximal covariances' $\{c(j)\}$.

The following suggests that the condition

$$c(j) = O(j^{-1-w}) \tag{4.5}$$

where $w > 0$ implies that

$$r(j) = O(j^{-w}), \tag{4.6}$$

and hence that (4.4) holds.

Example 4.5. According to p. 292 of Taqqu (1975) for $v > 0$, $c(j) = (1 + |j|)^{-v}$ is a correlation kernel. But if $0 < v - 1 = k + \beta$ where $k = 0, 1, \ldots$ and $0 \leqq \beta < 1$, then $f^{(k)}$ satisfies a Lipschitz condition of order β where f is the spectral density of $\{c(j) = |j|^{-v}\}$. This is because by (1) p. 10 of Erdelyi (1954),

$$f(x) = 2 \int_0^\infty y^{-v} \cos x y \, dy = (2\pi)^{-1} \Gamma(v)^{-1} \sec(v\pi/2) x^{v-1}.$$

Of course the condition (4.5) implies that σ_N^2/n is bounded – and is also bounded away from 0, for example if $f(0-) > 0$ or $f(0+) > 0$ by Theorem 4.1(c). This suggests that (4.6) with $w > 0$ actually implies that $h(.)$ is bounded away from 0, ∞ for nearly second-order stationary samples. (This *is* true under (4.1) by (c) and (d) of Theorem 4.1.)

The following results of Sarason (1971) is interesting, but of doubtful practical value.

Example 4.6. Under (4.1), $r(j) \to 0$ as $j \to \infty$ if and only if

$$f(\lambda) = |P(\lambda)|^2 \exp\{u(\lambda) + \tilde{v}(\lambda)\}$$

where P is a polynomial, $\tilde{v}$ is the Hilbert transform of v, and u, v are real functions continuous on the unit circle.

We conjecture that when $\tilde{c}(k)$ decreases moderately slowly - as in Example 4.5 with $v>2$, then $r^{**}(k)$, $r^{*}(k)$, $r(k)$ are all equivalent to $k\tilde{c}(k)$, but that if $\tilde{c}(k)$ decreases rapidly then $r^{**}(k)$, $r^{*}(k)$, $r(k)$ are all equivalent to $\tilde{c}(k)$. (The latter holds for example if $\tilde{c}(k)=O(e^{-\lambda k})$ for λ suitably large and (4.1) holds and $0<\operatorname{covar}(X_0, X_k)$, since then for k sufficiently large

$$0<\ldots<\operatorname{corr}(X_{-1}+aX_0, X_k+bX_{k+1})<\operatorname{corr}(X_{-1}+aX_0, X_k)<\operatorname{corr}(X_0, X_k).)$$

The proof of Theorem 4.1(a) is completed by

Lemma 4.1. *Suppose that* (4.1) *holds and* $\{X_j\}$ *has spectral density* $f(\lambda)$. *Set*

$$\psi(x)=\int_{-\pi}^{\pi}\frac{\sin(x\lambda/2)^2}{\sin(\lambda/2)^2}f(\lambda)\,d\lambda,$$

$$h(x)=\psi(x)/x.$$

Then

(a) $\sigma_N^2=\psi(n)$.

(b) $\psi(ax)\leqq a^2\psi(x)$ *for* a *an integer.*
Suppose that $\sigma_1^2<\infty$.

(c) *If* $\sigma_N^2\to\infty$ *as* $N\to\infty$, *then*

(4.7) $$\psi(x)/\psi([x])\to 1 \quad \text{as } x\to\infty.$$

(d) *If* $\sigma_N^2\to\infty$ *and* $r^{***}(j)\to 0$ *as* $j\to\infty$, *then*

(4.8) *h is slowly varying of integral argument*

and

(4.9) *there exists* $c_0>0$ *such that for* $0<c<c_0$, $h(jc)/h(j)<c^{\frac{1}{2}}$, *if* j *and* jc *are integers.*

(e) *Conditions* (4.7) *and* (4.8) *imply*

(4.10) $$h(\lambda x)/h(x)\to 1 \quad \text{as} \quad x\to\infty \quad \text{for } 0<\lambda \text{ rational.}$$

(f) *Conditions* (4.7) *and* (4.9) *imply*

(4.11) $$\psi_2(\lambda)\leqq\lambda(2/c_0)^{3/2} \quad \text{where} \quad \psi_2(\lambda)=\limsup_{x\to\infty}\psi(\lambda x)/\psi(x)$$

and

(4.12) $$\psi_1 \text{ and } \psi_2 \text{ are continuous where } \psi_1(\lambda)=\liminf_{x\to\infty}\psi(\lambda x)/\psi(x).$$

(g) *Conditions* (4.10) *and* (4.12) *imply that* h *is slowly varying.*

Proof. (a) is well known.

(b) $|\sin(ax)|\leqq|a\sin x|$.

(c) The last two lines of p. 329 of IL are incorrect: in the second to last line a factor $1-2\sin^2(ax\lambda/2)$ should be inserted in the first integrand, and hence

$$(4.13)\qquad |\psi((a+\varepsilon)x)-\psi(ax)|\leqq\psi(\varepsilon x)+\psi(2\varepsilon x)^{\frac{1}{2}}\psi(2ax)^{\frac{1}{2}}/2$$
$$\leqq\psi(\varepsilon x)+\psi(\varepsilon x)^{\frac{1}{2}}\psi(2ax)^{\frac{1}{2}} \quad \text{by (b).}$$

Put $a=1$, $\varepsilon=[x]/x-1$, $\gamma^2=(\pi/2)^2\psi(1)$. Then for $|\theta|\leqq 1$, $\psi(\theta)\leqq\gamma^2$. Hence

$$|\psi([x])-\psi(x)|\leqq\gamma^2+2\gamma\psi(x)^{\frac{1}{2}}.$$

Hence $\psi(x)\approx\psi([x])$, as asserted without proof on p. 329 of IL.

(d) and (e) are proved on pp. 326–330 of IL.

(f) If p is a positive integer and $x\to\infty$ then by (b),

$$h(\lambda x)\leqq 2^p h(2^{-p}\lambda x)\approx 2^p h([2^{-p}\lambda x])\leqq h([x])\,2^p K^{-\frac{1}{2}}$$

if $K<c_0$ where $K=[2^{-p}\lambda x]/[x]\approx 2^{-p}\lambda$. Given $\lambda>0$ choose p so that $c_0/2\leqq 2^{-p}\lambda<c_0$. Hence for x large,

$$h(\lambda x)/h(x)\leqq 2^{3p/2}\lambda^{-\frac{1}{2}}(1+o(1))\leqq\lambda(2/c_0)^{3/2}(1+o(1))$$

so that (4.11) holds. By (4.13) for $j=1, 2$

$$|\psi_j(a+\varepsilon)-\psi_j(a)|\leqq\psi_2(\varepsilon)+\psi_2(\varepsilon)^{\frac{1}{2}}\psi_2(2a)^{\frac{1}{2}}\to 0$$

as $\varepsilon\to 0$ by (4.11).

(g) By (4.10), $\psi_1(\lambda)=\psi_2(\lambda)=\lambda$ for λ rational and hence for all $\lambda>0$. □

§5. Conditions for a Linear Process to be ℓ-Mixing

Gorodetskii (1977) and Withers (1981) give conditions for a linear process to be strong-mixing. This section gives conditions for a linear process to be strongly ℓ-mixing. These conditions are shown to be much less stringent than those required for strong-mixing. In particular the densities of the random variables generating the process need not exist.

We consider the general linear process

$$(5.1)\qquad X_{jN}=Z_{jN}+g_{1N}Z_{j-1,N}+g_{2N}Z_{j-2,N}+\dots,\qquad 1\leqq j\leqq n,\ N\geqq 1$$

where $\{g_{jN}\}$ are constants and $\{Z_{jN}\}$ independent real r.v.s such that this sum converges in probability. This is true for example, if for some $\delta>0$

$$\max_j E|Z_{jN}|^\delta<\infty \quad\text{and}\quad \sum_j |g_{jN}|^\delta<\infty.$$

Theorem 5.1. For $0<\delta\leqq 1$

$$\ell_N(k,u)\leqq 2\,|u|^\delta\sigma_N^{-\delta}\left\{\sum_{j=1}^{n-k} j\,|g_{k+j-1,N}|^\delta+(n-k)\sum_{j=n}^{\infty}|g_{jN}|^\delta\right\}\max_j E\,|Z_{jN}|^\delta.$$

Proof. Let $F=\sigma_N^{-1}\sum_{m+k}^{n} b_j X_j$ where $b_j=0$ or 1 and we drop the subscript N. Then $F=F_0+F_1$ where F_1 depends on $Z_{m+1},\ldots,Z_{m+k}$ and

$$\begin{aligned}\sigma_n F_0 = {} & b_{m+k}(g_k Z_m + g_{k+1} Z_{m-1}+\ldots)+b_{m+k+1}(g_{k+1} Z_m + g_{k+2} Z_{m-1}+\ldots)\\ & +\ldots+b_n(g_{n-m} Z_m + g_{n-m+1} Z_{m-1}+\ldots).\end{aligned}$$

Hence F_1 is independent of (F_0, P) where $P=\sigma_N^{-1}\sum_1^n a_j X_j$ and $\{a_j\}$ are arbitrary constants. But for Y_1 independent of (Y_0, X),

$$\operatorname{covar}(e^{iX}, e^{-iY_0-iY_1}) = E e^{iY_1} E e^{iX}(e^{iY_0}-E e^{iY_0})$$

is bounded by $E|e^{iY_0}-E e^{iY_0}| \leqq 2E|Y_0|^\delta$.

Hence $\ell_N(k,u) \leqq 2|u|^\delta E|F_0|^\delta$. Now use $|\sum a_i|^\delta \leqq \sum |a_i|^\delta$. □

From Theorem 5.1 one may easily obtain

Corollary 5.1. *Suppose that for some δ in* $(0,1]$

$$\max_{jN} E|Z_{jN}|^\delta < \infty \tag{5.2}$$

and n/σ_N^2 is bounded. If

$$\max_N |g_{kN}| = O(k^{-\lambda}) \quad \text{as} \quad k\to\infty \tag{5.3}$$

then one may choose

$$\ell_N(k) = \begin{cases} n^{2-\delta(\lambda+\frac{1}{2})}, & \lambda<2/\delta\\ n^{-\delta/2}\,\ell n(n), & \lambda=2/\delta\\ n^{-\delta/2} k^{2-\delta\lambda}, & \lambda>2/\delta\end{cases}$$

and

$$\ell(k) = \begin{cases} k^{2-\delta(\lambda+\frac{1}{2})}, & \lambda\neq 2/\delta\\ k^{-\delta/2}\,\ell n(k), & \lambda=2/\delta.\end{cases}$$

If $\max_N |g_{k,N}| = O(e^{-\lambda k})$ *as* $k\to\infty$, *one may choose*

$$\ell_N(k) = n^{-\delta/2} e^{-\lambda\delta k},$$

and

$$\ell(k) = k^{-\delta/2} e^{-\lambda\delta k}.$$

Corollary 5.1 illustrates that the conditions of Gorodetskii (1977) and Withers (1981) for a linear process to be strong-mixing are much stronger than the condition that it be strongly ℓ-mixing. In particular under (5.2) and (5.3) the linear process is strongly ℓ-mixing if $\lambda>2/\delta-\frac{1}{2}$, whereas the above conditions for strong-mixing require $\lambda>2/\delta+1$.

The above choice of $\ell(k)$ should be improveable since Diananda (1953) proved the C.L.T. for the linear process with $\{g_{jN}\equiv g_j\}$ for the case of $\{Z_{jN}\equiv Z_j\}$ stationary and m-dependent when $\sum|g_j|<\infty$.

Corollary 5.1 also illustrates that one can often obtain a bound of the form $\ell_N(k,u) \leqq K(u)\,\ell_N(k)$ where

$$\ell_N(k) = n^{-a} M(k)$$

where $a > 0$ so that one may choose

$$\ell(k) = k^{-a} M(k).$$

It may be thought that considerable information is being lost in passing from $\ell_N(k)$ to $\ell(k)$. However, this form of $\ell_N(k)$ does not enable Theorem 2.1, for example, to be improved.

The following result gives a class of first-order autoregressive schemes which are strongly ℓ-mixing – in fact with $\ell(k)$ decreasing exponentially, but are neither strong-mixing nor absolutely regular.

Theorem 5.2. *Consider the autoregressive process*

$$X_{jN} = Z_{jN} + (-)^{a_{jN}} p^{-1} X_{j-1,N}$$

where $p > 1$ *is a positive integer,* $\{a_{jN}\}$ *are given integers, and* $\{Z_{jN}\}$ *are independent r.v.s taking only the values* $0, 1, 2, \ldots, p-1$ *and such that*

$$\max_{jN} E\,|Z_{jN}| < \infty.$$

Then one may choose

$$\ell_N(k) = n^{-\frac{1}{2}} p^{-k}, \qquad \ell(k) = k^{-\frac{1}{2}} p^{-k}.$$

However, if for some k $P(Z_{0N} = k)$ *is bounded away from* 0 *and* 1, *say* $\varepsilon \leqq P(Z_{0N} = k) \leqq 1 - \varepsilon$ *where* $\varepsilon > 0$, *then* $\alpha(k) \geqq \alpha_N(k) \geqq \varepsilon - \varepsilon^2$ *and* $\beta(k) \geqq \beta_N(k) \geqq 2(\varepsilon - \varepsilon^2)$.

Proof. The first part follows from Corollary 5.1 with $\delta = 1$, $e^{\lambda} = p$. To prove the second part, note that $(Z_{jN}, Z_{j-1,N}, \ldots, Z_{0N}, \ldots)$ is a generalisation of the p-adic expansion of X_{jN}, and is obtainable uniquely from X_{jN}. Hence for $k = 0, 1, \ldots, p-1$, $j \leqq 1$ the event

$$A = \{Z_{jN} = k\} \in \mathcal{M}(X_{1N}) \cap \mathcal{M}(\{X_{1+k,N}, \ldots, X_{nN}\})$$

so that $\alpha_N(k) \geqq \max\limits_{j \leqq 1} \max\limits_{k=0}^{p-1} p_{jN}(k)(1 - p_{jN}(k)) \geqq \varepsilon - \varepsilon^2$ where $p_{jN}(k) = P(Z_{jN} = k)$. The same is true for $\beta_N(k)/2$. □

Since an autoregressive scheme is a linear process it follows that not all linear processes are strong-mixing. The latter is also true for $\mathcal{X}$ a stationary Gaussian process with spectral density f: it is a linear process if and only if f is absolutely continuous (by Theorem 16.2.1 of IL), while it is strong-mixing if and only if the conditions of Example 4.6 hold.

§6. Further Comparison of ℓ-Mixing and α-Mixing

The proof of the C.L.T. when the observations are α-mixing typically rests on the following choice of bounds:

(6.1) $$\tilde{c}_N(k) \leqq 10 K_{Np}^2 \alpha_N(k)^{1-2/p} \leqq 10 K_p^2 \alpha(k)^{1-2/p};$$

(6.2) $$c_N^{(13)}(k), \quad c_N^{(31)}(k) \leqq 10 K_{Np}^4 \alpha_N(k)^{1-4/p} \leqq 10 K_p^4 \alpha(k)^{1-4/p};$$

(6.3) $$\ell_N(k,u) \leqq 16 \alpha_N(k) \leqq 16 \alpha(k)$$

where $K_{Np} = \max_{j=1}^{n} \|X_{jN}\|_p$, $K_p = \max_N K_{Np}$, – (6.2) being used to verify the moment inequality via Proposition 2.2; c.f. IL.

We have shown that the above bounds for $\ell_N(k,u)$ may be very crude. The same is true of the bounds in (6.1), (6.2) – even if $K_p < \infty$ for $p < \infty$. This is true even for those rather exceptional processes – such as Gaussian processes – for which $\alpha(k)$ and $r(k)$ are equivalent. If $\{X_{jN}\}$ is Gaussian, then

$$c_N^{(13)}(k), \quad c_N^{(31)}(k) \leqq 3 \tilde{c}_N(o) \tilde{c}_N(k)$$

and

$$K_{Np} \leqq M p^{\frac{1}{2}} \tilde{c}_N(o)^{\frac{1}{2}}$$

where $M = \sup_{p \geqq 1} \|N(0,1)\|_p / p^{\frac{1}{2}} < \infty$.

If $\{X_{jN}\}$ is Gaussian then, as noted in §1, $\rho(k) = r(k)$ so that one may choose $\ell(k) = r(k)$. The next result improves this.

Theorem 6.1. *If $\{X_{jN}\}$ is Gaussian then $\ell(k,u) \leqq r^*(k)$.*

Proof. $|\mathrm{covar}(e^{iuP}, e^{-iuF})| \leqq \rho_c(P,F) = r(P,F)$ by Theorem 1.1, and Kolmogorov and Rozanov (1960). Hence $\ell_N(k,u) \leqq r_N^*(k)$. □

A comparison of $r^{***}(k)$ and $\alpha(k)$ is afforded by

Theorem 6.2. *Suppose $\{X_{jN} \equiv X_j\}$ is second-order stationary and $n_N \equiv N$. Let $F_N(x) = P((S_N - ES_N)/\sigma_N \leqq x)$,*

$$A(z) = \limsup_{N \to \infty} \int_{|x|>z} x^2 \, dF_N(x)$$

and

$$G_p(\varepsilon) = 4 \varepsilon^{1-2/p} + 6 A(\varepsilon^{-1/p})^{\frac{1}{2}}.$$

Then $r^{***}(k) \leqq \inf_p G_p(\alpha(k))$. Hence if (2.5) *holds, that is, if $A(z) \to 0$ as $z \to \infty$, then $r^{***}(k) \to 0$ as $\alpha(k) \to 0$.*

If the C.L.T. holds then there exists $\alpha_0 > 0$ such that for $\alpha(k) < \alpha_0$

(6.4) $$r^{***}(k) \leqq 219 \alpha(k) \log \alpha(k)^{-1}.$$

Proof. Set $\xi = \sum_1^m X_j$, $\zeta = \sum_{m+k}^{m+k+n} X_j$, $\xi_N = \xi 1(|\xi| \leqq N)$, $\zeta_M = \zeta 1(|\zeta| \leqq M)$ where $1(.)$ is the indicator function. By the analog of (17.2.5) of IL – which should have $|E \bar{\xi}_N \zeta_N|$ added to its R.H.S. – and the equations following applied with $\delta = 0$, one obtains

$$|\mathrm{covar}(\xi, \zeta)| \leqq 4 MN\alpha + 3 \sigma_n \sigma_m \{A_m(M/\sigma_m)^{\frac{1}{2}} + A_n(N/\sigma_n)^{\frac{1}{2}}\}$$

where $A_N(z) = \int_{|x|>z} x^2 \, dF_N(x)$ and $\alpha = \alpha(k)$.

Setting $M=\sigma_m \alpha^{-1/p}$, $N=\sigma_n \alpha^{-1/p}$ yields

$$r^{***}(k) \leqq G_p(\alpha).$$

Now set $X=\log \alpha^{-1}$, $\theta=2/(4+\log X)$, $p=X\theta$. Hence $\alpha^{1-2/p}=e^4 \alpha X$. Suppose that the C.L.T. holds, so that

$$A(z)=2z(2\pi)^{-\frac{1}{2}} e^{-z^2/2}(1+o(1)) \quad \text{as} \quad z \to \infty.$$

Hence

$$\begin{aligned} r^{***}(k) &\leqq 4e^4 \alpha(X+o(1)) \qquad \alpha \to 0 \\ &\leqq 219 \alpha X \qquad \text{for } \alpha \leqq \alpha_0, \text{ say.} \quad \square \end{aligned}$$

It seems plausible that if $\mathscr{Y}$ is a Gaussian array with covariances the same as $\mathscr{X}$, then

$$\alpha(k, \mathscr{Y}) \leqq \alpha(k, \mathscr{X}) \quad \text{where} \quad \alpha(k, \mathscr{X})=\alpha(k).$$

This would imply quite generally that

$$r(k) \leqq 2\pi \alpha(k),$$

– a stronger relationship than (6.4). We now show that

$$r(k)=O(\alpha(k)^{1-2/v})$$

for a large class of processes satisfying $K_v < \infty$.

Theorem 6.3. *Suppose that for some positive even integer* v, $f_v(.)$ *satisfies* (2.10) *and*

$$\sum_1^\infty k^{v/2-1} f_v(k) < \infty.$$

Suppose also that $\max_{jN} E X_{jN}^v < \infty$, *and*

$$\inf_{N, \{c_j\}} \left\| \sum_1^n c_j X_{jN} \right\|_2 \Big/ \left(\sum_1^n c_j^2 \right)^{\frac{1}{2}} > 0. \tag{6.5}$$

Then $r(k)=O(\alpha(k)^{1-2/v})$.

Note 6.1. Assumption (6.5) holds provided not 'too many' covariances of $\mathscr{X}$ are 'large' and negative.

Proof. Under the conditions of Lemma 2.2 with $q=1$ one actually has the stronger result:

$$\begin{aligned} \left\| \sum_1^n c_j X_{jN} \right\|_v &\leqq (v!\,\beta(n,1)+D_v)^{1/v} K_v \left(\sum_1^n c_j^2 \right)^{\frac{1}{2}} \\ &= O\left(\left(\sum_1^n c_j^2 \right)^{\frac{1}{2}} \right) \text{ by assumption, so that by (6.5)} \end{aligned}$$

$$M_{Nv} = \sup_{\{c_j\}} \left\| \sum_1^n c_j X_{jN} \right\|_v \Big/ \left\| \sum_1^n c_j X_{jN} \right\|_2 \quad \text{is bounded.}$$

But $r(k) \leqq 10\,\alpha(k)^{1-2/v} M_{Nv}^2$. $\square$

Corollary 6.1. *Suppose that for some positive even integer ν and some $p>\nu$*

$$\max_{jN} E|X_{jN}|^p < \infty \quad \textit{and} \quad \sum_1^\infty k^{\nu/2-1}\alpha(k)^{1-\nu/p} < \infty$$

and (6.5) *holds.*

Then $r(k)=O(\alpha(k)^{1-2/\nu})$.

Proof. Take $f_\nu(k)=10K_\nu^{-\nu}K_p^\nu\alpha(k)^{1-\nu/p}$. □

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Received July 25, 1979; in revised form May 15, 1981