WAVE FLOW REGIMES OF A THIN LAYER OF VISCOUS FLUID SUBJECT TO GRAVITY

V. Ya. Shkadov

Izv. AN SSSR. Mekhanika Zhidkosti i Gaza, Vol. 2, No. 1, pp. 43–51, 1967

The studies of Kapitsa initiated the detailed experimental and theoretical study of the flow of a thin layer of viscous liquid (liquid film) over a solid surface [1–2]. Extensive experimental data on this question have now been accumulated. As a rule, the existing theories are based on linearization of the problem and diverge considerably from the experimental results. The present paper is also addressed to the theoretical solution of this problem. The solution method used enables consideration of the wave flow of the liquid as a nonlinear problem and on this basis permits determining all the parameters of the wave regime—amplitude, wavelength, wave propagation speed, frequency.

1. Consider a thin layer of viscous liquid that flows along a vertical surface under the influence of the gravity force. We shall assume that the liquid surface is free, i. e., the air friction force does not act on the surface. We direct the x-axis along the surface in the direction of action of the gravitational force, and the y-axis toward the liquid; in this coordinate system the liquid motion is described by the following system of Navier-Stokes and continuity equations:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x^2}\right)+g,$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}\right),$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{1.1}$$

Let $y = a(x, t)$ be the equation of the layer free surface. To system (1) we must also add the equation

$$\frac{\partial a}{\partial t}+\frac{\partial Q}{\partial x}=0, \qquad Q=\int_0^a u\,dy, \tag{1.2}$$

expressing the condition that the surface consists of streamlines. In view of no liquid slip at the wall we have the two boundary conditions

$$u=0, \quad v=0 \quad \text{for} \quad y=0. \tag{1.3}$$

If only surface tension and the constant pressure p_0 act on the liquid surface, then in the case of plane flow with $y = a(x, t)$ the following relations will be satisfied [3]:

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{2b}{1-b^2}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)=0, \qquad b=\frac{\partial a}{\partial x},$$

$$p+\sigma\frac{1}{R}-\mu b\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-2\mu\frac{\partial v}{\partial y}=p_0,$$

$$\frac{1}{R}=\frac{\partial b/\partial x}{(1+b^2)^{3/2}}. \tag{1.4}$$

Here σ is the surface tension coefficient and R is the radius of curvature of the surface.

In the following we shall consider a liquid layer which is not bounded along the x-axis with a time-average thickness a_0 and shall seek solutions periodic with respect to x in the form of traveling waves. Such solutions may be represented in the form of functions of the variables y and $\xi = (n/a_0)(x - \omega t)$.

Parameters of Optimal Wave Regimes

R	λ, cm	$10\,a_0$, cm	ρ	ω, cm/sec	n	$10\,\rho^2\varphi_{20}$	$10\,\rho^2\varphi_{21}$	z
9	1.196	0.106	0.080	9.530	0.055	−0.159	0.036	2.945
15	0.980	0.124	0.174	12.700	0.079	−0.328	0.159	2.760
20	0.912	0.134	0.241	14.513	0.093	−0.435	0.292	2.569
25	0.886	0.143	0.294	15.936	0.101	−0.520	0.424	2.401
30	0.865	0.150	0.334	17.191	0.109	−0.561	0.529	2.268
35	0.859	0.157	0.365	18.321	0.115	−0.601	0.627	2.161
40	0.851	0.163	0.390	19.401	0.120	−0.618	0.705	2.078
45	0.845	0.168	0.410	20.439	0.125	−0.626	0.773	2.010
50	0.841	0.173	0.427	21.442	0.129	−0.628	0.832	1.955
55	0.837	0.178	0.441	22.418	0.133	−0.626	0.885	1.909
60	0.834	0.182	0.452	23.370	0.137	−0.620	0.931	1.871
65	0.830	0.187	0.462	24.313	0.141	−0.611	0.972	1.839
70	0.830	0.191	0.471	25.226	0.144	−0.605	1.014	1.810
75	0.829	0.195	0.478	26.121	0.148	−0.598	1.053	1.786
85	0.825	0.202	0.490	27.888	0.154	−0.575	1.116	1.746

Then a_0n will be the characteristic dimension along x and a_0 will be the same along y.

Let us assume that the condition $n \ll 1$ is satisfied. Physically this means that the wavelengths are considerably longer than the average layer thickness. From experiments [2], it follows that n in the case of periodic wave motions is on the order of 0.1; with an average layer thickness on the order of 1 mm we observe wavelengths of about 1 cm. The calculated values of n shown in the table are of the same order.

The condition $n \ll 1$ enables us to estimate the terms in Eqs. (1.1) and the boundary conditions (1.4) and immediately simplify them, which facilitates the solution considerably. If we assume that $u \sim V_0$, from the continuity equation it follows that $v \sim nV_0$.

As a result of this, from the second equation of motion we obtain

$$\frac{a_0}{\rho V_0^2}\frac{\partial p}{\partial y}=o\left(n\frac{\nu}{a_0V_0}\right)+o(n^2).$$

Here the first term denotes the viscous terms and the second term the inertial.

The quantity $R = 3a_0V_0\nu^{-1}$ is the Reynolds number; it will be seen later than R is on the order of 10 and the ratio n/R is small except for the case of very small flow rates.

Thus, with an error whose order does not exceed the larger of the numbers n^2 and $3nR^{-1}$, the pressure may be considered constant across the layer and equal to the pressure at the surface.

Simplifying with the same accuracy the boundary conditions (1.4) and neglecting in the first equation of motion the derivative

$$\frac{\partial^2 u}{\partial x^2} \qquad \left(\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial^2 u}{\partial y^2}\right)^{-1} \sim n^2\right),$$

we obtain the following problem:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} + g,$$

$$p = p_0 - \sigma\frac{\partial^2 a}{\partial x^2}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \frac{\partial a}{\partial t} + \frac{\partial Q}{\partial x} = 0,$$

$$u = 0, \quad v = 0 \quad \text{for } y = 0,$$

$$\partial u / \partial y = 0 \quad \text{for } y = a(x, t) \qquad (1.5)$$

Now we replace the equations of system (1.5) by equations integrated with respect to the variable y.

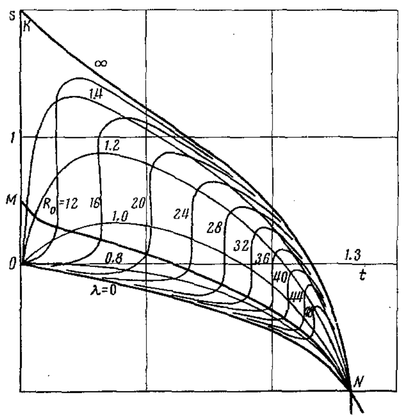

Fig. 1

This replacement may be considered as the first step in sequential application of the direct method. For this we select a complete system of functions $W_i(y)$ satisfying the boundary conditions; we represent the velocity u in the form

$$u = \sum_i b_i(x,t) W_i(y),$$

and from the continuity equation we find

$$v = -\sum_i \partial b_i/\partial x \int_0^y W_i(y)\,dy.$$

Substituting u, v into Eq. (1.5), we require that the resulting expression on the segment $(0, a)$ be orthogonal to the complete system of functions $V_i(y)$, and from the orthogonality conditions we obtain the equations for $b_i(x,t)$.

In limiting ourselves to the first term, we avoid verifying the rapidity of the convergence of this process and we judge the accuracy thus obtained only in comparison with the experimental data. We set

$$u = 3U(x,t)\,[y/a - {}^1\!/_2 (y/a)^2],$$

which coincides with the exact solution for the laminar flow regime of the layer. Integrating the first of Eqs. (1.5) with respect to y from 0 to a and introducing the variables

$$\xi, \quad \tau = n\frac{U_0}{a_0}t, \quad h = \frac{a}{a_0}, \quad q = \frac{Q}{a_0 U_0}, \qquad (1.6)$$

we obtain the equations for determining the dimensionless thickness h and the flow q:

$$\frac{\partial h}{\partial \tau} + \frac{\partial}{\partial \xi}(q - zh) = 0$$

$$\frac{1}{h}\frac{\partial q}{\partial \tau} - \left(z - \frac{12}{5}q\right)\frac{1}{h}\frac{\partial q}{\partial \xi} - \frac{6}{5}\frac{q^2}{h^3}\frac{\partial h}{\partial \xi} -$$

$$- G\frac{\partial^3 h}{\partial \xi^3} - H + E\frac{q}{h^3} = 0, \qquad (1.7)$$

$$\left(G = \frac{\gamma n^2}{V_0^2 \rho a_0}, \quad H = \frac{g a_0}{V_0^2 n}, \quad E = \frac{3\nu}{V_0 a_0 n}\right).$$

Here V_0 is the characteristic value of the velocity, which, generally speaking, may be selected arbitrarily. In place of V_0, a_0 it is more convenient to use the dimensionless parameters R and $R_0 = g a_0^3 \nu^{-2}$, with whose aid the coefficients of Eqs. (1.7) may be expressed as follows:

$$G = \frac{9n^2\gamma R_0^{1/3}}{R^2}, \quad H = 9\frac{R_0}{R^2 n},$$

$$E = 9\frac{1}{Rn}, \quad \gamma = \frac{\sigma}{\rho}(\nu^4 g)^{-1/3}.$$

The first of these parameters characterizes the flow rate, and the second is uniquely associated with the film thickness. If as V_0 we select the average velocity for laminar flow of a layer of thickness a_0, then, as follows from the second of Eqs. (1.7), $R_0 = R$ in this case and the average value q_0 of the flow rate differs from unity. If we take as V_0 the average velocity in that section where the film thickness is a_0 for wave flow, then $q_0 = 1$ but $R_0 \neq R$.

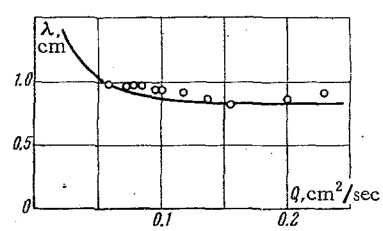

Fig. 2

Only four parameters appear in system (1.7): R_0, R, z, and n or R, z, n, q_0 if $R_0 = R(z = \omega/V_0)$. From the physical aspect of the considered problem and from the condition of unique solvability for the periodic solution, two of these parameters may be considered known, while the other two must be determined in the course of the solution. For example, we may consider that R and λ are given, i.e., the flow rate and the disturbance wavelength, and determine the corresponding average thickness and wave phase velocity, or else specify R_0 and λ. The quantities γ, ν characterize the physical properties of the liquid and are considered given. In the following calculations we take $\nu = 0.0114$ cm^2/sec, g = 981 cm/sec^2, $\tau = 2850$ (water at 15° C).

In the case of the steady-state traveling wave regime, $\partial/\partial\tau = 0$; therefore, from the first of Eqs. (1.7) we easily find that

$$q = zh + q_0 - z. \qquad (1.8)$$

We select V_0 so that $q_0 = 1$, and we set $h = 1 + \varphi$, where φ represents the disturbance of the surface of the downflowing liquid caused by the wave formation. Then with the aid of (1.8), we obtain the equation for φ:

$$(1+\varphi)^3\varphi''' + [\Lambda - B\varphi(2+\varphi)]\varphi' +$$

$$+ A\varphi^2(3+\varphi) + D\varphi + r = 0,$$

$$\Lambda = \frac{5z^2 - 12z + 6}{45\gamma R_0^{1/3} n^2} R^2, \qquad r = \frac{R_0 - R}{\gamma R_0^{1/3} n^3},$$

$$D = \frac{3R_0 - zR}{\gamma R_0^{1/3} n^3},$$

$$A = \frac{R_0^{2/3}}{\gamma n^3}, \quad B = \frac{z^2}{5z^2 - 12z + 6}\Lambda. \qquad (1.9)$$

The study of the periodic solutions of (1.9) forms the subject of further consideration. This equation was first obtained in [1] (the term $v\partial u/\partial y$ was not considered in the derivation). In the form written here it was considered in [4], where for the particular value $\Lambda = 1$ the periodic solution was constructed for small values of the wave amplitude by a series expansion.

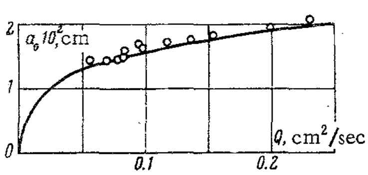

Fig. 3

2. We shall seek the periodic solution of (1.9) in the form of the Fourier series

$$\varphi = \rho \sin\xi + \rho^2(\varphi_{20}\sin 2\xi + \varphi_{21}\cos 2\xi) + \ldots, \qquad (2.1)$$

in which, without loss of generality, we can drop $\cos\xi$, since ξ does not appear explicitly in (1.9), and therefore solution (2.1) is determined only with accuracy to within an arbitrary shift along ξ. If we substitute (2.1) into the left side of (1.9), we obtain a nonlinear expression containing powers of the trigonometric functions. The basic idea of the solution method is to transform this expression and represent it in the form of a Fourier series as well. It is convenient to make this transformation as follows. We write (2.1) in the form of an expansion in powers of $\sin\xi$ and $\cos\xi$, using for this the relations expressing the trigonometric functions of a multiple angle in terms of the function of a single angle. We substitute the resulting series for φ into (1.9) and collect terms of like order in s, c (where $s = \sin\xi$, $c = \cos\xi$).

The left side of (1.9) will represent the sum of polynomials homogeneous in s, c

$$Q_i = Q_{i0}s^i + Q_{i1}s^{i-1}c + \ldots + Q_{ii}c^i \quad (i = 0, 1, 2, \ldots)$$

of zero, first, etc. orders. We add to this sum the differences

$$\Omega_i - \Omega_i(s^2 + c^2),$$

$$\Omega_i = \Omega_{i0}s^i + \Omega_{i1}s^{i-1}c + \ldots + \Omega_{ii}c^i.$$

These differences are identically zero in view of the relation $s^2 + c^2 = 1$. We then define the coefficients Ω_{ik} in terms of Q_{ik} so that the expressions $Q_2 - \Omega_0(s^2 + c^2)$, $Q_3 - \Omega_1(s^2 + c^2)$, ... convolute into the corresponding harmonics $M_{k0}\sin k\xi + M_{k1}\cos k\xi$.

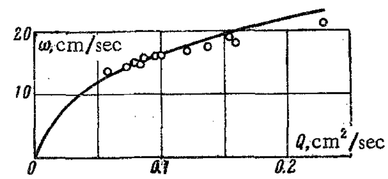

Fig. 4

Here in order to reduce the second harmonic to the normal form, we must introduce Ω_0, for the third harmonic we introduce Ω_1, etc. After the left side of (1.9) is transformed into a Fourier series, it remains to equate its coefficients to zero.

The resulting system of equations will be infinite and must be truncated for practical consideration. If we limit ourselves to the consideration of the i + 1 harmonic and set all Ω, beginning with Ω_{i+1}, equal to zero, then we obtain a closed finite system of equations. Its solution will represent the solution of the considered problem in the i-th approximation. In order to write out this system, we must find concrete expressions for Q_{ik} in terms of φ_{ik} by expanding Q in a series.

In the first approximation we set $\Omega_k = 0$ (k = 2, 3, ...). Equating to zero the coefficients of the first two harmonics and the free term of the expansion of the left side of (1.9), we obtain the following system after some transformations:

$$r + {}^3/_2 A\rho^2 = 0,$$
$$4(1-\Lambda)\rho^{-2} + 3 + B - 12A\varphi_{20} - 4(13+B)\varphi_{21} = 0,$$
$$4D\rho^{-2} + 3A - 12A\varphi_{21} + 4(13+B)\varphi_{20} = 0,$$
$$2(4-\Lambda)\varphi_{21} + D\varphi_{20} - {}^1/_3(3+2B) = 0,$$
$$D\varphi_{21} - 2(4-\Lambda)\varphi_{20} - {}^3/_2 A = 0. \qquad (2.2)$$

Here the unknowns will be ρ, φ_{20}, φ_{21} and any two of the four parameters R_0, R, z, n. We introduce the quantities t, s, w by the relations

$$t = 3 - z, \quad w^{-1} = t^{-1}(1 - R/R_0), \quad s = \Lambda - 1, \qquad (2.3)$$

and take as the unknowns A and w, while we consider t and s known. From the first of Eqs. (2.2) we find

$$\rho^2 = {}^2/_3\, t/w. \qquad (2.4)$$

Excluding φ_{20}, φ_{21} from the four remaining equations, we obtain the equations for determining A and w:

$$18(2-s)w^2 - [63 + 6s + 18B - (3+B)t + $$
$$+ 18(3-t)(2-s)]\, w - t(3-t)(3+B) = 0, \quad (2.5)$$

$$6[3 - t/w(3-t-w)(5/2 - t - w)]A^2 -$$
$$-(21+2B)(3+2B) +$$
$$+ 2(3-s)(3+B) - 12(3-s)swt^{-1} = 0, \quad (2.6)$$
$$B = \frac{(3-t)^2}{15 - 18t + 5t^2}(1+s).$$

If we now specify the values of t and s, from (2.5) it is easy to find w, and then from (2.6) and (2.4) we find A and ρ^2.

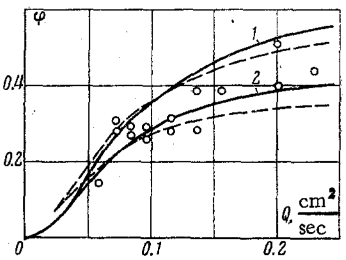

Fig. 5

Using (1.9), (2.3) it is easy to find the expressions for all the physical parameters in terms of t, s, A, w:

$$45^3\gamma R_0^{-11/3} = (1+s)^{-3}(1+t/w)^6 A^2 (5t^2 - 18t + 15)^3,$$
$$n^3 = R_0^{2/3}/\gamma A, \quad R = R_0(1 - t/w), \quad a_0^3 = R_0\nu^2/g,$$
$$\omega = (3-t)R\nu/3a_0, \quad \lambda = 2\pi a_0/n. \quad (2.7)$$

We see from (2.4) that real solutions of the considered problem exist for t/w > 0. From (2.5) follows that if t, s are not too large, then w will be positive and therefore we must have t > 0. Recalling the definition of t, we find that the ratio z of the wave propagation velocity to the average liquid velocity is always less than 3. We determine completely the region of existence of the wave regimes in the t, s plane if we assume that the inequalities $\infty > A^2 \geq 0$ must always be satisfied.

In accordance with (2.6), the limiting values of A^2 are reached on lines whose equations are

$$3 - t/w(3-t-w)(5/2 - t - w) = 0,$$
$$(21+2B)(3+2B) - 2(3-s)\times$$
$$\times(3+B) - 12(3-s)sw/t = 0. \quad (2.8)$$

These lines are shown in Fig. 1. The values of t, s included between these lines and the t = 0 axis are admissible, i.e., for any of these values the periodic solution of (1.9) in the first approximation exists and may be calculated with the aid of (2.4)–(2.7). The entire existence region consists of two parts having a common point with the coordinates t = 1.3101, s = −1 at which the boundary curves intersect. The portion of the region to the right of this point represents a very narrow lune whose vertex has the coordinates t = 1.4908, s = −∞. The values of t, s lying therein correspond to large flow rates; they are not used for comparison with the experimental data and therefore are not examined in detail.

Each periodic solution may be characterized by the number R_0 and the wavelength λ; in Fig. 1 this is the point of intersection of lines of constant values of R_0 and λ. All lines R_0 = const, λ = const begin at the point O and pass through the point N. The solutions represented by the point O have zero amplitude. In essence they were considered in [1], while the solutions constructed in [4] are represented by the segment of the s = 0 axis beginning at O. These solutions are bounded by the value R = 14.75 which is obtained as t → +0. The portion of the s-axis from O to K corresponds to zero flow rate.

For small values of t and s ≠ 0, i.e., small flowrates, it is not difficult to obtain approximate expressions for all the problem parameters if we limit ourselves to the first terms of the expansion in t

$$R^{11/3} = +27\gamma\frac{30}{101}\frac{(1+s)^3(1 - {}^1/_2 s)}{s(3-s)(1 - 62/303s)}t,$$

$$a_0 = \left(\frac{\nu^2}{g}\right)^{1/3} R^{1/3}, \quad \lambda = 2\pi\left(\frac{\nu^2}{g}\right)^{1/3}\frac{1}{\omega}R^{-1/3},$$

$$\omega = \nu\left(\frac{\nu^2}{g}\right)^{-1/3} R^{2/3}$$

$$n = dR^{5/6}, \qquad d = [3\gamma(1+s)]^{-1/2}. \quad (2.9)$$

All the periodic solutions for given R_0 are represented in Fig. 1 by the lines R_0 = const. With increase of the distance l from the point O along each of these lines the amplitude ρ increases monotonically, while the number R first increases and then decreases. At the point where R reaches a maximum, the derivative $\partial R/\partial l$ vanishes. Since R_0 and R are functions of t and s, this condition may be written

$$\frac{\partial R}{\partial t}\frac{\partial R_0}{\partial s} - \frac{\partial R}{\partial s}\frac{\partial R_0}{\partial t} = 0. \quad (2.10)$$

We introduce w and A in place of R and R_0; then the condition (2.1) may be written as

$$\frac{1}{A^2}(1+s)\left[\frac{\partial A^2}{\partial s} + \frac{t}{w}\left(\frac{\partial A^2}{\partial s}\frac{\partial w}{\partial s} - \frac{\partial w}{\partial t}\frac{\partial A^2}{\partial s}\right)\right] +$$

$$+ 3\frac{t}{w}\frac{\partial w}{\partial t} - 3 - 6\frac{t}{w}\frac{9-5t}{5t^2 - 18t + 15}(1+s)\frac{\partial w}{\partial s} = 0. \quad (2.11)$$

Here the derivatives are found with (2.5) and (2.6). The set of all such points constitutes the line MN in Fig. 1. In view of the symmetry of Eq. (2.10), a minimum value of R_0 for a given R is also reached on this line. Thus, MN is the line of optimum periodic solutions. Along this line the value of R increases monotonically: R = 0 for t = 0, and R = 95 for t = = 1.282. The calculation of these solutions may be carried out by joint solution of Eqs. (2.5), (2.6), (2.11); the results of the calculations are presented in the table.

In order to evaluate the accuracy of the solution constructed, let us consider the second approximation. We set $\Omega_i = 0$ (i = 3, 4, ...) and equate to zero the first seven coefficients of the Fourier series for the left side of Eq. (1.9). If we introduce the variables t, s, w, A^2, as was done in the first approximation, after lengthy calculations we obtain the equations for A^2 and w

$$6(3+\delta_0)(2-s)w^2 - [63+18B+6s-(3+B)t+$$
$$+6(3-t)(3+\delta_0)(2-s)-6\delta_1-$$
$$(21+2B)\delta_2]w - t(3-t)(3+B) = 0,$$
$$(21+2B)(3+2B-\delta_1)-2(3-s)\times$$
$$\times(3+B)+4(3-s)(3+\delta_0)swt^{-1}-$$
$$-6\{3-\delta_2-tw^{-1}(3-t-w)\times$$
$$\times[(1+{}^1/_3\delta_0)(3-t-w)-{}^1/_2]\}A^2 = 0,$$
$$\delta_0 = 2\rho^2[4^{-1}A^{-1}(18+B)\varphi_{20}-$$
$$-{}^3/_4\varphi_{21}+{}^3/_2\varphi_{20}^2-6A^{-1}\varphi_{20}\varphi_{21}+{}^3/_2\varphi_{21}^2],$$
$$\delta_1 = {}^1/_2\rho^2[-1+6A\varphi_{20}+4(12+B)\varphi_{21}+$$
$$+4(39+2B)\varphi_{30}-12A\varphi_{31}],$$
$$\delta_2 = \rho^2[-2A^{-1}(12+B)\varphi_{20}+3\varphi_{21}+$$
$$+6\varphi_{30}+2A^{-1}(39+2B)\varphi_{31}], \qquad (2.12)$$

and the coefficients of the expansion (1.9) are expressed as

$$A^{-1}\varepsilon\varphi_{20} = {}^1/_2f(3+2B-\delta_1)-(3-s)(3-\delta_2),$$
$$\varepsilon\varphi_{21} = (3-s)(3+2B-\delta_1)+{}^1/_2(3-\delta_2)fA^2,$$
$$\varepsilon_1\varphi_{30} = L_1fA^2-3(8-s)L_2,$$
$$A^{-1}\varepsilon_1\varphi_{31} = 3(8-s)L_1+fL_2,$$
$$f = -tw^{-1}(3-t-w),$$
$$\varepsilon = (fA)^2+4(3-s)^2, \quad \varepsilon_1 = 9(8-s)^2+(fA)^2,$$
$$L_1 = {}^1/_4[1+6(9+2B)A^{-1}\varphi_{20}-12\varphi_{21}],$$
$$L_2 = {}^1/_4[-(3+B)+12A\varphi_{20}+6(9+2B)\varphi_{21}].$$

If we set $\delta_0 = 0$, $\delta_1 = 0$, $\delta_2 = 0$ here, then we obtain the equations of the first approximation considered above.

The relations derived are convenient to use for calculations by iterations. In the second approximation the set of admissible values of t, s changes, and solution of (2.12) will not exist in some portion of the region shown in Fig. 1, and in the case in which solutions exist in both approximations the degree of difference between them will not be the same in various parts of this region.

Calculations show that for values of R of the order of 30 the difference between the first and second approximations with regard to all the parameters is completely negligible for the optimal regimes. This difference increases in the direction of small flow-rates and particularly in the direction of large rates. For values of R of order 50 this difference amounts to about 10%. If we construct graphs of the variation of a_0, λ, ω with flow rate for values of s, t lying on the line MN, as is done in Figs. 2–4, then we obtain curves which practically coincide. The difference is in the wave shape and, in particular, in the values of the smallest and largest oscillation amplitudes is more marked. This difference may be seen in Fig. 5, where the solid lines correspond to the first approximation and the dashed lines to the second (1 is the wave trough, 2 the crest).

3. Let us apply the constructed solution to the explanation of the experimental data, considering primarily the results of [2]. First of all we note that Eq. (1.9), whose periodic solution was studied, was obtained from the governing system of equations using certain simplifications; therefore, it describes the wave motion of a viscous liquid in a thin layer approximately. We can judge the accuracy by comparison with experimental data; however, some preliminary statements may be made. The accuracy will be poorer for small λ, comparable with the layer thickness a_0, since in this case the boundary layer approximation used becomes incorrect. For very large λ the wave profile has a complex form, as calculations show, and its representation by two or even three harmonics of the Fourier series is not exact; in this case the solution method used is not sufficiently accurate. Therefore the cases of long and short wavelengths must be excluded from consideration. We see in Fig. 1 that the lines of constant values of λ cluster together near the boundaries of the existence region, near which their density increases sharply. The major portion of this region is included between the lines λ = 0.7 cm and λ = 1.4 cm; the simplifications made are best justified for the wave regimes corresponding to this portion of the region.

Theoretically, for a given flow rate there exists an infinite number of wave regions which differ in wavelength, and there is no a priori indication of which will be observed experimentally. In reality it is found that if special measures are not taken the wave flow of the liquid layer will be unsteady and waves of different length are observed simultaneously. Clear-cut periodic motion is realized only in the case in which periodic disturbances of a frequency which is unique for the given flow rate are imposed on the flow. In this case a wavelength may be associated with each flow rate. Experiment does not indicate in what way this wavelength differs from the others and why this specific wavelength is realized from among the infinite set of possible regimes. However the theoretical examination leads to the conclusion that among all possible regimes there actually exist those which are in a definite sense exceptional regimes—these are the optimum regimes. It is natural to suggest that it is precisely these regimes which are experimentally observed. In Fig. 2 (and in the remaining figures) the solid curve shows the theoretical variation of wavelength with flow rate, and the dashed curves show the experimental results [2]. The close correspondence of the theoretical and experimental data indicate that the assumption concerning the optimum regimes will be correct or very close thereto.

In Fig. 1 the optimum regimes are represented by the points of the line MN. They exist in a large range of flows from R = 0 to R ≈ 100. The experimentally observed regimes [2] occupy only the portion of this range from about R = 20 to R = 60, i.e., the region of medium flow rates. A comparison of the theoretical and experimental data is shown in Figs. 3–5. Good agreement is obtained with respect to the values of the average thickness (in Fig. 3 the curve is for $1.07a_0$, since the experimental points represent the quantity $1.07 \cdot (a_{max} + a_{min})/2$) and the wave propagation velocity and somewhat poorer agreement with respect to the amplitude values, although nearly all the experimental points fall in the band between the maximum and minimum amplitudes or very close thereto. On the whole, this comparison confirms the rather good accuracy of the constructed solution.

REFERENCES

1. P. L. Kapitsa, "Wave flow of thin layers of a viscous liquid," Zh. eksperim. i teor. fiz., vol. 18, 1948.

2. P. L. Kapitsa and S. P. Kapitsa, "Wave flow of thin layers of a viscous liquid," Zh. eksperim. i teor. fiz., vol. 19, 1949.

3. L. D. Landau and E. M. Lifshits, The Mechanics of Continuous Media [in Russian], Gostekhizdat, 1954.

4. L. N. Maurin and V. S. Sorokin, "On the wave flow of thin layers of a viscous liquid," PMTF, 4, 1962.

14 October 1966 Moscow