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Analyticity Properties and Power Law Estimates of Functions in Percolation Theory

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We consider percolation on the sites of a graph $\mathcal{G}$, e.g., a regular d-dimensional lattice. All sites of $\mathcal{G}$ are occupied (vacant) with probability p (respectively, $q = 1 - p$), independently of each other. W denotes the cluster of occupied sites containing a fixed site (which will usually be taken to be the origin) and $\# W$ the cardinality of W. The percolation probability θ is the probability that $\# W = \infty$, i.e., $\theta(p) = P_p\{\# W = \infty\}$. Some critical values of p, p_H and p_T, are defined, respectively, as the smallest value of p for which $\theta(p) > 0$, and for which the expectation of $\# W$ is infinite. Formally, $p_H = \inf\{p : \theta(p) > 0\}$ and $p_T = \inf\{p : E_p\{\# W\} = \infty\}$. We show for fairly general graphs $\mathcal{G}$ that if $p < p_T$, then $P_p\{\# W \geqslant n\}$ decreases exponentially in n. For the special cases $\mathcal{G} = \mathcal{G}_0 =$ the simple quadratic lattice and $\mathcal{G} = \mathcal{G}_1 =$ the graph which corresponds to bond-percolation on $\mathbb{Z}^2$, we obtain upper and lower bounds for $\theta(p)$ of the form $C|p - p_H|^{\alpha}$, and bounds for $E_p\{\# W\}$ of the form $C|p - p_H|^{-\alpha}$. We also investigate smoothness properties of $\Delta(p) = E_p\{\text{number of clusters per site}\} = E_p\{(\# W)^{-1}; \# W \geqslant 1\}$. This function was introduced by Sykes and Essam, who assumed that $\Delta(\cdot)$ has exactly one singularity, namely, at $p = p_H$. For the graphs $\mathcal{G}_0$ and $\mathcal{G}_1$ (i.e., site or bond percolation on $\mathbb{Z}^2$) we show that $\Delta(p)$ is analytic at $p \neq p_H$ and has two continuous derivatives at $p = p_H$. The emphasis is on rigorous proofs.

KEY WORDS: Percolation theory; cluster size distribution; exponential decay; power laws; number of clusters per site; critical probability; analyticity and smoothness properties.

1. INTRODUCTION AND STATEMENT OF RESULTS

Consider a periodic graph $\mathcal{G}$ in $\mathbb{R}^d$. By this we mean that the vertices and edges of $\mathcal{G}$ are embedded in $\mathbb{R}^d$ and that there exist d linearly independent vectors $U_1, \ldots, U_d \in \mathbb{R}^d$ such that the set of vertices and edges of $\mathcal{G}$ is

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invariant under translation by any of the U_i. Two vertices v_1 and v_2 of $\mathcal{G}$ will be called *adjacent* or *neighbors of each other* if there is an edge of $\mathcal{G}$ between v_1 and v_2. A *path* is a sequence $v_0, \ldots, v_n$ of vertices for which v_{i+1} is adjacent to v_i for $0 \leqslant i \leqslant n-1$. *Unless stated otherwise a path is assumed to be self-avoiding*, i.e., to have $v_i \neq v_j$ for $i \neq j$. A set of vertices is *connected* if for every pair of vertices v_1, v_2 in the set there is a path which starts with v_1 and ends with v_2. *Throughout we only consider periodic* $\mathcal{G}$ *with the following properties*:

$$\text{There exists a } z < \infty \text{ such that each vertex is an end point of at most } z \text{ edges} \tag{1.1}$$

$$\text{Each compact set of } \mathbb{R}^d \text{ contains at most finitely many vertices of } \mathcal{G} \tag{1.2}$$

and

$$\mathcal{G} \text{ is connected} \tag{1.3}$$

In site percolation all vertices (or sites) of $\mathcal{G}$ are chosen, independently of each other, to be occupied or vacant. Except in Section 2 we only consider the case where all vertices have the same probability, p, of being occupied. The corresponding probability measure is denoted by P_p, and E_p denotes expectation with respect to P_p. A convenient way of describing the configuration of occupied sites is by means of the random variables

$$X(v) = \begin{cases} +1 & \text{if } v \text{ is occupied} \\ -1 & \text{if } v \text{ is vacant} \end{cases}$$

Each configuration can now be specified by specifying all $X(v)$, i.e., by specifying a point of $\Omega = \prod_{v \in \mathcal{G}} \{-1, +1\}$. P_p can then be identified with the product measure on Ω with each marginal distribution given by $P_p\{X(v) = +1\} = p$, $P_p\{X(v) = -1\} = 1 - p$. The best-known example is site percolation on the square lattice. In this case the sites of $\mathcal{G}$ are the points of $\mathbb{Z}^2 = \{(i_1, i_2) : i_1, i_2 \text{ integral}\}$ and two points (i_1, i_2) and (j_1, j_2) of $\mathbb{Z}^2$ have an edge between them if and only if

$$|i_1 - j_1| + |i_2 - j_2| = 1 \tag{1.4}$$

We shall denote this graph by $\mathcal{G}_0$.

Another common variant of percolation deals with bond percolation. In this model, the edges (or bonds) of $\mathcal{G}$ are chosen independently to be open or blocked. It is well known[1,2] that bond percolation can be viewed as a special case of site percolation, by going over to a so-called covering lattice. Each edge of $\mathcal{G}$ is viewed as a vertex of the covering lattice $\tilde{\mathcal{G}}$, and two vertices of $\tilde{\mathcal{G}}$ are adjacent if their corresponding edges in $\mathcal{G}$ have a vertex in common. An open (blocked) edge of $\mathcal{G}$ corresponds to an occupied (vacant) vertex of $\tilde{\mathcal{G}}$. Because of the greater generality of the site problem we shall phrase everything in this paper in terms of site percolation (even though this does make the formulation less conceptual for some examples). The classical example of bond percolation on the square lattice ($\mathbb{Z}^2$) is by the above translation scheme equivalent to the site problem on the following graph $\mathcal{G}_1$: the vertices of $\mathcal{G}_1$ are the points of $\mathbb{Z}^2$ and the points (i_1, i_2) and (j_1, j_2) are adjacent on $\mathcal{G}_1$ when (1.4) holds or when

$$|i_1 - j_1| = |i_2 - j_2| = 1, \text{ together with } i_1 + i_2 \text{ odd or } i_1 - i_2 \text{ even} \tag{1.5}$$

$\mathcal{G}_1$ is obtained from $\mathcal{G}_0$ by adding as edges the diagonals in alternating squares. Bond percolation on the square lattice will be referred to here simply as percolation on $\mathcal{G}_1$.

A set of vertices will be called occupied when all vertices in the set are occupied. An *occupied cluster* is a maximal connected occupied set of vertices. W_v will denote the occupied cluster containing v; $W_v = \emptyset$ if v is vacant. $\# W_v$ denotes the cardinality of W_v. W will be the occupied cluster of some singled out vertex w_0. Usually we take w_0 to be the origin. In many examples all vertices of $\mathcal{G}$ play the same role so that it makes no difference how we choose w_0. In any case, it is known[3] [compare also (2.42) and (2.43) below] that the critical probabilities which we define now are independent of the choice of w_0. The *percolation probability* is

$$\theta(p) = P_p\{\# W = \infty\} \tag{1.6}$$

The critical probability of most interest is

$$p_H = \inf\{p : \theta(p) > 0\} \tag{1.7}$$

In addition one also uses

$$p_T = \inf\{p : E_p\{\# W\} = \infty\} \tag{1.8}$$

and a further critical probability p_S, defined in terms of "sponge crossing probabilities."[4,5] To define these, take $U_1, \ldots, U_d$ as the basis for $\mathbb{R}^d$ so that each vertex v of $\mathcal{G}$ can be written uniquely as

$$v = \sum_{i=1}^{d} \lambda_i(v) U_i \tag{1.9}$$

The "sponge" $T_0(n; i)$ is the parallelepiped

$$T_0(n; i) = \{v \text{ a vertex of } \mathcal{G} : 0 \leqslant \lambda_j(v) \leqslant 3n, j \neq i, \text{ and } 0 \leqslant \lambda_i(v) \leqslant n\} \tag{1.10}$$

We call a path[2] $v_0, v_1, \ldots, v_r$ an *i-crossing* of $T_0(n; i)$ if each v_s is occupied,

[2] Note that the initial point v_0 and the end point v_r may be outside $T_0(n; i)$.

$0 \leqslant s \leqslant r$, and for $0 < s < r$

$$v_s \in T_0(n; i) \text{ as well as } \lambda_i(v_0) \leqslant 0 < \lambda_i(s) < n \leqslant \lambda_i(v_r) \tag{1.11}$$

The crossing probability of $T_0(n; i)$ is now

$$\tau_0(n; i) = \tau_0(n; i, p) = P_p\{\exists\ i\text{-crossing of } T_0(n; i)\} \tag{1.12}$$

The critical probability p_S is defined by[3]

$$p_S = \inf\left\{ p : \limsup_{n\to\infty} \max_{1 \leqslant i \leqslant d} \tau_0(n; i, p) > 0\right\} \tag{1.13}$$

We need two more lattice constants:

$$\mu = \#\{v \in \mathcal{G} : 0 \leqslant \lambda_i(v) < 1, 1 \leqslant i \leqslant d\} \tag{1.14}$$

and Λ is an integer such that

$$|\lambda_i(v') - \lambda_i(v'')| \leqslant \Lambda \text{ for all pairs of adjacent vertices } v', v'' \in \mathcal{G} \quad \text{and} \quad 1 \leqslant i \leqslant d \tag{1.15}$$

It is clear that Λ can be taken finite by virtue of the periodicity of $\mathcal{G}$ and $\mu < \infty$ [see (1.2)].

For instance if $\mathcal{G} = \mathcal{G}_0$ we can take $U_1 = (1, 0)$, $U_2 = (0, 1)$, i.e., the usual coordinate vectors. With this choice $\lambda_i(x)$ is simply the ith (Cartesian) coordinate of x, $\mu = 1$ and $\Lambda = 1$. τ_0 is now the probability of crossing an $n \times 3n$ rectangle in the short direction, and p_S is the smallest value of p for which this τ_0 does not tend to zero.

The following corollary is a special case of Theorem 1, and is valid for any periodic graph $\mathcal{G}$.

Corollary 1

$$p_T = p_S \leqslant p_H \tag{1.16}$$

Set

$$\kappa = \kappa(d) = d^{-1}(2e7^d)^{-13^d} \tag{1.17}$$

Then for $p > p_S$ and $N \geqslant \Lambda$ we must have

$$\max_{1 \leqslant i \leqslant d} \tau_0(N; i, p) > \kappa \tag{1.18}$$

Moreover, if (1.18) fails for some $N \geqslant \Lambda$ and some p (in particular for $p < p_S = p_T$), then there exist constants $0 < C_i = C_i(p, \mathcal{G}) < \infty$ such that

$$P_p\{\# W \geqslant n\} \leqslant C_1 e^{-C_2 n}, \qquad n \geqslant 0 \tag{1.19}$$

[3] It is important to note that the present definition of p_S differs from those used in Refs. 4–6, since those references consider the crossing probabilities of squares rather than rectangles. Only for graphs with sufficient symmetry properties can we prove that the two definitions lead to the same value of p_S. This is for instance true when $\mathcal{G} = \mathcal{G}_0$ or $\mathcal{G} = \mathcal{G}_1$.

Theorem 1 itself is more general in that it allows different vertices to have different probabilities of being occupied.

We note that (1.19) shows that for $p < p_T$ not only is the first moment of $\# W$ finite, but all moments of $\# W$ are finite for such p. At p_T we have the following general result:

Corollary 2

$$E_{p_T}\{\# W\} = \infty \tag{1.20}$$

(See Theorem 3 for further information for the special graphs $\mathcal{G}_0$ and $\mathcal{G}_1$).

There is good reason to believe that $p_T = p_S = p_H$ in general, but so far this has only been proved for bond and site percolation on $\mathbb{Z}^2$ (i.e., $\mathcal{G} = \mathcal{G}_0$ or $\mathcal{G}_1$)[7,8] and for bond percolation on the triangular and hexagonal lattice.[9] $p_T = p_H$ has also been proved by Griffeath,[10] Section 10, for oriented site percolation on the first quadrant of $\mathbb{Z}^2$. All but the last case are examples of graphs in $\mathbb{R}^2$ which have a matching lattice in the sense of Sykes and Essam.[11] We shall use $\mathcal{G}^*$ to denote the matching lattice when it exists. Sykes and Essam tried to characterize the critical probability as the singularitity of the "average number of occupied clusters per site." It was shown by Grimmett[12] and Wierman[13] that this average number of occupied clusters per site can be defined as a thermodynamic limit, and that it equals

$$\Delta(p) \equiv \frac{1}{\mu} \sum_v {}^* E_p \left\{ \frac{1}{\# W_v} ; \# W_v \geqslant 1 \right\} \tag{1.21}$$

where $\sum_v^*$ runs over those $v \in \mathcal{G}$ with $0 \leqslant \lambda_i(v) < 1, 1 \leqslant i \leqslant d$, and μ is the number of such v [see (1.14)]. Sykes and Essam proved the remarkable result that for any pair of matching graphs $\mathcal{G}$ and $\mathcal{G}^*$ there exists a polynomial in p, $\Phi(p)$, such that

$$\Delta(p) - \Delta^*(1-p) = \Phi(p) \tag{1.22}$$

Of course $\Delta^*(p)$ is defined by (1.21) but with $\mathcal{G}$ replaced by $\mathcal{G}^*$; in general, if A is a quantity defined for percolation on $\mathcal{G}$, then A^* denotes the same quantity for percolation on $\mathcal{G}^*$. Sykes and Essam conjectured that $\Delta(p)$ has only one singularity as a function of p, and that this occurs at $p = p_H$. They then used (1.22) and other relations to calculate p_H for various lattices. The next theorem confirms their conjecture for the examples listed above as having $p_T = p_S = p_H$ (excluding the directed percolation example).

Theorem 2. Let $\mathcal{G}$ and $\mathcal{G}^*$ be a pair of matching (in the sense of[11]) periodic graphs in $\mathbb{R}^2$. Then $\Delta(p)$ is an analytic function of p for all p outside $[p_T, 1 - p_T^*]$. For site or bond percolation on $\mathbb{Z}^2$ (i.e., $\mathcal{G} = \mathcal{G}_0$ or $\mathcal{G}_1$)

$\Delta(p)$ is analytic for $p \neq p_H$ and twice continuously differentiable for all $p \in [0,1]$.

We note that Theorem 2 says nothing about a singularity of $\Delta(\cdot)$ at $p = \frac{1}{2}$ for $\mathcal{G} = \mathcal{G}_1$, or at $p = p_H$ in general. At present we do not know whether such a singularity exists[4]; see end of Section 4 for some further details.

In analogy with various other models in statistical mechanics one might hope that several functions of p with a singularity at p_H behave asymptotically as a power of $|p - p_H|$ as $p \to p_H$ from above and/or below (see Refs. 14–17). In Section 4 we obtain estimates for $\theta(p)$ and $E_p\{\#W\}$ in powers of $|p - p_H|$ for $\mathcal{G} = \mathcal{G}_0$ or $\mathcal{G} = \mathcal{G}_1$.

Theorem 3. When $\mathcal{G} = \mathcal{G}_0$ or $\mathcal{G} = \mathcal{G}_1$, there exist constants $0 < C_i < \infty$ and $0 < \alpha_i < \infty$ such that

$$C_3(p - p_H)^{\alpha_3} \leqslant \theta(p) \leqslant C_4(p - p_H)^{\alpha_4}, \qquad p > p_H \tag{1.23}$$

$$C_5(p_H - p)^{-\alpha_5} \leqslant E_p\{\#W\} \leqslant C_6(p_H - p)^{-\alpha_6}, \qquad p < p_H \tag{1.24}$$

and

$$C_7(p - p_H)^{-\alpha_7} \leqslant E_p\{\#W; \#W < \infty\} \leqslant C_8(p - p_H)^{-\alpha_8}, \qquad p > p_H \tag{1.25}$$

Moreover, uniformly for $0 \leqslant p \leqslant 1$

$$P_p\{N \leqslant \#W < \infty\} \leqslant C_9 N^{-\alpha_9} \tag{1.26}$$

and

$$E_p\{(\#W)^{\alpha_9/2}; \#W < \infty\} \leqslant C_{10} \tag{1.27}$$

Remarks. (i) Of course $p_H = \frac{1}{2}$ in the above theorem when $\mathcal{G} = \mathcal{G}_1$. Also, since it is known[1,8,18] that $\theta(p) = 0$ for $p \leqslant p_H$ for $\mathcal{G} = \mathcal{G}_0$ or $\mathcal{G}_1$, (1.27) implies

$$E_p\{(\#W)^{\alpha_9/2}\} \leqslant C_{10} \qquad \text{for } p \leqslant p_H$$

Theorem 3 will be proven in Section 3 before Theorem 2, because the proof contains some estimates on the distribution of $\#W$ which are useful for Theorem 2. It appears that the method of proof will also work for other graphs.

(ii) As we shall see in Section 4 it is easy to see from Theorem 1 that $\Delta(p)$ and $E_p\{(\#W)^m\}$ are analytic in $p < p_T$ for any $\mathcal{G}$ and any $m \geqslant 0$.

[4] We had previously claimed that $\Delta(\cdot)$ did not have a fourth derivative at $p = \frac{1}{2}$ when $\mathcal{G} = \mathcal{G}_1$. However, our "proof" of this contained an error.

However, we know nothing about analyticity of $\Delta(p)$ for $p > p_T$ except for matching pairs of graphs. Similarly we do not know whether $\theta(p)$ is analytic for any $p > p_H$. [It is, of course, analytic for $p < p_H$ and not at $p = p_H$ because $\theta(p) = 0$ for $p < p_H$ and $\theta(p) > 0$ for $p > p_H$.]

2. THE EXPONENTIAL DECAY OF $P_p\{\# W \geq n\}$

In this section we consider a slightly more general setup than discussed in the introduction. We allow $P\{X(v) = 1\} = P\{v \text{ is occupied}\}$ to vary with v. However, to preserve the periodicity we insist that

$$P\{X(v_1) = 1\} = P\{X(v_2) = 1\} \tag{2.1}$$

whenever v_2 is obtainable from v_1 by a translation through a multiple of any U_i. Thus (2.1) must hold, whenever

$$v_2 - v_1 = \sum_{i=1}^{d} k_i U_i \qquad \text{for some } k_i \in \mathbb{Z}$$

Since there are only finitely many vertices of $\mathcal{G}$ in the compact set $\{v : 0 \leqslant \lambda_i(v) \leqslant 1, 1 \leqslant i \leqslant d\}$ $P\{X(v) = 1\}$ can take only finitely many different values. These values will be fixed throughout this section and will not be indicated explicitly in the notation. P will denote the corresponding probability measure when all $X(v)$ are again independent, and E will denote expectation with respect to P. For $\bar{n} = (n_1, \ldots, n_d)$ with each n_i a positive integer we introduce the parallelepipeds

$$T(\bar{n}; i) = \{v \in \mathcal{G} : 0 \leqslant \lambda_j(v) \leqslant 3n_j, j \neq i, \text{ and } 0 \leqslant \lambda_i(v) \leqslant n_i\} \tag{2.2}$$

As in (1.11) an i-crossing of $T(\bar{n}; i)$ is a path[5] $v_0, \ldots, v_r$ with all v_s occupied $0 \leqslant s \leqslant r$, and for $0 < s < r$ $v_s \in T(\bar{n}; i)$, as well as $\lambda_i(v_0) \leqslant 0 < \lambda_i(v_s) < n_i \leqslant \lambda_i(v_r)$. The crossing probability for the ith direction of $T(\bar{n}; i)$ is

$$\tau(\bar{n}; i) = P\{\exists \text{ an } i\text{-crossing of } T(\bar{n}; i)\} \tag{2.3}$$

We remind the reader that Λ was defined in (1.15) and $W = W_{w_0}$ for some fixed site w_0.

Theorem 1. Let

$$\kappa = \kappa(d) = d^{-1}(2e7^d)^{-13^d} \tag{2.4}$$

If there exists an $\bar{N} = (N_1, \ldots, N_d)$ such that

$$N_i \geqslant \Lambda \qquad \text{and } \tau(\bar{N}; i) \leqslant \kappa \quad \text{for } i = 1, \ldots, d \tag{2.5}$$

[5] Again v_0 and v_r may be outside $T(\bar{n}; i)$.

then there exist constants $0 < C_1, C_2 < \infty$ such that

$$P\{\# W \geqslant n\} \leqslant C_1 e^{-C_2 n}, \qquad n \geqslant 0 \tag{2.6}$$

[see (2.28)–(2.30) for the values of C_1, C_2]. If $P\{v \text{ is occupied}\} >$ for all v and

$$E\{\# W\} < \infty \tag{2.7}$$

then

$$\tau(\bar{n}; i) \to 0, \qquad 1 \leqslant i \leqslant d \tag{2.8}$$

as $\bar{n} \to \infty$ along the diagonal [i.e., n of the form $(n, n, \ldots, n)$] and hence (2.6) holds.

Kunz and Souillard[19] already proved (2.6) if $P\{X(v) = 1\} < (z-1)^{-1}$ for all v, where z satisfies (1.1). The present proof is a reduction of condition (2.6) to the case of small $P\{X(v) = 1\}$ by a block approach; the parallelepiped $T(\bar{n})$ and suitable translates of it are viewed as vertices of an auxiliary graph $\mathcal{L}$. A similar construction of an auxiliary graph also occurs in the closely related Lemma 2 of Ref. 8. The remainder of this section is devoted to the details of the proof of Theorem 1. The proof is broken up into several lemmas. As in Ref. 19 we bring in the number of connected sets of a given size and given boundary size and containing w_0. For any set $\mathcal{C}$ of vertices of $\mathcal{G}$, $\#\mathcal{C}$ denotes the cardinality of $\mathcal{C}$. $\partial\mathcal{C}$ is the set of vertices of $\mathcal{G} \backslash \mathcal{C}$ which are adjacent to some vertex of $\mathcal{C}$, and $\#\partial\mathcal{C}$ its cardinality. We set $a(0, l) = \delta_{1,l}$ and for $n \geqslant 1$

$$\begin{aligned} a(n,l) = {} & \text{number of connected sets } \mathcal{C} \text{ containing } w_0 \\ & \text{with } \#\mathcal{C} = n \quad \text{and} \quad \#\partial\mathcal{C} = l \end{aligned} \tag{2.9}$$

Lemma 1. For any $0 \leqslant p \leqslant 1$, $q = 1 - p$,

$$\sum_{n=0}^{\infty} \sum_{l \geqslant 0} a(n,l) p^n q^l = 1 - \theta(p) \leqslant 1 \tag{2.10}$$

Consequently

$$a(n,l) \leqslant \left(\frac{n+l}{n}\right)^n \left(\frac{n+l}{l}\right)^l \tag{2.11}$$

Also

$$\begin{aligned} \sum_{l \geqslant 0} a(n,l) & \leqslant (z-1)^{2+(z-1)n} (z-2)^{-(z-2)n} \\ & \leqslant (z-1)^2 e^n (z-1)^n \end{aligned} \tag{2.12}$$

and for some universal constant $\epsilon_0 > 0$, and all $0 \leqslant p \leqslant 1$, $q = 1 - p$,

$0 \leqslant x \leqslant \epsilon_0$,

$$\sum_{l \text{ with } |pl-qn| \geqslant xnpq} a(n,l)p^n q^l \leqslant \{(z-2)n+2\}\exp\left(-\frac{x^2p^2q}{3}n\right) \quad (2.13)$$

Proof. (2.10) is well known and immediate from

$$P_p\{W = \mathcal{C}\} = p^n q^l \quad (2.14)$$

for any connected $\mathcal{C}$ containing w_0 with $\#\mathcal{C} = n$, $\#\partial\mathcal{C} = l$; the left-hand side of (2.10) simply equals $P_p\{W = \mathcal{C}$ for some finite $\mathcal{C}\}$. (2.11) follows from (2.10) by taking $p = n/(n+l)$, $q = l/(n+l)$. Also (2.12) follows from (2.10) by taking $p = (z-1)^{-1}$, $q = (z-2)(z-1)^{-1}$ and the observation that for any connected $\mathcal{C}$

$$1 \leqslant \#\partial\mathcal{C} \leqslant (\#\mathcal{C})(z-2)+2 \quad (2.15)$$

(cf. Ref. 5, pp. 141, 142; (2.15) can easily be proved by induction on $\#\mathcal{C}$). As a consequence of (2.15) the sums in (2.10) and (2.12) over l can be restricted to $1 \leqslant l \leqslant n(z-2)+2$. Finally, by virtue of (2.15) and (2.11), the left-hand side of (2.13) is bounded by

$$\sum_{|pl-qn| \geqslant xnpq} a(n,l)\left\{\frac{(n+l)p}{n}\right\}^n \left\{\frac{(n+l)q}{l}\right\}^l \left(\frac{n}{n+l}\right)^n \left(\frac{l}{n+l}\right)^l$$

$$\leqslant \{(z-2)n+2\}\max\left\{\frac{(n+l)p}{n}\right\}^n \left\{\frac{(n+l)q}{l}\right\}^l \quad (2.16)$$

where the maximum in the right-hand side is over all $0 \leqslant p \leqslant 1$ and $1 \leqslant l \leqslant (z-2)n+2$ with $|pl-qn| \geqslant xnpq$. Now fix n and $1 \leqslant l \leqslant (z-2)$ $n+2$ and consider

$$f(p) = n\log\frac{n+l}{n} + n\log p + l\log\frac{n+l}{l} + l\log q \quad (2.17)$$

One easily sees that f is increasing in p for $pl - qn < 0$ and decreasing for $pl - qn > 0$. It follows that its maximum over the set $\{p : |pl-qn| \geqslant xnpq\}$ is taken on when $pl - qn = \pm xnpq$. One easily sees that when x is small, and $pl - qn = \pm xnpq$, then $f(p) = -\frac{1}{2}nx^2p^2q[1 + O(x)]$. (2.13) follows. ∎

We now assume that $\bar{N}$ is such that (2.5) holds. We introduce an auxiliary graph $\mathcal{L}$ and a percolation problem on $\mathcal{L}$. The vertices of $\mathcal{L}$ are the points of $\mathbb{Z}^d$, and two such points $\bar{k} = (k_1, \ldots, k_d)$ and $\bar{l} = (l_1, \ldots, l_d)$ are connected by an edge of $\mathcal{L}$, or adjacent in $\mathcal{L}$, if $|k_i - l_i| \leqslant 3$ for $i = 1, \ldots, d$. The vertices of $\mathcal{L}$ are again divided into two classes, white and black, say; the vertex $\bar{k}$ is colored white when at least one of the events $E(\bar{k}, i) = E(\bar{k}, i; \bar{N})$ occurs, and black otherwise. Here

$$E(\bar{k}, i) = \left\{ \text{there exists an } i\text{-crossing of } T(\bar{N}; i) + \sum_{j=1}^{d} k_j N_j U_j \right\} \tag{2.18}$$

Of course an i-crossing of $T(\bar{N}; i) + \sum k_j N_j U_j$ is an occupied path $v_0, \ldots, v_r$ with $k_j N_j \leqslant \lambda_j(v_s) \leqslant (k_j + 3)N_j$ for $j \neq i$ and $\lambda_i(v_0) \leqslant k_i N_i < \lambda_i(v_s) < (k_i + 1)N_i \leqslant \lambda_i(v_r)$, $0 < s < r$. The event $E(\bar{k}, i)$ is simply obtained by "shifting" $E(\bar{0}, i)$ by $\sum k_j N_j u_j$, where $\bar{0}$ stands for $(0, \ldots, 0)$. Thus, by (2.1)

$$P\left\{E(\bar{k}, i)\right\} = P\left\{E(\bar{0}, i)\right\} = \tau(\bar{N}; i) \tag{2.19}$$

The following lemma describes a crucial relation between W, the occupied cluster of w_0 on $\mathcal{G}$, and certain clusters on $\mathcal{L}$. $\bar{\nu} = (\nu_1, \ldots, \nu_d)$ is defined by

$$\nu_j N_j \leqslant \lambda_j(w_0) < (\nu_j + 1)N_j \tag{2.20}$$

and μ is defined in (1.14).

Lemma 2. Assume W contains a vertex v with

$$k_j N_j \leqslant \lambda_j(v) < (k_j + 1)N_j, \qquad 1 \leqslant j \leqslant d \tag{2.21}$$

and

$$k_m \leqslant \nu_m - 2 \quad \text{or} \quad k_m \geqslant \nu_m + 2 \qquad \text{for some } m \tag{2.22}$$

Then there exists a self-avoiding path $\bar{k}_0, \ldots, \bar{k}_r$ of white points on $\mathcal{L}$ such that $\bar{k}_0$ is adjacent to $\bar{k} = (k_1, \ldots, k_d)$ and $\bar{k}_r$ is adjacent to $\bar{\nu}$ on $\mathcal{L}$. Furthermore, if $\tilde{W}(\bar{l})$ is the white cluster on $\mathcal{L}$ containing $\bar{l}$, then

$$\max_{\substack{\bar{l} \text{ adjacent} \\ \text{to } \bar{\nu}}} \# \tilde{W}(\bar{l}) \geqslant 7^{-2d}\left(\# W - \mu 4^d \prod_{j=1}^{d} N_j\right) \Big/ \left(\mu \prod_{j=1}^{d} N_j\right) \tag{2.23}$$

Proof. Assume $v \in W$ satisfies (2.21). Then there exists a path $v_0 = v, v_1, \ldots, v_s = w_0$ on $\mathcal{G}$, with all v_j occupied. If also (2.22) holds, then $v_s = w_0$ does not belong to the set

$$\{v \in \mathcal{G} : (k_j - 1)N_j < \lambda_j(v) < (k_j + 2)N_j, 1 \leqslant j \leqslant d\} \tag{2.24}$$

Since v_0 does belong to this set by (2.21), there is a smallest index b for which v_b is outside the set (2.24). For the sake of argument let

$$\lambda_i(v_b) \geqslant (k_i + 2)N_i$$

Since $\lambda_i(v_0) = \lambda_i(v) \leqslant (k_i + 1)N_i$, $b > 0$ and there is a last index $a \leqslant b$ with

$$\lambda_i(v_a) \leqslant (k_i + 1)N_i$$

Moreover, for $t < b$, v belongs to (2.24), whence

$$(k_j - 1)N_j < \lambda_j(v_t) < (k_j + 2)N_j, \qquad a < t < b, \qquad 1 \leqslant j \leqslant d$$

Thus the path $v_a, \ldots, v_b$ is an i-crossing of

$$T(\bar{N}; i) + \sum_{j=1}^{d} k_{0j} N_j U_j \quad \text{with} \qquad k_{0j} = k_j - 1, \qquad j \neq i, \qquad k_{0i} = k_i + 1 \tag{2.25}$$

We take $\bar{k}_0 = (k_{01}, \ldots, k_{0d})$. Note that

$$|k_j - k_{0j}| \leqslant 1, \qquad 1 \leqslant j \leqslant d \tag{2.26}$$

and, by construction, $\bar{k}_0$ is white and adjacent to $\bar{k}$. We now try to repeat the above process with v replaced by v_1. Since $N_j \geqslant \Lambda$, and v_1 is adjacent to v, we have

$$l_j N_j \leqslant \lambda_j(v_1) < (l_j + 1)N_j, \qquad 1 \leqslant j \leqslant d$$

for some $\bar{l} = (l_1, \ldots, l_d)$ with $|k_j - l_j| \leqslant 1$. If (2.22) holds with k_m replaced by l_m, then the above construction yields a white point $\bar{k}_1 = (k_{11}, \ldots, k_{1d})$ with $|k_{1j} - l_j| \leqslant 1$. By virtue of (2.26) $|k_{1j} - k_{0j}| \leqslant 3$ so that $\bar{k}_1$ is adjacent to $\bar{k}_0$ on $\mathfrak{L}$. Also, part of $v_1, \ldots, v_s$ will be an m crossing of $T(\bar{N}; m) + \sum k_{1j} N_j U_j$ for some m. If possible we now replace v by v_2 etc. The process can be continued until we arrive at a white point $\bar{k}_r$ and a crossing of $T(\bar{N}; m) + \sum k_{rj} N_j U_j$, corresponding to the point v_r of the original path, while for some l

$$l_j N_j \leqslant \lambda_j(v_{r+1}) < (l_j + 1)N_j, \qquad 1 \leqslant j \leqslant d$$

and

$$|l_m - k_{rm}| \leqslant 2 \text{ as well as } |l_m - \nu_m| \leqslant 1 \text{ for all } 1 \leqslant m \leqslant d \tag{2.27}$$

(2.27) implies that $\bar{k}_r$ is adjacent to $\bar{\nu}$. Note that there must be such a v_{r+1}, so that the process will stop, by virtue of (2.20). The constructed sequence $\bar{k}_0, \ldots, \bar{k}_r$ consists of white points on $\mathfrak{L}$ with successive points adjacent to each other and $\bar{k}_0$ ($\bar{k}_r$) adjacent to $\bar{k}$ (respectively, $\bar{\nu}$). However, it may fail to be self-avoiding. In that case it can be made self-avoiding by removal of loops, so that the first part of the lemma is proved.

The second part now follows easily. Each point v of W with $|\lambda_i(v) - \lambda_i(w_0)| \geqslant 2N_i$ for some i satisfies (2.21) and (2.22) for some $\bar{k}$. For fixed $\bar{k}$, there are at most $\mu N_1 \cdots N_d$ vertices v which satisfy (2.21). Thus, if $\# W = M$, then there are at least

$$\left\{ M - \mu \prod_{1}^{d} (4N_j) \right\} \{ \mu N_1 \cdots N_d \}^{-1}$$

distinct values of $\bar{k}$ for which there exists a $v \in W$ such that (2.21) and

(2.22) are satisfied. For each such $\bar{k}$ we can find a white path $\bar{k}_0, \ldots, \bar{k}_r$ on $\mathfrak{L}$ starting at a neighbor $\bar{k}_0$ of $\bar{k}$ and ending at a neighbor of $\bar{\nu}$. Since each point of $\mathfrak{L}$ has fewer than 7^d neighbors, a given value of $\bar{k}_0$ can be used for at most 7^d distinct values of $\bar{k}$. Also $\bar{k}_r$ can have at most 7^d values. Thus, there are at least

$$7^{-2d}\left\{M - \mu\prod_1^d(4N_j)\right\}\{\mu N_1 \cdots N_d\}^{-1}$$

distinct white points $\bar{k}_0$ connected by white paths on $\mathfrak{L}$ to some white neighbor $\bar{l}$ of $\bar{\nu}$. This proves (2.23). ■

Lemma 3. (2.5) implies (2.6) with

$$A = 7^{-2d}(\mu N_1 \cdots N_d)^{-1} \tag{2.28}$$

$$C_1 = e^{-1}7^{2d}\left\{\sum_i \tau(\bar{N};i)\right\}^{-13^{-d}} \times\left[1 - e7^d\left\{\sum_i \tau(\bar{N},i)\right\}^{13^{-d}}\right]^{-1} \leqslant 7^{2d}\left\{\sum_i \tau(\bar{N};i)\right\}^{-13^{-d}} \tag{2.29}$$

$$e^{-C_2} = (e7^d)^A\left\{\sum_i \tau(\bar{N};i)\right\}^{A13^{-d}} \leqslant 2^{-A} \tag{2.30}$$

Proof. By Lemma 2

$$P\{\#W \geqslant n\} \leqslant \sum_{\substack{\bar{l}\text{ adjacent}\\ \text{to }\bar{\nu}}} P\left\{\#\tilde{W}(\bar{l}) \geqslant An - 1\right\} \tag{2.31}$$

Put

$\tilde{a}(m)$ = number of connected sets on $\mathfrak{L}$ of m vertices and containing $\bar{\nu}$

and let $\tilde{\mathcal{C}}$ stand for a generic connected set on $\mathfrak{L}$. Then the right-hand side of (2.31) is bounded by

$$7^d \sum_{m \geqslant An-1} \tilde{a}(m) \max_{\#\tilde{C}=m} P\{\text{all points of }\tilde{\mathcal{C}}\text{ are white}\} \tag{2.32}$$

Here we used the fact that $\bar{\nu}$ has fewer than 7^d neighbors $\bar{l}$ on $\mathfrak{L}$, and that the number of connected sets of size m and containing a given $\bar{l}$ is the same for each $\bar{l}$ [i.e., $= \tilde{a}(m)$]. To estimate the probability in (2.32) we observe that strictly speaking we are not dealing with a percolation problem on $\mathfrak{L}$ because the colors of different vertices are not independent. Nevertheless, the color of the vertex $\bar{k} = (k_1, \ldots, k_d)$ depends only on the occupancy of the vertices v with

$$(k_j - 1)N_j \leqslant k_j N_j - \Lambda \leqslant \lambda_j(v) \leqslant (k_j + 3)N_j + \Lambda \leqslant (k_j + 4)N_j \qquad \text{for all } 1 \leqslant j \leqslant d$$

Thus, if $\bar{k}_1, \ldots, \bar{k}_t$ are vertices of $\mathcal{L}$ such that for each $r \neq s$ there exists an i with $|k_{ri} - k_{si}| \geqslant 6$, then the colors of $\bar{k}_1, \ldots, \bar{k}_t$ depend on disjoint sets of vertices of $\mathcal{G}$, and hence are independent. Now, if $\tilde{\mathcal{C}}$ is a given connected set of vertices of $\mathcal{L}$ with $\#\tilde{\mathcal{C}} = m$, then we can choose $\bar{k}_1, \ldots, \bar{k}_t$ in $\tilde{\mathcal{C}}$ with the above property for some $t \geqslant 13^{-d}m$. With $\bar{k}_1, \ldots, \bar{k}_t$ chosen in this way we have, by virtue of (2.19),

$$P\{\text{all points of } \tilde{\mathcal{C}} \text{ are white}\} \leqslant P\{\bar{k}_1, \ldots, \bar{k}_t \text{ are white}\} \leqslant \left\{\sum_{i=1}^{d} \tau(\bar{N}; i)\right\}^t$$

Substituting this estimate with $t = 13^{-d}m$ into (2.32) we obtain

$$P\{\#W \geqslant n\} \leqslant 7^d \sum_{m \geqslant An-1} \tilde{a}(m)\left\{\sum_{i=1}^{d} \tau(\bar{N}; i)\right\}^{13^{-d}m} \tag{2.33}$$

Finally, (2.12) applied to $\mathcal{L}$ with 7^d for z shows

$$\tilde{a}(m) \leqslant 7^{2d}(7^d e)^m$$

This together with (2.33) yields (2.6) with the values (2.29), (2.30) of C_i. ■

Lemma 4. (2.7) implies (2.8).

Proof. When $d = 2$ this is easy and is proved for instance in Ref. 5 (cf. proof of Theorem 3.1). It also follows from Proposition 1 of Ref. 20. Most of the proof of this proposition actually works for any d. We indicate here the necessary modifications to make the proof go through for $d > 2$. Lemma 1 of Ref. 20 needs to be replaced by the following argument. For $v, w \in \mathcal{G}$ and n, N any positive integers make the following definitions:

$$A(v,n) = \{\exists \text{ an occupied path } v_0, v_1, \ldots, v_r \text{ with } v_0 \text{ adjacent to } v \text{ and } \lambda_1(v_r) \geqslant n\}$$

$$S_0 = S_0(v,N) = \{w \in \mathcal{G} : |\lambda_i(w) - \lambda_i(v)| \leqslant N, 1 \leqslant i \leqslant d\} \tag{2.34}$$

$$S_1 = S_0 \cup \partial S_0 = \{w \in \mathcal{G} : w \in S_0 \text{ or } w \text{ adjacent to a point of } S_0\}$$

$$g(v,w,N) = P\{\exists \text{ selfavoiding occupied path } v_0, v_1, \ldots, v_r \text{ which passes through } w, \text{ has } v_0 \text{ adjacent to } v, \text{ and } v_r \notin S_0\}$$

Here, and below, we say that the path $v_0, \ldots, v_r$ passes through w, if w is one of the v_j.

We claim that if $\lambda_1(v) < n - N$, then

$$P\{A(v,n)\} \leqslant \sum_{w \in S_1(v,N)} g(v,w,N) P\{A(w,n)\} \tag{2.35}$$

and that there exists an N_0 such that

$$\sum_{w \in S_1(v,N)} g(v,w,n) \leqslant \tfrac{3}{4}, \qquad v \in \mathcal{G}, \qquad N \geqslant N_0 \tag{2.36}$$

Once (2.35) and (2.36) have been proved it is easy to show [e.g., by specializing the proof of Proposition 1 of Ref. 20 with (2.35) replacing Lemma 1 and (2.36) replacing (2.8) of Ref. 20] that $P\{A(v,n)\}$ decreases exponentially in n, uniformly in $\lambda_1(v) \leqslant 0$. Since for $\bar{n} = (n, n, \ldots, n)$

$$\tau(\bar{n}; 1) = O\Big[n^{d-1} \max_{\lambda_1(v) \leqslant 0} P\{A(v,n)\} \Big] \tag{2.37}$$

this will prove (2.8) for $i = 1$, and the proof for $i = 2, \ldots, d$ is the same.

We turn to the proof of (2.35). Assume $\lambda_1(v) < n - N$ and $A(v,n)$ occurs. Then there exists an occupied path $v_0, \ldots, v_r$ with v_0 adjacent to v and $\lambda_1(v_r) \geqslant n > \lambda_1(v) + N$, and hence $v_r \notin S_0$. Thus, there is a lowest index a such that $v_a \notin S_0$. Now let

$$R = \{ w \in S_1 : \exists \text{ a path } w_0, w_1, \ldots, w_s \text{ which passes through } w, \text{ such that } w_0 \text{ is adjacent to } v,\ w_s \notin S_0, \text{ but } w_t \in S_0 \text{ for } t < s \}$$

In other words, R is the random set of occupied points of S_1 through which there exists an occupied path from a neighbor of v to the complement of S_0, which, except for its endpoint lies in S_0. By choice of a, $v_a \in R$. Let $b \geqslant a$ be the last index with $v_b \in R$ and consider the path $(v_{b+1}, \ldots, v_r)$. This is an occupied path lying entirely outside R, starting at the neighbor v_{b+1} of v_b and ending at v_r with $\lambda_1(v_r) \geqslant n$. Thus, taking into account all possibilities for v_b, we have

$$\begin{aligned} P\{A(v,n)\} &\leqslant \sum_{w \in S_1} P\{ w \in R \text{ and } \exists \text{ an occupied path } w_0, w_1, \ldots, w_s \\ &\qquad \text{with } w_0 \text{ adjacent to } w,\ \lambda_1(w_s) \geqslant n \\ &\qquad \text{and } w_t \notin R \text{ for } 0 \leqslant t \leqslant s\} \\ &= \sum_{w \in S_1} \sum_{\substack{\mathcal{C} \subset S_1 \\ w \in \mathcal{C}}} P\{ R = \mathcal{C} \text{ and } \exists \text{ an occupied path} \\ &\qquad w_0, w_1, \ldots, w_s \text{ with } w_0 \text{ adjacent to } w, \\ &\qquad \lambda_1(w_s) \geqslant n \text{ and } w_t \notin \mathcal{C} \text{ for } 0 \leqslant t \leqslant s\} \end{aligned} \tag{2.38}$$

We now fix $w \in S_1$ and a set $\mathcal{C} \subset S_1$, containing w, and estimate the probability in the last member of (2.38). We observe that $R = \mathcal{C}$ can occur only if all vertices of $\mathcal{C}$ are occupied, and that the indicator function of $\{R = \mathcal{C}\}$ is decreasing in all the $X(u)$, $u \notin \mathcal{C}$. This is so because $R = \mathcal{C}$ occurs if there exist suitable occupied paths in $\mathcal{C}$ through every point of $\mathcal{C}$, and in addition any path $u_0, \dots, u_s$, which starts at a neighbor u_0 of v and ends at u_s outside S_0 and has $u_t \in S_0$ for $t < s$ and has some point u_t outside $\mathcal{C}$, is not entirely occupied. Therefore

$$I[R = \mathcal{C}] = I[\mathcal{C} \text{ is occupied}]J$$

for some decreasing function J of the $\{X(u) : u \notin \mathcal{C}\}$. It now follows from the independence of the vertices in and outside $\mathcal{C}$ and the FKG inequality (see Ref. 5, Section 2.2) that the probability in the right-hand side of (2.38) is at most

$$\begin{aligned} &P\{\mathcal{C} \text{ is occupied}\}E\{J\}P\{\exists \text{ an occupied path } w_0, w_1, \dots, w_s \\ &\quad \text{with } w_0 \text{ adjacent to } w,\ \lambda_1(w_s) \geqslant n \text{ and } w_t \notin \mathcal{C} \text{ for } 0 \leqslant t \leqslant s\} \\ &\quad \leqslant P\{R = \mathcal{C}\}P\{A(w,n)\} \end{aligned} \tag{2.39}$$

Substitution of (2.39) into (2.38) yields

$$\begin{aligned} P\{A(v,n)\} &\leqslant \sum_{w \in S_1} \sum_{\substack{\mathcal{C} \in S_1 \\ w \in \mathcal{C}}} P\{R = \mathcal{C}\}P\{A(w,n)\} \\ &= \sum_{w \in S_1} P\{w \in R\}P\{A(w,n)\} \\ &\leqslant \sum_{w \in S_1} g(v,w,N)P\{A(w,n)\} \end{aligned}$$

This proves (2.35). (2.36) is easy because any path from a neighbor u of v to the complement of $S_0(v,N)$ contains at least N/Λ points. Thus

$$\begin{aligned} g(v,w,N) \leqslant P\{&w \in W_u \text{ and } \# W_u \geqslant N/\Lambda \\ &\text{for some neighbor } u \text{ of } v\} \end{aligned} \tag{2.40}$$

and

$$\sum_{w \in S_1(v,N)} g(v,w,N) \leqslant \sum_{\substack{u \text{ adjacent} \\ \text{to } v}} E\{\# W_u; \# W_u \geqslant N/\Lambda\} \tag{2.41}$$

For each $u \in \mathcal{G}$, by virtue of the FKG inequality,

$$\begin{aligned} P\{\# W \geqslant n\} &\geqslant P\{w_0 \text{ is connected to } u \text{ by an occupied} \\ &\qquad \text{path and } \# W_u \geqslant n\} \\ &\geqslant P\{w_0 \text{ is connected to } u \text{ by an occupied path}\} \\ &\qquad P\{\# W_u \geqslant n\} \end{aligned} \tag{2.42}$$

so that (2.7) implies for each u

$$\sum_{n=1}^{\infty} n P\{\# W_u = n\} = \sum_{n=1}^{\infty} P\{\# W_u \geqslant n\}$$
$$= E\{\# W_u\} < \infty \tag{2.43}$$

Both series in (2.43) converge uniformly in u, because of the invariance of the probability measure P under translation by any U_i. Thus (2.36) is immediate from (2.41). As pointed out before this implies the lemma. ∎

Theorem 1 is just a combination of Lemmas 3 and 4. Corollary 1 and 2 of the introduction are almost immediate. Indeed in the special case when all vertices have the same probability p of being occupied, then for $p < p_S$, by definition

$$\max_{1 \leqslant i \leqslant d} \tau_0(n; i, p) = \max_{1 \leqslant i \leqslant d} \tau(\bar{n}; i) \to 0$$

so that (2.6) holds for such p. Clearly (2.6) shows that $E_p\{\# W\} < \infty$ and that no percolation takes place. Consequently $p < p_S$ implies $p \leqslant p_T$ and $p \leqslant p_H$. Conversely, if $p < p_T$, then (2.7) and hence (2.6) holds. In turn (2.6) immediately shows that $\tau_0(n; i, p) \to 0$ $(n \to \infty)$ [compare (2.37) and Ref. 5 p. 32]. This proves (1.16). The proof of (1.20) is essentially in Remark 4 of Ref. 20. For any p with $E_p\{\# W < \infty\}$ there exists an n such that $\tau(\bar{n}; i) = \tau_0(n; i, p) \leqslant \frac{1}{2}\kappa(d)$, by Lemma 4. But for fixed n $\tau_0(n; i, p)$ is continuous in p, hence $\tau_0(n; i, \tilde{p}) < \kappa(d)$ for $\tilde{p} \leqslant p + \delta$ for some $\delta > 0$. From (2.6) it then follows that also

$$E_{\tilde{p}}\{\# W\} < \infty \quad \text{for } \tilde{p} \leqslant p + \delta \tag{2.44}$$

By definition of p_T (2.44) cannot hold for $p = p_T$, whence (1.20).

All other statements in Corollaries 1 and 2 are obvious.

3. POWER LAW ESTIMATES

In this section we prove Theorem 3. Again the proof is split into several lemmas. The upper bound for θ is fairly simple (see Lemma 6). All the other estimates are based on Lemma 7, which is a sharper form of the main argument in Ref. 7. Once Lemma 7 has been proved we quickly obtain in Proposition 1 a lower bound for the probability that the origin is connected to a square of size N, when $p = p_H$. The bound is of the order $N^{-1+\alpha}$ for some $\alpha > 0$; the known lower bound of $(2N)^{-1}$ for this probability given for $\mathcal{G}_1$ in Ref. 5, p. 61, and Ref. 20, Remark 4 is insufficient for our purposes. Proposition 1 also gives an upper bound for sponge crossing probabilities for $p < p_H$. Theorem 3 follows fairly easily from Proposition 1 and Theorem 1.

Before the proofs proper we collect some facts from Refs. 4, 6, and 8. Unless otherwise specified $\mathcal{G}$ in this section is any one of the four graphs $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_0^*, \mathcal{G}_1^*$. When necessary we shall indicate the dependence of various quantities on $\mathcal{G}$ in a self-explanatory manner; e.g., $p_H(\mathcal{G})$ will be the critical probability p_H for the graph $\mathcal{G}$ · $\mathcal{G}_0$ and $\mathcal{G}_1$ where described in the introduction. Both have vertex set $\mathbb{Z}^2$ and are invariant under translations by multiples of the vectors $U_1 = (2,0)$ and $U_2 = (0,2)$. With this choice of U_1, U_2 the point $(v_1, v_2) \in \mathbb{Z}^2$ has $\lambda_1(v) = i_1/2$, $\lambda_2(v) = i_2/2$. These facts also apply to $\mathcal{G}_0^*$ and $\mathcal{G}_1^*$. Two points (i_1, i_2) and (j_1, j_2) of $\mathbb{Z}^2$ are adjacent on $\mathcal{G}_0^*$ if

$$|i_1 - j_1| + |i_2 - j_2| = 1 \quad \text{or} \quad |i_1 - j_1| = |i_2 - j_2| = 1 \tag{3.1}$$

On $\mathcal{G}_1^*$ (i_1, i_2) and (j_1, j_2) are adjacent if

$$\begin{gathered} |i_1 - j_1| + |i_2 - j_2| = 1 \quad \text{or} \quad |i_1 - j_1| = |i_2 - j_2| = 1 \\ \text{together with } i_1 + i_2 \text{ even or } i_1 - i_2 \text{ odd} \end{gathered} \tag{3.2}$$

Thus $\mathcal{G}_1^*$ is isomorphic to $\mathcal{G}_1$ and $\mathcal{G}_0^*$ is obtained from $\mathcal{G}_0$ by adding as edges all diagonals of lattice squares. In all four graphs all points play the same role so that we shall henceforth take $w_0 = 0$, and W as the occupied component of the origin. We shall drop the subscript 0 in (2.34). Thus

$$S(v, N) = \{ w \in \mathcal{G} : |\lambda_i(w) - \lambda_i(v)| \leqslant N, i = 1, 2 \} \tag{3.3}$$

We shall also need the rectangles

$$S(v, N, k) = \{ w : |\lambda_1(w) - \lambda_1(v)| \leqslant kN, |\lambda_2(w) - \lambda_2(v)| \leqslant N \} \tag{3.4}$$

and the annuli

$$R(v, k) = S(v, 3^k) / S(v, 3^{k-1}) \tag{3.5}$$

An occupied *left–right crossing of* $S(v, N, k)$ is an occupied path $v_0, \ldots, v_r$ with

$$\begin{gathered} -kN = \lambda_1(v_0) < \lambda_1(v_s) < kN = \lambda_1(v_r), \qquad 1 \leqslant s \leqslant r-1, \\ \text{and } -N \leqslant \lambda_2(v_t) \leqslant N, \qquad 0 \leqslant t \leqslant r \end{gathered}$$

Note that we can take $\lambda_1(v_0) = -kN$ and $\lambda_1(v_r) = kN$ because for the present graphs $\mathcal{G}$ a path cannot cross a line $x_1 = \pm kN$ in $\mathbb{R}^2$ without passing through a vertex on this line. Up–down crossings and vacant crossings are defined similarly. The crossing probabilities of interest are

$$\tau_1(N; k; p) = \tau_1(N, k; \mathcal{G}, p) = P_p\{\exists \text{ an occupied left–right crossing of } S(0, N, k) \text{ on } \mathcal{G}\} \tag{3.6}$$

Since for our graphs $\mathcal{G}$ a rotation over 90° takes $\mathcal{G}$ into a graph isomorphic to $\mathcal{G}$ we have also

$$\tau_1(N, 1; p) = P_p\{\exists \text{ an occupied up–down crossing of } S(v, N) \text{ on } \mathcal{G}\} \quad (3.7)$$

It follows from Refs. 4, 6, and 8 that there exists a $\lambda_1 > 0$ such that

$$\tau_1(N, 1; \mathcal{G}, p_H(\mathcal{G})) \geqslant \gamma_1 > 0, \qquad N \geqslant 1 \quad (3.8)$$

For $\mathcal{G} = \mathcal{G}_1$ or $\mathcal{G}_1^*$ (3.8) is formula (11) in Ref. 4 since $\mathcal{G}_1$ and $\mathcal{G}_1^*$ are isomorphic and $p_H(\mathcal{G}_1) = p_H(\mathcal{G}_1^*) = \frac{1}{2}$. For $\mathcal{G} = \mathcal{G}_0$ or $\mathcal{G}_0^*$ (3.8) follows from Ref. 8. Indeed, if $\tau_1(N, 1; \mathcal{G}_0, p_H(\mathcal{G}_0)) \to 0$ along some sequence of N's, then along the same sequence $\tau_1(N, 1; \mathcal{G}_0^*, 1 - p_H(\mathcal{G}_0)) = \tau_1(N, 1; \mathcal{G}_0^*, p_H(\mathcal{G}_0^*)) \to 1$ because $p_H(\mathcal{G}_0^*) = 1 - p_H(\mathcal{G}_0))$ (see Ref. 8) and if there is no occupied left–right crossing of $S(0, N)$ on $\mathcal{G}_0$, then there must be a vacant up–down crossing of $S(0, N)$ on $\mathcal{G}_0^*$ (compare Ref. 4, Theorem 4.1). But then also $\tau_1(N, 3; \mathcal{G}_0^*, p_H(\mathcal{G}_0^*)) \to 1$ (by Lemma 1 of Ref. 8) and we know that this is not the case (by Lemma 2 of Ref. 8). Thus (3.8) holds for $\mathcal{G}_0$, and the proof for $\mathcal{G}_0^*$ is the same. It follows as in Ref. 4 Lemmas 5.2–5.4, Ref. 6 Lemmas 3 and 4, and Ref. 8 Lemma 1 that there exist constants $\gamma_k > 0$ such that

$$\tau_1(N, k; \mathcal{G}, p_H(\mathcal{G})) \geqslant \gamma_k > 0, \qquad N \geqslant 1 \quad (3.9)$$

There also exists a $\gamma_0 > 0$ such that

$$P_{p_H(\mathcal{G})}\{\exists \text{ an occupied circuit on } \mathcal{G} \text{ surrounding } v \text{ inside the annulus } R(v, k)\} \geqslant \gamma_0 > 0 \quad (3.10)$$

for all $v \in \mathcal{G}$, $k \geqslant 1$. Here a circuit is a path $v_0, \ldots, v_r$ with $v_0 = v_r$. It is said to surround v if it is impossible to connect v to ∞ by a continuous path in $\mathbb{R}^2$ without intersecting one of the edges from v_i to v_{i+1}, $0 \leqslant i < r$.

Last we introduce a notation for the perimeter of a square, and the event that 0 is connected to the perimeter of $S(0, N)$:

$$\Delta S(v, N) = \{w : |\lambda_i(w) - \lambda_i(v)| \leqslant N \text{ for } i = 1, 2 \text{ with equality for at least one } i\} \quad (3.11)$$

$$B(N) = \{\exists \text{ an occupied path } v_0 = 0, v_1, \ldots, v_r \text{ in } S(0, N) \text{ starting at the origin and with endpoint } v_r \in \Delta S(0, N)\} \quad (3.12)$$

Lemma 5. Set

$$\beta_0 = -\frac{1}{\log 3} \log(1 - \gamma_0)$$

There exists a $C_{11} < \infty$ for which

$$P_{p_H}\{B(N)\} \leqslant C_{11} N^{-\beta_0}, \qquad N \geqslant 1 \quad (3.13)$$

Moreover, for any $\beta < \beta_0/2$

$$E_{p_H}\left\{(\# W)^{\beta}\right\} < \infty \tag{3.14}$$

Proof. It is used repeatedly in Refs. 4–6 and 8 (see Lemma 5.6 of Ref. 4, Theorem 2 of Ref. 6, as well as Corollary 3.7 and Lemma 3.11 of Ref. 5) that if there exists a vacant circuit on $\mathcal{G}^*$ in $R(v,k)$, then W_v does not contain any vertex of $\mathcal{G}$ outside $S(v, 3^k)$. Here $(\mathcal{G}_i^*)^* = \mathcal{G}_i$. This fact implies as in the above references

$$\begin{aligned} P_{p_H(\mathcal{G})}\{B(3^n)\} \leqslant P_{p_H(\mathcal{G})}\{&\text{there does not exist a vacant circuit} \\ &\text{surrounding the origin in } R(0,j) \text{ on} \\ &\mathcal{G}^* \text{ for all } 1 \leqslant j \leqslant k\} \\ \leqslant (1-\gamma_0)^k \end{aligned}$$

Consequently, for $k_0 = [\log N/\log 3]$

$$P_{p_H(\mathcal{G})}\{B(N)\} \leqslant P_{p_H(\mathcal{G})}\{B(3^{k_0})\} \leqslant (1-\gamma_0)^{k_0}$$

which immediately implies (3.13).

(3.14) is immediate from (3.13), because if $B(3^k)$ fails, then $W \subset S(0, 3^k)$ and hence $\# W \leqslant (2.3^k + 1)^2$. Thus, for some constant C

$$\begin{aligned} E_p\left\{(\# W)^{\beta}\right\} &\leqslant C \sum_{k=0}^{\infty} 3^{2k}(2.3^k+1)^{2(\beta-1)} P_p\left\{\# W > (2.3^k+1)^2\right\} \\ &= O\left(\sum_{k=0}^{\infty} 3^{2k\beta} P_p\left\{B(3^k)\right\}\right) < \infty \end{aligned}$$

at $p = p_H$, as soon as $\beta < \beta_0/2$ [compare proof of Lemma (5.6) in Ref. 4]. ■

We now use a general argument to derive a bound for $P_p\{B(N)\}$ from (3.13) when $p > p_H$. This immediately gives the upper bound in (1.23) on $\theta(p)$.

Lemma 6. For $p_2 \geqslant p_1$

$$P_{p_1}\{B(N)\} \geqslant (p_1/p_2)^{(4N+1)^2} P_{p_2}\{B(N)\} \tag{3.15}$$

and there exists a $0 < C_4 < \infty$ (independent of p) such that

$$\theta(p) \leqslant C_4(p - p_H)^{\beta_0/2}, \qquad p \geqslant p_H \tag{3.16}$$

Proof. $B(N)$ depends only on the occupancy of the $(4N+1)^2$ sites in $S(0,N)$. Now let $p_1 \leqslant p_2$ and construct P_{p_1} in two stages. In the first stage choose the occupancy of all sites according to P_{p_2}, i.e., independent of each other and with

$$P_{p_2}\{v \text{ is occupied}\} = p_2$$

Next, if v came out vacant in this first stage, leave it vacant. If it was occupied in the first stage make it occupied (vacant) in the second stage with conditional probability p_1/p_2 (respectively, $1-p_1/p_2$), and do this independently for all sites which are occupied in the first stage. One easily sees that at the end of the second stage the occupancy of the sites is distributed according to P_{p_1}. Now, since $S(0,N)$ contains $(4N+1)^2$ sites

$$P_{p_1}\{B(N)\} \geqslant P\{\exists \text{ occupied path } v_0=0, v_1, \ldots, v_r \text{ in } S(0,N) \text{ from } 0 \text{ to } \Delta S(0,N) \text{ in the first stage and each } v_i \text{ is still occupied in the second stage}\}$$

$$\geqslant (p_1/p_2)^{(4N+1)^2} P_{p_2}\{B(N)\} \tag{3.17}$$

This proves (3.15). We now take $p_1 = p_H$, $p \geqslant p_H$ and use the obvious inequality $\theta(p) \leqslant P_p\{B(N)\}$ for any N. Together with (3.13) this yields

$$\theta(p) \leqslant P_p\{B(N)\} \leqslant (p/p_H)^{(4N+1)^2} C_{11} N^{-\beta_0} \tag{3.18}$$

This gives (3.16) by taking

$$N = \left[\{\log p/p_H\}^{-1/2}\right] \sim (p-p_H)^{-1/2} p_H^{1/2} \quad \blacksquare$$

Remark (iii). Instead of (3.15) we can use Lemma 3 of Ref. 8. For our situation this lemma gives

$$\frac{d}{dp} P_p\{B(N)\} \leqslant \text{number of vertices in } S(0,N) = (4N+1)^2 \tag{3.19}$$

This together with (3.13) shows that for $p \geqslant p_H$

$$\begin{aligned}\theta(p) \leqslant P_p\{B(N)\} &\leqslant P_{p_H}\{B(N)\} + (4N+1)^2(p-p_H)\\ &\leqslant C_{11}N^{-\beta_0} + (4N+1)^2(p-p_H)\end{aligned} \tag{3.20}$$

If we now choose

$$N = \left[(p-p_H)^{-1/(\beta_0+2)}\right]$$

we obtain (3.16) with $\beta_0/2$ replaced by $\beta_0/(\beta_0+2)$.

We need some preparations for the next lemma, which is quite similar to the proof of Proposition 1 of Ref. 7. A minor complication arises because $\mathcal{G}$ is not necessarily planar. The polygonal curves obtained by connecting successive points of a path by line segments can intersect at a point which is not a vertex of $\mathcal{G}$. However, such an intersection would have to be the center of a lattice square of $\mathbb{Z}^2$, i.e., the intersection of the two diagonals of the lattice square. For this reason we shall bring in an auxiliary planar graph $\mathcal{G}_p$. $\mathcal{G}_p$ is obtained from $\mathcal{G}$ by adding a site to $\mathcal{G}$ at the center of each lattice square of $\mathbb{Z}^2$ where two edges of $\mathcal{G}$ cross. The new site at the center of square S will be adjacent on $\mathcal{G}_p$ exactly to the four corners of S. Thus a single diagonal edge on $\mathcal{G}$ is divided on $\mathcal{G}_p$ into two edges. Now let $U = (u_0, \ldots, u_t)$ be a path on $\mathcal{G}$. πU is called a *polygon through* U (on $\mathcal{G}$) if it is a polygonal curve along edges of $\mathcal{G}_p$ such that the successive sites of $\mathcal{G}$ through which it passes are exactly the points of U. A side of the polygonal curve πU is therefore a straight line segment between two neighbors on $\mathbb{Z}^2$, or between a point $(i_1, i_2) \in \mathbb{Z}^2$ and $(i_1 \pm \frac{1}{2}, i_2 \pm \frac{1}{2})$. We can make any polygon into a non-self-intersecting polygon by the usual procedure of loop removal. If the polygon goes through a point d twice we remove the piece between the first and last passage through d. It is not difficult to check that one loop removal will take πU into a polygon $\pi \tilde{U}$ through some path $\tilde{U}$ of the form $(u_0, \ldots, u_i, u_j, \ldots, u_t) \subset U$, but with the same end points u_0 and u_t. $u_{i+1}, \ldots, u_{j-1}$ are the sites of U which lie on the removed piece of πU (including d once if d is one of the u's). By repeating this procedure we will end up with a polygon with the same initial and end point as πU, but without double points. From now on we therefore restrict ourselves to non-self-intersecting polygons. *Polygon will mean a polygon without double points.*

Now let a and Θ be vertices of $\mathcal{G}_p$ and $k, N \geqslant 0$ such that

$$S(a, 2^k) \subset S(\Theta, N) \tag{3.21}$$

Also, let $V = (v_0, \ldots, v_r)$ and $W = (w_0, \ldots, w_s)$ be two paths in $S(\Theta, N)$ with end point on $\Delta S(\Theta, N)$ and let πV (πW) be a polygon through V (respectively, W) starting at a and ending at v_r (respectively, w_s) and contained in $S(\Theta, N)$. Thus we have

$$v_t \in S(\Theta, N) \setminus \Delta S(\Theta, N), \qquad 0 \leqslant t < r$$
$$v_r \in \Delta S(\Theta, N) \tag{3.22}$$
$$w_t \in S(\Theta, N) \setminus \Delta S(\Theta, N), \qquad 0 \leqslant t < s$$
$$w_s \in \Delta S(\Theta, N) \tag{3.23}$$
$$V \subset \pi V, \qquad \pi V \text{ goes from } a \text{ to } v_r, \qquad \pi V \subset S(\Theta, N)$$
$$W \subset \pi W, \qquad \pi W \text{ goes from } a \text{ to } w_s, \qquad \pi W \subset S(\Theta, N) \tag{3.24}$$

In addition we assume

$$\pi V \cap \pi W = \{a\} \tag{3.25}$$

In this situation $\pi V \cup \pi W$ is a simple curve which divides the interior of $S(\Theta, N)$ into two components, each bounded by $\pi V \cup \pi W$ and one of the arcs on $\Delta S(\Theta, N)$ between v_r and w_s (the end points of πV and πW). Denote these components (in any order) by $S'(\Theta) = S'(\Theta, \pi V, \pi W)$ and $S''(\Theta) = S''(\Theta, \pi V, \pi W)$. Let R be some subset of $S(\Theta, N)$. We shall say that v_t is *connected to W in R* if there exists an occupied path $u_1, \ldots, u_l$ on $\mathcal{G}$ such that all $u_j \in R$, u_1 is adjacent on $\mathcal{G}$ to v_t, and u_l is adjacent on $\mathcal{G}$ to some w_j, $0 \leqslant j \leqslant s$. We include in this the case where v_t is adjacent to some w_j, in which case no u's are needed. Now set

$$Y(v_t, k, \pi V, \pi W) = \begin{cases} 1 & \text{if } v_t \text{ is connected to } W \text{ in } S'(\Theta) \cap S(a, 2^k) \\ 0 & \text{otherwise} \end{cases} \tag{3.26}$$

Finally

$$z(k) = \min_{\Theta, N, a, \pi V, \pi W, \epsilon} E_{p_H} \left\{ \sum_{v_t \in S(a, 2^k)} Y(v_t, k, \pi V, \pi W) \,\middle|\, X(v) = \epsilon(v), v \in \bar{S}''(\Theta) \right\} \tag{3.27}$$

The minimum in (3.27) is over all $\Theta, N, a, \pi V, \pi W$ which satisfy (3.21)–(3.25) and all choices of $\epsilon(v) = +1$ or -1 for[6] $v \in \bar{S}''(\Theta)$; $X(v)$ is as in the introduction. Note that the sum in (3.27) is simply the number of sites of V in $S(a, 2^k)$ which are connected to W in $S'(\Theta) \cap S(a, 2^k)$. This sum is actually independent of all $X(v)$ with $v \in \bar{S}''(\Theta)$, so that the conditioning in (3.27) is superfluous; it is nevertheless useful for the proof of (3.28) and (3.29) below to introduce this conditioning.

We shall also need a $z^*(k)$ which is defined analogously to $z(k)$, but with the requirement that some paths are on $\mathcal{G}^*$ instead of $\mathcal{G}$. Specifically, we take for W a path on $\mathcal{G}^*$ and for πW a polygon through W on $\mathcal{G}^*$, and now call v_t $*$-connected to W in R if there exists a path $u_1^*, \ldots, u_l^*$ on $\mathcal{G}^*$, such that all $u_j^* \in R$, all u_j^* *vacant*, u_1^* adjacent on $\mathcal{G}^*$ to v_t and u_l^* adjacent on $\mathcal{G}^*$ to some w_j, $0 \leqslant j \leqslant s$. In (3.26) we now replace connected by $*$-connected and denote the resulting random variable by $Y^*(v_t, k, \pi V, \pi W)$. $z^*(k)$ is given by (3.27) with Y replaced by Y^*.

Lemma 7. There exist constants $0 < C_{12}, \alpha_{12} < \infty$ such that

$$z(k) \geqslant C_{12} 2^{\alpha_{12} k}, \qquad k \geqslant 0 \tag{3.28}$$

and

$$z^*(k) \geqslant C_{12} 2^{\alpha_{12} k}, \qquad k \geqslant 0 \tag{3.29}$$

Proof. We restrict ourselves to (3.28), since the proof of (3.29) is practically identical. (3.28) will be proved by showing

$$z(k) \geqslant (1 + \beta_1) z(k-3), \qquad k \geqslant 3 \tag{3.30}$$

[6] For any set $B \subset \mathbb{R}^2$, $\bar{B}$ denotes its closure.

for some constant $\beta_1 > 0$. To prove (3.30) fix $\Theta, N, a, k, \pi V$, and πW such that (3.21)–(3.25) hold and consider the square $S(a, 2^{k-3}) \subset S(a, 2^k)$. Then (3.21)–(3.25) certainly hold with $S(a, 2^k)$ replaced by $S(a, 2^{k-3})$, and trivially $Y(v_t, k, \pi V, \pi W) \geqslant Y(v_t, k-3, \pi V, \pi W)$. Thus

$$E_{P_H}\left\{ \sum_{v_t \in S(a, 2^{k-3})} Y(v_t, k, \pi V, \pi W) \,|\, X(v) = \epsilon(v), v \in \bar{S}''(\Theta) \right\}$$
$$\geqslant E_{P_H}\left\{ \sum_{v_t \in S(a, 2^{k-3})} Y(v_t, k-3, \pi V, \pi W) \,|\, X(v) = \epsilon(v), v \in \bar{S}''(\Theta) \right\}$$
$$\geqslant z(k-3) \tag{3.31}$$

Next consider the annulus

$$A \equiv \text{closure of } \left\{ S(a, 2^{k-1} + 2^{k-3}) \backslash S(a, 2^{k-1} - 2^{k-3}) \right\}$$
$$\subset S(a, 2^k) \subset S(\Theta, N)$$

Further, let $U = (u_1, \ldots, u_l)$ be a path on $\mathcal{G}$ such that

$$U \subset S'(\Theta) \cap A, u_1 \text{ is adjacent on } \mathcal{G} \text{ to some } v_\nu \in V$$
$$\text{and } u_l \text{ is adjacent on } \mathcal{G} \text{ to some } w_\rho \in W \tag{3.32}$$

In (3.32) we allow the possibility that $U = \emptyset$ if $v_\nu \in V$ is adjacent to $w_\rho \in W$. Now let πU be a polygon through U on $\mathcal{G}$ such that

$$\pi U \subset S'(\Theta) \cap A \text{ except for its end points, } \xi \text{ and } \tau \text{ say,}$$
$$\text{which lie on } \pi V \cap A \text{ respectively, } \pi W \cap A \tag{3.33}$$

If $U = \emptyset$ we interpret (3.33) as meaning that πU is a polygon on $\mathcal{G}_p$ which (with the possible exception of its end points) does not contain any vertex of $\mathcal{G}$, but connects a point $\xi \in \pi V \cap A$ to a point $\tau \in \pi W \cap A$ while $\pi U \backslash \{\xi, \tau\} \subset S'(\Theta) \cap A$. Note that in any case the initial point ξ is either a vertex of $\mathcal{G}$ on πV, i.e., one of the v_j, or a new vertex of $\mathcal{G}_p$, i.e., a center of a lattice square. In the latter case ξ must be between two vertices v_ν and $v_{\nu+1}$ on πV, i.e., πV contains the segments $[v_\nu, \xi]$ and $[\xi, v_{\nu+1}]$. Moreover if $U \neq \emptyset$, then πU begins with the segment $[\xi, u_1]$ and u_1 is adjacent to v_ν and $v_{\nu+1}$ on $\mathcal{G}$. In fact, v_ν, $v_{\nu+1}$ and u_1 must be three corners of a lattice square with ξ at its center. A similar comment applies to τ.

When (3.33) holds, then πU divides $S'(\Theta)$ into two components which we shall call the inner component [denoted by $S_i(\pi U)$] and the exterior component [denoted by $S_e(\pi U)$]. $S_i(\pi U)$ is bounded by the piece of πV between a and ξ, πU, and the piece of πW between τ and a (see Fig. 1). $S_e(\pi U)$ is bounded by the piece of πV between v_r and ξ, πU and the piece of πW between τ and w_s and the arc of $\Delta S(\Theta, N)$ between w_s and v_r which

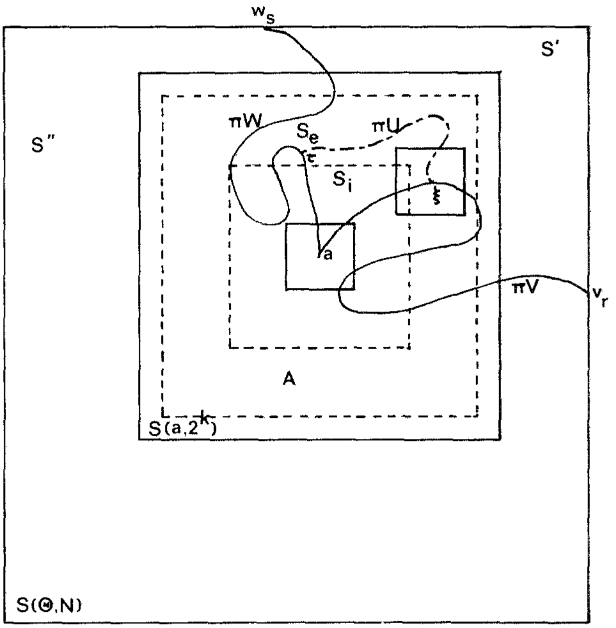

Fig. 1. A is the annulus between the --- lines. The small squares centered at a and ξ are $S(a, 2^{k-3})$ and $S(\xi, 2^{k-3})$.

also bounds $S'(\Theta)$ (see Fig. 1). For two polygons π' and π'' of the form πU discussed above we shall say that π' precedes π'' if $S_i(\pi') \subset S_i(\pi'')$. We now consider the collection $\mathcal{C} = \mathcal{C}(\pi V, \pi W)$ of all polygons πU satisfying (3.33) through some occupied path U satisfying (3.32). A variation of the proof of Lemma 1 in Ref. 7 shows that if there exists any such πU, then there exists a unique first one, i.e., one with minimal $S_i(\pi U)$. If U and πU satisfy (3.32), (3.33), then we denote by $F(\pi U)$ the event

$$F(\pi U) = \{U \text{ is occupied and } \pi U \text{ is the first polygon in the collection } \mathcal{C}\} \tag{3.34}$$

The occurrence of $F(\pi U)$ depends only on (see footnote 6 on p. 738)

$$\left\{X(v) : v \text{ a vertex of } \mathcal{G} \text{ in } \bar{S}_i(\pi U) \cap S'(\Theta)\right\} \tag{3.35}$$

because $F(\pi U)$ occurs if and only if U is occupied, and for any $\tilde{U}$ and corresponding $\pi\tilde{U}$ satisfying (3.32), (3.33), and $S_i(\pi\tilde{U}) \subset S_i(\pi U)$, $\pi\tilde{U} \neq \pi U$, at least one of the vertices of $\tilde{U}$ must be vacant. But $S_i(\pi\tilde{U}) \subset S_i(\pi U)$ implies that all points of $\tilde{U}$ are vertices of $\mathcal{G}$ in $\bar{S}_i(\pi U)$. They must also lie in S', because $\tilde{U}$ is supposed to satisfy (3.32), and therefore appear in (3.35).

Now assume that $F(\pi U)$ occurs for some specific $U = (u_1, \ldots, u_l)$ and πU with initial point $\xi \in \pi V$ and end point $\tau \in \pi W$. If $\xi = v_{\nu+1} \in V$ or if ξ lies between v_ν and $v_{\nu+1}$ on πV, then take

$$V_1 = (v_{\nu+1}, \ldots, v_r)$$

and

$$\pi V_1 \text{ the piece of } \pi V \text{ from } \xi \text{ to } v_r \tag{3.36}$$

W_1 (and πW_1) will be made up of U and a piece of W (respectively, πU and a piece of πW). Let τ, the end point of πU on πW, be equal to $w_{\rho+1} \subset W$ or lie between w_ρ and $w_{\rho+1}$ on πW. Then take

$$W_1 = (u_1, \ldots, u_l, w_{\rho+1}, \ldots, w_s)$$

$$\pi W_1 = \pi U \text{ followed by the piece of } \pi W \text{ from } \tau \text{ to } w_s \tag{3.37}$$

$V_1, \pi V_1$ and $W_1, \pi W_1$ satisfy (3.22)–(3.25) with a subscript 1 added, and the point a replaced by ξ. $\pi V_1 \cup \pi W_1$ again divides the interior of $S(\Theta, N)$ into two components, $S'(\Theta, \pi V_1, \pi W_1)$ and $S''(\Theta, \pi V_1, \pi W_1)$. One of these components is bounded by $\pi V_1 \cup \pi W_1$ and the arc of $\Delta S(\Theta, H)$ between w_s and v_r which also bounds $S'(\Theta)$. Since $\pi V_1 \cup \pi W_1$ is equal to

$$(\text{the piece of } \pi V \text{ from } v_r \text{ to } \xi) \cup \pi U \cup (\text{the piece of } \pi W \text{ from } \tau \text{ to } w_s)$$

this is the same boundary as that of $S_e(\pi U)$. In other words one of $S'(\Theta, \pi V_1, \pi W_1)$ or $S''(\Theta, \pi V_1, \pi W_1)$ is precisely $S_e(\pi U)$. We index them in such a way that

$$S'(\Theta, \pi V_1, \pi W_1) = S_e(\pi U) \tag{3.38}$$

We now show how this allows us to choose a square around ξ which plays the same role as $S(a, 2^{k-3})$ and whose sites which are on V_1 will contribute the term $\beta_1 z(k-3)$ in (3.30). Specifically we consider $S(\xi, 2^{k-3})$. By (3.33) ξ, the first point of U, is in A so that

$$|\lambda_j(\xi) - \lambda_j(a)| \geqslant 2^{k-1} - 2^{k-3} \qquad \text{for } j = 1 \text{ or } 2$$

Consequently

$$S(\xi, 2^{k-3}) \cap S(a, 2^{k-3}) = \emptyset$$

Similarly $A \subset S(a, 2^{k-1} + 2^{k-3})$ implies

$$S(\xi, 2^{k-3}) \subset S(a, 2^k) \subset S(\Theta, N) \tag{3.39}$$

Consequently

$$\sum_{v_t \in S(a, 2^k)} Y(v_t, k, \pi V, \pi W) \geqslant \sum_{v_t \in S(a, 2^{k-3})} + \sum_{v_t \in S(\xi, 2^{k-3})} Y(v_t, k, \pi V, \pi W) \tag{3.40}$$

The expectation of the first sum in the right-hand side of (3.40) was already estimated in (3.31). For the second sum we make two observations. First set

$$\tilde{Y}(v_t, k-3, \pi V_1, \pi W_1) = \begin{cases} 1 & \text{if } v_t \text{ is connected to } W_1 \text{ in} \\ & S'(\Theta, \pi V_1, \pi W_1) \cap S(\xi, 2^{k-3}) \\ 0 & \text{otherwise} \end{cases}$$

$\tilde{Y}$ is the analog of Y, when $S'(\Theta, \pi V, \pi W)$ is replaced by $S'(\Theta, \pi V_1, \pi W_1)$ and a by ξ. $\tilde{Y}(v_t, k-3, \pi V_1, \pi W_1) = 1$ means that there exists an occupied path $Q = (q_1, \ldots, q_m)$ on $\mathcal{G}$ with

$q_j \in S'(\Theta, \pi V_1, \pi W_1) \cap S(\xi, 2^{k-3})$, $1 \leqslant j \leqslant m, q_1$ adjacent to v_t, and q_m adjacent to some point of W_1

W_1 consist of U plus the points $w_{\rho+1}, \ldots, w_s$ of W. If q_m is adjacent to one of these w_j with $j \geqslant \rho + 1$, then Q actually connects v_t to W in

$$S'(\Theta, \pi V_1, \pi W_1) \cap S(\xi, 2^{k-3}) \subset S'(\Theta, \pi V, \pi W) \cap S(a, 2^k) \qquad (3.41)$$

[by (3.38) and (3.39)]. Thus in this case

$$Y(v_t, k, \pi V, \pi W) = 1 \qquad (3.42)$$

If q_m is not adjacent to a point of W, then it must be adjacent to u_j, for some $1 \leqslant j \leqslant l$. In this case the path $Q_1 = (q_1, \ldots, q_m, u_j, u_{j+1}, \ldots, u_l)$ ends in a point adjacent to $w_{\rho+1}$, and by (3.41), (3.32) and the inclusion $A \subset S(a, 2^k)$, Q_1 is still contained in

$$S'(\Theta, \pi V, \pi W) \cap S(a, 2^k)$$

Thus (3.42) holds again, and for any $v_t \in S(\xi, 2^{k-3})$

$$Y(v_t, k, \pi V, \pi W) \geqslant \tilde{Y}(v_t, k-3, \pi V_1, \pi W_1)$$

The second observation is that

$$\bar{S}''(\Theta, \pi V, \pi W) \cup \bar{S}_i(\pi U) = \bar{S}''(\Theta, \pi V_1, \pi W_1)$$

which is equivalent to (3.38), since

$$S'(\Theta, \pi V, \pi W) \backslash \pi U = S_i(\pi U) \cup S_e(\pi U)$$

These observations, together with the fact that $F(\pi U)$ is determined by the sites of $\mathcal{G}$ in $\bar{S}_i(\pi U) \cap S'(\Theta, \pi V, \pi W)$, which is disjoint from $\bar{S}''(\Theta, \pi V,$

$\pi W)$ [see (3.35)], show that

$$E_{p_H}\left\{\sum_{v_t\in S(\xi,2^{k-3})} Y(v_t,k,\pi V,\pi W)\,|\,X(v)=\epsilon(v), v\in \bar{S}''(\Theta,\pi V,\pi W)\right\}$$

$$=\sum_{\mathcal{C}} P_{p_H}\{F(\pi U)\,|\,X(v)=\epsilon(v), v\in \bar{S}''(\Theta,\pi V,\pi W)\}$$

$$\times E_{p_H}\left\{\sum_{v_t\in S(\xi,2^{k-3})} Y(v_t,k,\pi V,\pi W)\,|\,X(v)\right.$$

$$\left.=\epsilon(v), v\in \bar{S}''(\Theta,\pi V,\pi W), F(\pi U)\right\}$$

$$\geqslant \sum_{\mathcal{C}} P_{p_H}\{F(\pi U)\}\min_{\eta} E_{p_H}\left\{\sum_{v_t\in S(\xi,2^{k-3})} \tilde{Y}(v_t,k-3,\pi V_1,\pi W_1)\,|\,X(v)\right.$$

$$\left.=\eta(v), v\in \bar{S}''(\Theta,\pi V_1,\pi W_1)\right\}$$

$$\geqslant z(k-3)\sum_{\mathcal{C}} P_{p_H}\{F(\pi U)\} \tag{3.43}$$

In (3.43) $\sum_{\mathcal{C}}$ denotes the sum over all πU in $\mathcal{C}(\pi V,\pi W)$; the min over η is over $\eta(v)=+1$ or -1. The last inequality in (3.43) is direct from the definition, since $\xi,\pi V_1,\pi W_1$ is a permissable choice for $a,\pi V,\pi W$ in the min in (3.27).

This practically completes the proof of (3.30). Indeed (3.40), (3.31), and (3.43) prove

$$z(k)\geqslant z(k-3)\left\{1+\sum_{\mathcal{C}} P_{p_H}\{F(\pi U)\}\right\}$$

and it therefore suffices to prove that

$$\sum_{\mathcal{C}} P_{p_H}\{F(\pi U)\} \tag{3.44}$$

is bounded away from zero. But (3.44) is precisely the probability that some πU is the first one with an occupied U among all the ones which satisfy (3.32) and (3.33). This is simply the probability that $\mathcal{C}(\pi V,\pi W)$ is not empty. However, just as in Lemma 3 of Ref. 7 we can see that $\mathcal{C}(\pi V,\pi W)$ $\neq \emptyset$ as soon as there exists a circuit $q_0,\ldots,q_m$ on $\mathcal{G}$ surrounding a inside A, and such that all q_i which lie in $S'(\Theta)$ are occupied. As in step (ii) of Proposition 1 in Ref. 7 it follows that (3.44) is bounded below by

$$P_{p_H}\{\exists \text{ an occupied circuit on } \mathcal{G} \text{ surrounding } a \text{ inside } A\} \tag{3.45}$$

Finally, A is built up of four rectangles isomorphic to

$$S(0,2^{k-3},5)=\{w:|\lambda_1(w)|\leqslant 2^{k-1}+2^{k-3}=5.2^{k-3},\ |\lambda_2(w)|\leqslant 2^{k-3}\}$$

and by (3.9), the probability of an occupied left–right crossing of this rectangle is at least γ_5. As in Lemma 5.4 of Ref. 4, Lemma 3.5 of Ref. 5, or Theorem 2 of Ref. 6 it follows from the FKG inequalities that (3.45), and hence (3.44), is bounded below by $\beta_1=(\gamma_5)^4$. This proves (3.30). (3.28) immediately follows from (3.30) with

$$\alpha_{12}=\frac{\log(1+\beta_1)}{3\log 2}\quad\text{and}\quad C_{12}=2^{-3\alpha_{12}}z(0)>0\quad\blacksquare$$

We remind the reader that $B(N)$ was defined in (3.12), τ_0 in (1.12), and τ_1 in (3.6).

Proposition 1. There exist constants $0<C_{13},C_{14}<\infty$ such that on $\mathcal{G}=\mathcal{G}_0,\mathcal{G}_1,\mathcal{G}_0^*$ or $\mathcal{G}_1^*$

$$P_{p_H}\{B(N)\}\geqslant C_{13}N^{\alpha_{12}-1}\tag{3.46}$$

and for $p\leqslant p_H$, $i=1,2$,

$$\tau_1(2^{k-1},1,p)\leqslant\tau_0(2^k,i;p)\leqslant\exp-\{C_{14}e^{\alpha_{12}k}(p_H-p)\}\tag{3.47}$$

Proof. Fix N and take $2^{k+1}\leqslant N<2^{k+2}$, and $\Theta=$ the origin. Now consider the collection of occupied up–down crossings on $\mathcal{G}$ of the rectangle $[-2^k,2^k]\times[0,N]$, i.e., the collection of all occupied paths $W=(w_0,\ldots,w_s)$ which satisfy

$$|\lambda_1(w_t)|\leqslant 2^k,\qquad 0\leqslant t\leqslant s$$

and

$$\lambda_2(w_0)=0<\lambda_2(w_t)<N=\lambda_2(w_s),\qquad 1\leqslant t\leqslant s-1\tag{3.48}$$

For each W satisfying (3.48) consider all polygons πW through W on $\mathcal{G}$ which start at w_0 and end at w_s. As in Lemma 1 of Ref. 7, if there is any occupied path satisfying (3.48) there is among all polygons through all these paths a unique one that is "furthest to the left." Denote by $G(\pi W)$ the event

$$G(\pi W)=\{\pi W \text{ is the left-most polygon through any occupied path satisfying (3.48)}\}$$

Note that all permissible πW are contained in $S(0,N)$. Now, to apply Lemma 7 take $a=w_0$, $V=(v_0,v_1,\ldots,v_r)$ the path given by $\lambda_2(v_i)=0$, $\lambda_1(v_i)=\lambda_1(w_0)+i/2$, until $\lambda_1(v_r)=N$. Thus, V consists of the sites of $\mathcal{G}$ in $S(0,N)$ on the first coordinate axis to the right of w_0 (including w_0). For πV we take the straight line segment $[\lambda_1(w_0),N]\times\{0\}$ through these points.

For $S' = S'(\Theta, \pi V, \pi W)$ we take the "upper right-hand corner" of $S(0, N)\setminus \pi V \cup \pi W$, i.e., the component bounded by πV, $\{N\} \times [0, N]$ = upper half of the right edge of $S(0, N)$, $[\lambda_1(w_s), N] \times \{N\}$ = part of the upper edge of $S(0, N)$, and finally πW. Now notice that any v_t which is connected to W in $S' \cap S(a, 2^k)$ [in the sense used in (3.26)] is also connected by an occupied path in $S(0, N)$ to the upper edge of $S(0, N)$. Indeed, we can connect v_t to this upper edge by first following its connecting path to W and then continuing along W to the upper edge of $S(0, N)$. Consequently

$$\begin{aligned} &E_{p_H}\{\#\{k \in [-N, +N] : (k, 0) \text{ is connected by} \\ &\qquad \text{an occupied path to the line } \{w : \lambda_2(w) = N\}\}\} \\ &\qquad \sum_{\pi W} P_{p_H}\{G(\pi W)\}\ E_{p_H}\{\text{number of } v_t \text{ in } S(w_0, 2^k) \text{ connected} \\ &\qquad\qquad \text{to } W \text{ in } S' \cap S(w_0, 2^k) \mid G(\pi W)\} \end{aligned} \tag{3.49}$$

As in Ref. 7, or in (3.35) above, the occurrence of $G(\pi W)$ depends only on

$$\{X(v) : v \in \bar{S}''(0, \pi V, \pi W)\}$$

[in fact only on the $X(v)$ with v on or "to the left" of πW]. Consequently, the last conditional expectation in (3.49) is by virtue of (3.26)–(3.28) at least

$$\begin{aligned} &\min_{\epsilon} E_{p_H}\left\{ \sum_{v_t \in S(w_0, 2^k)} Y(v_t, k, \pi V, \pi W) \mid X(v) = \epsilon(v), v \in \bar{S}''(0, \pi V, \pi W) \right\} \\ &\quad \geqslant z(k) \geqslant C_{12} 2^{\alpha_{12} k} \geqslant C_{12} 2^{-2\alpha_{12}} N^{\alpha_{12}} \end{aligned}$$

It follows that the left-hand side of (3.49) is at least

$$\begin{aligned} &C_{12} 2^{-2\alpha_{12}} N^{\alpha_{12}} P_{p_H}\{\exists \text{ occupied path } W \text{ satisfying } (3.48)\} \\ &\quad \geqslant C_{12} 2^{-2\alpha_{12}} N^{\alpha_{12}} \tau_1(2^k, 2; \mathcal{G}, p_H(\mathcal{G})) \\ &\quad \geqslant C_{12} 2^{-2\alpha_{12}} \gamma_2 N^{\alpha_{12}} \end{aligned} \tag{3.50}$$

[see (3.9) and use the symmetry between the horizontal and vertical direction]. On the other hand, it is immediate from the definition (3.12) that the left-hand side of (3.49) is at most

$$(4N + 1) P_{p_H}\{B(N)\}$$

This, together with (3.50) yields (3.46).

To prove (3.47) fix k and $p \leqslant p_H$. Consider all occupied left–right crossings of $T = [0, 2^k] \times [-1, 2^{k+2}]$ which lie in the "lower three quarters" of this rectangle, and above the line $\{w : \lambda_2(w) = 0\}$, i.e., occupied paths $U = (u_0, u_1, \ldots, u_t)$ which satisfy

$$0 = \lambda_1(u_0) < \lambda_1(u_l) < 2^k = \lambda_1(u_t), \qquad 1 < l < t, \text{ and}$$

$$0 \leqslant \lambda_2(u_l) \leqslant 3.2^k, \qquad 0 \leqslant l \leqslant t \tag{3.51}$$

Again for each such path U consider all polygons πU through U from u_0 to u_t. As in Lemma 1 of Ref. 7, if there are any occupied paths satisfying (3.51), then among all polygons so obtained, there is a unique "lowest" one. We denote by $H(\pi U)$ the event

$$H(\pi U) = \{\pi U \text{ is the lowest polygon through any occupied path satisfying (3.51)}\}$$

Given πU we denote by $T^+(\pi U)$ and $T^-(\pi U)$ the components above and below πU of $T \backslash \pi U$ (compare Ref. 7, Lemma 1).

Now assume that $H(\pi U)$ occurs and consider weak cut sets (in the terminology of Ref. 7) in the strip $[1, 2^{k-1}] \times [0, 2^{k+2}]$, which lies in the left half of T. In the present notation these are *vacant* paths $W = (w_1, \ldots, w_s)$ on $\mathcal{G}^*$ such that

$$w_1 \text{ is adjacent on } \mathcal{G}^* \text{ to some } u_j \in U,$$
$$w_1, \ldots, w_{s-1} \in T^+(\pi U) \text{ and,}$$
$$1 \leqslant \lambda_1(w_i) \leqslant 2^{k-1}, \qquad 1 \leqslant i \leqslant s, \qquad \lambda_2(w_2) = 2^{k+2} \tag{3.52}$$

Again with each such W we associate all polygons πW through W, starting at a point a on πU, ending at w_s, and having only the point a in common with πU. We denote by $G(\pi U, \pi W)$ the event that a given πW is the left-most πW obtainable in this way. If there is any W satisfying (3.52) there is indeed a unique left-most polygon of this form, just as in Lemma 2 of Ref. 7.

We are again going to apply Lemma 7, but this time we use (3.29). If $G(\pi U, \pi W)$ occurs, take

$$a = \pi U \cap \pi W = \text{initial point of } \pi W$$
$$\pi V = \text{piece of } \pi U \text{ from } a \text{ to the right edge of } T$$

If $a = u_{\nu+1}$, or a lies between u_ν and $u_{\nu+1}$, then πV is a polygon through $V = (u_{\nu+1}, \ldots, u_t)$. Furthermore, take Θ to be the point with

$$\lambda_1(\Theta) = 2^k - 2^{k+2}, \qquad \lambda_2(\Theta) = 0$$

and take $N = 2^{k+2}$. Then

$$S(a, 2^{k-2}) \subset S(\Theta, N)$$

since $-1 \leqslant \lambda_2(a) \leqslant 3.2^k + 1$, $0 \leqslant \lambda_1(a) \leqslant 2^{k-1} + 1$ by (3.51) and (3.52). Also (3.22)–(3.25) hold. As in the first part of this lemma we take $S' =$

$S'(\Theta, \pi V, \pi W)$ to be the upper right corner of $S(\Theta, N)$. Next we need an interpretation of the conclusion (3.29) for the present situation. Assume that $Y^*(u_i, k-2, \pi V, \pi W) = 1$ for some $u_i \in S(a, 2^{k-2})$, $i \geqslant \nu + 1$. Then there exists a path $(q_1, \ldots, q_m)$ on $\mathcal{G}^*$ with all q_i vacant, $q_i \in S(a, 2^{k-2}) \cap S'$, and q_1 adjacent on $\mathcal{G}^*$ to u_i, q_m adjacent on $\mathcal{G}^*$ to some w_j. Then the path $(q_1, \ldots, q_m, w_j, w_{j+1}, \ldots, w_s)$ is a vacant path on $\mathcal{G}^*$, which is a weak cut set (with respect to U) in the terminology of Ref. 7. In the terminology of Ref. 8 this implies that u_i is a "critical point" for the event

$$D = \{\exists \text{ occupied left–right crossing of } [0, 2^k] \times [0, 3.2^k]\}$$

Indeed, if $G(\pi U, \pi W)$ occurs, then certainly D must occur; but if also $Y^*(u_i, k-2, \pi V, \pi W) = 1$ and if u_i is changed from occupied to vacant, then D no longer occurs (compare Ref. 7, formula (2.31); in the terminology of Ref. 7, $(u_i, q_1, \ldots, q_m, w_j, \ldots, w_s)$ now becomes a strong cut set). By Lemma 3 of Ref. 8

$$\frac{d}{dp} P_p\{D\} \geqslant E_p\{\text{number of critical points for } D\} \tag{3.53}$$

By the above, the right-hand side of (3.53) is at least

$$\sum P_p\{G(\pi U, \pi W)\} E_p\left\{\sum_{\substack{u_i \in S(a, 2^{k-2}) \\ i \geqslant \nu+1}} Y^*(u_i, k-2, \pi V, \pi W) \mid G(\pi U, \pi W)\right\} \tag{3.54}$$

As before the summation here is over all possible $\pi U, \pi W$ for occupied U on $\mathcal{G}$ satisfying (3.51) and vacant W on $\mathcal{G}^*$ satisfying (3.52). Also as before, or as in Ref. 7, the occurrence of $G(\pi U, \pi W)$ depends only on

$$\{X(v) : v \in \bar{S}''(\Theta, \pi V, \pi W)\}$$

Moreover, the probability of any vertex being vacant decreases as p increases. Thus, the conditional expectation in (3.54) is for $p \leqslant p_H(\mathcal{G})$ at least

$$\min_{\epsilon} E_{p_H(\mathcal{G})}\left\{\sum_{\substack{u_i \in S(a, 2^{k-2}) \\ u_i \in \pi V}} Y^*(u_i, k-2, \pi V, \pi W) \mid X(v) \right.$$

$$\left. = \epsilon(v), v \in \bar{S}''(\Theta, \pi V, \pi W)\right\}$$

$$\geqslant z^*(k-2) \geqslant C_{12} 2^{\alpha_{12}(k-2)} \tag{3.55}$$

Also, as in step (i) of Proposition 1 in Ref. 7 for fixed πU,

$$\sum_{\substack{\text{all permissible}\\ \pi W}} P_p\{G(\pi U, \pi W) \mid H(\pi U)\}$$

$$\geqslant P_{p_H(\mathcal{G})}\{\exists \text{ vacant up–down crossing of } [1, 2^{k-1}] \times [0, 2^{k+2}] \text{ on } \mathcal{G}^*\}$$

$$\geqslant P^*_{1-p_H(\mathcal{G})}\{\exists \text{ occupied left–right crossing of } [0, 2^{k+2}] \times [1, 2^{k-1}] \text{ on } \mathcal{G}^*\}$$

$$\geqslant \gamma_{16} > 0 \tag{3.56}$$

Here P_q^* is the measure which makes each vertex of $\mathcal{G}^*$ occupied with probability q, independently for all vertices. In (3.56) we also used the symmetry between the horizontal and vertical direction, as well as the relation

$$p_H(\mathcal{G}) = 1 - p_H(\mathcal{G}^*) \tag{3.57}$$

which is known for our graphs,[7,8] and (3.9). Combining (3.53)–(3.56) we obtain

$$\frac{d}{dp} P_p\{D\} \geqslant \sum_{\substack{\text{all permissible}\\ \pi U}} P_p\{H(\pi U)\}$$

$$\gamma_{16} C_{12} 2^{\alpha_{12}(k-2)} = \gamma_{16} C_{12} 2^{\alpha_{12}(k-2)} P(D), \qquad p \leqslant p_H$$

Since $P_{p_H}(D) \leqslant 1$, integration of this inequality immediately gives

$$P_p\{D\} \leqslant \exp - \left\{\gamma_{16} C_{12} 2^{\alpha_{12}(k-2)}(p_H - p)\right\}, \qquad p \leqslant p_H$$

Since $P_p\{D\}$ is just what we formerly denoted by $\tau_0(2^k, 1; p)$, the second inequality of (3.47) for $i = 1$ has been proven. Since the horizontal and vertical direction play the same role for $\mathcal{G}$, we can also take $i = 2$ in (3.47). Finally, the first inequality in (3.47) is immediate from the definitions. ■

Proof of Theorem 3. The right-hand inequality in (1.23) is in Lemma 6. For the left-hand inequality we combine Theorem 1 and Proposition 1. First we observe that if the origin is connected on $\mathcal{G}$ to some vertex $v \in \Delta S(0, M)$ [in the notation of (3.11)] and if there is no vacant circuit on $\mathcal{G}^*$ surrounding both the origin and some site on $\Delta S(0, M)$, then 0 is not cut off from ∞, and $\# W = \infty$ (compare Theorem 2.1 of Ref. 5). Thus, for any M

$$\theta(p) \geqslant P_p\{0 \text{ is connected by an occupied path on } \mathcal{G} \text{ to } \Delta S(0, M) \text{ and there does not exist a vacant circuit on } \mathcal{G}^* \text{ surrounding 0 and some site on } \Delta S(0, M)\} \tag{3.58}$$

Next, by (3.46), for any $p \geqslant p_H$

$$P_p\{0 \text{ is connected by an occupied path on } \mathcal{G} \text{ to } \Delta S(0,M)\}$$
$$= P_p\{B(M)\} \geqslant P_{p_H}\{B(M)\} \geqslant C_{13}M^{\alpha_{12}-1} \tag{3.59}$$

Also, any vacant circuit on $\mathcal{G}^*$ surrounding 0 and a site on $\Delta S(0,M)$ must contain at least $4M$ vertices. Moreover, any such circuit containing l vertices must contain one of the points $(j,0)$, $1 \leqslant j \leqslant l$, on the x axis, and if this occurs, $W^-_{(j,0)}$, the vacant cluster on $\mathcal{G}^*$ containing $(j,0)$ contains at least l elements. This yields

$$P_p\{\exists \text{ vacant circuit on } \mathcal{G}^* \text{ surrounding 0 and a vertex of } \Delta S(0,M)\}$$
$$\leqslant \sum_{l=4M}^{\infty} \sum_{j=1}^{l} P_p\{\# W^-_{(j,0)} \geqslant l\}$$
$$= \sum_{l=4M}^{\infty} l P_q^*\{\# W^* \geqslant l\} \tag{3.60}$$

where P_q^* is as in (3.56), and W^* is the occupied cluster of 0 on $\mathcal{G}^*$. Finally, by the FKG inequality (Ref. 5, Section 2.2) the right-hand side of (3.58) is at least equal to the product of the left-hand side of (3.59) and {1 minus the left-hand side of (3.60)}. Hence, for any M

$$\theta(p) \geqslant C_{13}M^{\alpha_{12}-1}\left\{1 - \sum_{l=4M}^{\infty} l P_q^*\{\# W^* \geqslant l\}\right\} \tag{3.61}$$

It remains to choose M and to use (2.6) to estimate the series in (3.61).

We now take $p > p_H$ and

$$k = \left[C_{15}\log\frac{1}{p-p_H}\right] + C_{16} \tag{3.62}$$

where C_{15}, C_{16} are large enough so that [cf. (2.4)]

$$\exp - \left\{C_{14}2^{\alpha_{12}k}(p-p_H)\right\} \leqslant \kappa(2) \tag{3.63}$$

and $2^k \geqslant \Lambda$. For example, we can take

$$C_{15} = (\alpha_{12}\log 2)^{-1}$$
$$C_{16} = 2 + \left[(\alpha_{12}\log 2)^{-1}\log\frac{\log\kappa^{-1}}{C_{14}} + \frac{\log\Lambda}{\log 2}\right]$$

For this k, by (3.47) applied to $\mathcal{G}^*$, and (3.57),

$$\tau_0(2^k, i; \mathcal{G}^*, q) \leqslant \kappa(2), \qquad i = 1,2 \tag{3.64}$$

By Lemma 3 we then have

$$P_q^* \{ \# W^* \geqslant l \} \leqslant C_1 e^{-C_2 l} \tag{3.65}$$

with

$$C_1 = 7^4[2\kappa(2)]^{-13^{-2}} \quad \text{and} \quad e^{-C_2} = \left\{ e \cdot 49[2\kappa(2)]^{13^{-2}} \right\}^A$$

where

$$A = 7^{-4}\mu^{-1}2^{-2k}$$

[since (3.64) gives us (2.5) with $\bar{N} = (2^k, 2^k)$]. Putting these estimates into (3.65) produces the inequality

$$\begin{aligned} P_q^* \{ \# W^* \geqslant l \} &\leqslant C_1 e^{-C_{17} 2^{-2k} l} \\ &\leqslant C_1 \exp - \left\{ C_{18}(p - p_H)^{C_{19}} l \right\}, \quad p > p_H \end{aligned} \tag{3.66}$$

for suitable $0 < C_{17} - C_{19} < \infty$, independent of l and $p > p_H$. Finally we note that

$$\begin{aligned} \sum_{l=4M}^{\infty} l \exp - C_{18}(p - p_H)^{C_{19}} l &= - \frac{d}{dx} \left\{ \frac{e^{-4Mx}}{1 - e^{-x}} \right\}_{x = C_{18}(p - p_H)^{C_{19}}} \\ &\leqslant C_{20}(p - p_H)^{-C_{19}} \left\{ M + (p - p_H)^{-C_{19}} \right\} \\ &\quad \times \exp - \left\{ 4MC_{18}(p - p_H)^{C_{19}} \right\} \end{aligned} \tag{3.67}$$

Thus, there exists a constant C_{21} such that

$$M \geqslant C_{21}(p - p_H)^{-2C_{19}}$$

implies

$$\sum_{l=4M}^{\infty} l P_q^* \{ \# W^* \geqslant l \} \leqslant \tfrac{1}{2}$$

By virtue of (3.61) this proves

$$\theta(p) \geqslant \tfrac{1}{2} C_{13} C_{21}^{\alpha_{12}-1} (p - p_H)^{-2(\alpha_{12}-1)C_{19}}$$

This proves the left-hand inequality in (1.23). Practically the same estimates can be used for several of the other inequalities. First, the right-hand inequality in (1.25). Since $S(0, M)$ only contains $(4M + 1)^2$ sites, $\# W \geqslant (4M + 1)^2$ implies $W \cap \Delta S(0, M) \neq \emptyset$ and hence the occurrence of $B(M)$. By the argument leading to (3.58) and (3.60), (3.67) we therefore have for $p > p_H$

$$\begin{aligned}
P_p\left\{(4M+1)^2 \leqslant \# W < \infty\right\} &\leqslant P_p\{\text{there exists a vacant circuit on } \mathcal{G}^* \text{ surrounding } 0 \text{ and some site on } \Delta S(0, M)\} \\
&\leqslant \sum_{l=4M}^{\infty} l P_q^* \{\# W^* \geqslant l\} \\
&\leqslant C_1 C_{20} (p - p_H)^{-C_{19}} \left\{M + (p - p_H)^{-C_{19}}\right\} \\
&\quad \times \exp - \left\{4 M C_{18} (p - p_H)^{C_{19}}\right\} \qquad (3.68)
\end{aligned}$$

The right-hand inequality in (1.25) is immediate from this.

(3.68), when combined with (3.13) and (3.15), also gives us (1.26) for $p > p_H$. To see this, recall that $\# W \geqslant (4M + 1)^2$ implies the occurrence of $B(M)$. Thus, in addition to (3.68) we have the bound

$$\begin{aligned}
P_p\left\{(4M+1)^2 \leqslant \# W < \infty\right\} &\leqslant P_p\{B(M)\} \\
&\leqslant \left(\frac{p}{p_H}\right)^{(4M+1)^2} P_{p_H}\{B(M)\} \qquad [\text{by } (3.15)] \\
&\leqslant \left(\frac{p}{p_H}\right)^{(4M+1)^2} C_{11} M^{-\beta_0} \qquad [\text{by } (3.13)]
\end{aligned} \qquad (3.69)$$

We can use either (3.68) or (3.69) with $(4M + 1)^2 \leqslant N$ to estimate

$$P_p\{N \leqslant \# W < \infty\} \qquad (3.70)$$

We take $M_1 = \frac{1}{5} N^{1/2}$. Then, by (3.68), for $(p - p_H) \geqslant M_1^{-1/(2C_{19})}$, (3.70) is at most

$$3 C_1 C_{20} M_1^{3/2} \exp - \left\{4 C_{18} M_1^{1/2}\right\} \leqslant C_9 N^{-\alpha_9}$$

for any choice of $\alpha_9 > 0$, once N is sufficiently large. If, on the other hand,

$$0 < p - p_H < M_1^{-1/(2C_{19})} = C_{22} N^{-1/(4C_{19})} \qquad (3.71)$$

then we use (3.69) with

$$M = M_2 \equiv \tfrac{1}{5} N^{\min(1/2, 1/8C_{19})}$$

One easily sees that under (3.71)

$$(p/p_H)^{(4M_2+1)^2}$$

is bounded, so that (3.69) is at most $C_{23} M_2^{-\beta_0}$. This proves (1.26) for $p > p_H$

with $\alpha_9 = \beta_0 \min(1/2, 1/8C_{19})$, For $p \leqslant p_H$ (1.26) follows from (3.13) and

$$P_p\{N \leqslant \# W < \infty\} = P_p\{N \leqslant \# W\} \leqslant P_p\left\{B\left(\tfrac{1}{5}N^{1/2}\right)\right\}$$
$$\leqslant P_{p_H}\left\{B\left(\tfrac{1}{5}N^{1/2}\right)\right\}, \qquad p \leqslant p_H$$

(1.27) is immediate from (1.26).

Now we must prove the right-hand inequality in (1.24). This follows from (3.66) with $\mathcal{G}^*$ playing the role of $\mathcal{G}$, and p and q interchanged. With these changes (3.66) becomes

$$P_p\{\# W \geqslant l\} \leqslant C_1 \exp - \left\{C_{18}(p_H - p)^{C_{19}} l\right\}, \qquad p \leqslant p_H$$

Finally we must prove the left-hand inequalities in (1.24) and (1.25). These follow from the fact that $\# W \geqslant 3^k$ on $\{B(3^k)\}$. Thus for any k [cf. (3.5) for notation]

$$\begin{aligned} E_p\{\# W\} \geqslant E_p\{\# W; \# W < \infty\} &\geqslant 3^k P_p\{B(3^k) \text{ and there exists a vacant circuit on } \mathcal{G}^* \text{ surrounding } 0, \text{ outside } S(0, 3^k)\} \\ &\geqslant 3^k P_p\{B(3^k)\} \\ &\quad \times P_p\{\exists \text{ vacant circuit on } \mathcal{G}^* \text{ surrounding } 0 \text{ in } R(0, k+1)\} \end{aligned} \tag{3.72}$$

Now, for $p < p_H$, by (3.15) and (3.46)

$$P_p\left\{B(3^k)\right\} \geqslant (p/p_H)^{(4\cdot 3^k+1)^2} C_{13} 3^{k(\alpha_{12}-1)}$$

Also the last factor in (3.72) equals 1 for $p < p_H$ [since there exist infinitely many vacant circuits on $\mathcal{G}^*$ when $\theta(p) = 0$, cf. Ref. 18]. When $p \geqslant p_H$, then by (3.46)

$$P_p\left\{B(3^k)\right\} \geqslant C_{13} 3^{k(\alpha_{12}-1)}$$

while the last factor in (3.71) is, by the same argument which led to (3.15), at least

$$\begin{aligned} &(q/q_H)^{\# R(0,k+1)} P_{p_H}\{\exists \text{ vacant circuit on } \mathcal{G}^* \text{ surrounding } 0 \text{ in } R(0, k+1)\} \\ &\quad = (1/q_H)^{\# R(0,k+1)} P^*_{p_H(\mathcal{G}^*)}\{\exists \text{ occupied circuit on } \mathcal{G}^* \text{ surrounding } 0 \text{ in } R(0, k+1)\} \\ &\quad \geqslant \gamma_0 \left(\frac{q}{q_H}\right)^{(4\cdot 3^{k+1}+1)^2} \qquad [\text{by } (3.10)] \end{aligned}$$

Here $\# R(0, k+1)$ denotes the number of sites on $\mathcal{G}^*$ in $R(0, k+1)$, which is less than $(4 \cdot 3^{k+1} + 1)^2$ and P^*_q is as in (3.56). We obtain the left-hand inequality of (1.24), respectively (1.25), by taking 3^{2k} approximately $|p - p_H|^{-1/2}$. ■

4. SMOOTHNESS PROPERTIES OF Δ

In this section we prove Theorem 2. We note that the proof of the existence of $\Delta''(p)$ at $p = p_H$ depends on the hard estimate (1.26) for all p. This estimate is, however, quite easy for $p \leqslant p_H$ (see Lemma 5). It is therefore easy to use the proof below and Lemma 5 to show that Δ has two continuous left derivatives at $p = p_H$. By using this fact and (1.22) for $\mathcal{G}$ as well as $\mathcal{G}^*$, one obtains also the existence of two right derivatives. For $\mathcal{G} = \mathcal{G}_1$ we can use the fact that $\mathcal{G}_1$ is self matching, i.e., $\mathcal{G}_1^*$ is isomorphic to $\mathcal{G}_1$ and hence $\Delta(p) = \Delta^*(p)$, to derive from (1.22) that Δ'' is actually continuous at $p = p_H(\mathcal{G}_1) = \frac{1}{2}$. Thus Theorem 2 is quite easy for $\mathcal{G} = \mathcal{G}_1$. This simple approach does not seem to work for $\mathcal{G} = \mathcal{G}_0$, which is not self-matching.

Proof of Theorem 2. First we prove that $\Delta(p)$ is analytic in p for $p < p_T = p_S$ for any $\mathcal{G}$. To see this note first that $\Delta(p)$ is a finite sum of series of the following form [notation as in (2.9)]:

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{l} a(n,l) p^n q^l \tag{4.1}$$

Any such series can easily be handled by Corollary 1 and (2.15). Indeed for $p + \delta \leqslant p + \delta_0 < p_T$ and suitable $C_i = C_i(p + \delta_0, \mathcal{G})$

$$\sum_l a(n,l)(p+\delta)^n (1-p-\delta)^l = P_{p+\delta}\{\# W = n\}$$
$$\leqslant P_{p+\delta_0}\{\# W \geqslant n\} \leqslant C_1 e^{-C_2 n}$$

Thus, for any complex $\tilde{p}$ with $|\tilde{p} - p| \leqslant \delta \leqslant \delta_0$

$$\sum_l a(n,l) |\tilde{p}|^n |1-\tilde{p}|^l \leqslant \left(\frac{1-p+\delta}{1-p-\delta}\right)^{n(z-2)+2} \sum_l a(n,l)(p+\delta)^n (1-p-\delta)^l$$
$$\leqslant \left(\frac{1-p+\delta}{1-p-\delta}\right)^{n(z-2)+2} C_1 e^{-C_2 n} \tag{4.2}$$

When $p < p_T$ we can choose $\delta > 0$ such that the last member of (4.2) decreases exponentially fast with n. This implies the analyticity of $\Delta(p)$ for $p < p_T$, for any $\mathcal{G}$. If $\mathcal{G}$ and G^* are matching graphs this also shows that $\Delta^*(1-p)$ is analytic for $1 - p < p_T^* \equiv p_T(\mathcal{G}^*)$. By the Sykes and Essam relation (1.22) we then see that $\Delta(p)$ is analytic for $1 - p < p_T^*$ or $p > 1 - p_T^*$.

For the remainder of this section we take $\mathcal{G} = \mathcal{G}_0$ or $\mathcal{G}_1$. In this case[4,7,8] $p_T = p_H = 1 - p_T^*$ and hence Δ is analytic for $p \neq p_H$. Grimmett[21] already showed that Δ has one continuous derivative for all p. (This can also be rederived by the method below.) We therefore content ourselves with showing that the series obtained by differentiating (4.1) termwise twice converges uniformly on [0, 1]. For $\mathcal{G} = \mathcal{G}_0$ or $\mathcal{G}_1$ all vertices play the same role so that (1.21) reduces to

$$\Delta(p) = E_p\{1/\#W; \#W \geqslant 1\} = \sum_{n=1}^{\infty} \sum_{l} \frac{1}{n} a(n,l) p^n q^l$$

$[a(n,l)$ is now defined by (2.9) with $w_0 = 0]$. Thus, we have to verify the uniform convergence of.

$$\sum_{n=1}^{\infty} \sum_{l} \frac{1}{n} \left(\frac{n}{p} - \frac{l}{q} \right)^2 a(n,l) p^n q^l - \sum_{n=1}^{\infty} \sum_{l} \frac{1}{n} \left(\frac{n}{p^2} + \frac{l}{q^2} \right) a(n,l) p^n q^l \quad (4.3)$$

The second series in (4.3) was handled by Grimmett in Ref. 21, but its uniform convergence is also implied by (1.26). We restrict ourselves to the first series. We break the sum over l into two pieces. The first piece is over those l with $|l/q - n/p| \leqslant n^{1/2+\alpha_{10}} pq$, where

$$\alpha_{10} = \min(\tfrac{1}{4}\alpha_9, \tfrac{1}{4}) \qquad [\alpha_9 \text{ as in } (1.26)]$$

The tail sum for this piece is

$$\sum_{n=M}^{\infty} \sum_{|l/q - n/p| \leqslant n^{1/2+\alpha_{10}} pq} \frac{1}{n} \left(\frac{n}{p} - \frac{l}{q} \right)^2 a(n,l) p^n q^l$$
$$\leqslant \sum_{n=M}^{\infty} n^{2\alpha_{10}} \sum_{l} a(n,l) p^n q^l$$
$$\leqslant \sum_{n=M}^{\infty} n^{\frac{1}{2}\alpha_9} [P_p\{n \leqslant \#W < \infty\} - P_p\{n+1 \leqslant \#W < \infty\}] \quad (4.4)$$

(4.4) tends to zero uniformly in p as $M \to \infty$ by (1.26). The remaining values of l lead to the tail sum

$$\sum_{n=M}^{\infty} \frac{1}{n} \sum_{|l/p - n/q| > n^{1/2+\alpha_{10}} pq} \left(\frac{n}{p} - \frac{l}{q} \right)^2 a(n,l) p^n q^l$$
$$\leqslant 2\left(\frac{1}{p^2} + \frac{z}{q^2} \right) \sum_{n=M}^{\infty} n \sum_{|l/p - n/q| > n^{1/2+\alpha_{10}} pq} a(n,l) p^n q^l$$
$$\leqslant 2\left(\frac{1}{p^2} + \frac{z}{q^2} \right) z \sum_{n=M}^{\infty} n^2 \exp - \left\{ \frac{p^2 q}{3} n^{2\alpha_{10}} \right\} \qquad [\text{by } (2.15)]$$

[by (2.13) with $x = n^{-1/2+\alpha_{10}}$]. The last sum tends to zero as $M \to \infty$, uniformly for $p \in [\epsilon, 1-\epsilon]$, $\epsilon > 0$ fixed, but arbitrary. Of course we already know that $\Delta(p)$ is analytic for p close to 0 or 1, but it is in any case easy to see that the series

$$\sum \frac{1}{n} a(n,l) p^{n-2} q^{l-2} \{ n(n-1)q^2 - 2nlpq + l(l-1)p^2 \}$$

obtained by differentiating (4.1) twice termwise, converges uniformly, even for p close to 0 or 1. For $p \leqslant (2ez)^{-1}$ we obtain this from (2.12), whereas for p close to 1 we use (3.67) (see also Ref. 21, Theorem 3.1)

$$\begin{aligned} \sum_l a(n,l) p^n q^l &\leqslant P_p\{ n \leqslant \# W < \infty \} \\ &\leqslant \sum_{l \geqslant \sqrt{n}/2} l P_q^* \{ W^* \geqslant l \} \\ &\leqslant \sum_{l \geqslant \sqrt{n}/2} l \sum_{m=l}^{\infty} \sum_k a^*(m,k) q^m p^k \end{aligned} \tag{4.5}$$

where $a^*(m,k)$ is defined by (2.9), but with $\mathcal{G}$ replaced by $\mathcal{G}^*$. But then (2.15) and (2.12) applied to $\mathcal{G}^*$ show that (4.5) is $O\{ n(qez^*)^{\sqrt{n}/2} \}$, uniformly on $q = 1 - p \leqslant (2ez^*)^{-1}$, where z^* is the maximal coordination number for $\mathcal{G}^*$. ■

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