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Surface Tension, Percolation, and Roughening[1]

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We describe inequalities relating to the interface between coexisting phases of Ising ferromagnets. Some implications for the nature of the roughening transition are discussed.

KEY WORDS: Surface tension; percolation; roughening; long range order.

1. INTRODUCTION

We consider an Ising spin system ($\sigma_i = \pm 1$) on $\mathbb{Z}^d$ with nearest-neighbor interactions; the Hamiltonian in a box Λ with boundary conditions (b.c.) b specifying $\sigma_j = \tilde{\sigma}_j, j \notin \Lambda$, is

$$-H_\Lambda^b = J \sum_{\langle ij \rangle \subset \Lambda} \sigma_i \sigma_j + J \sum_{\substack{\langle ij \rangle \\ i \in \Lambda, j \in \Lambda^c}} \sigma_i \tilde{\sigma}_j$$

Λ will be taken to be a parallelipiped,

$$\Lambda_{L,M} = \left\{ i \in \mathbb{Z}^d \,|\, -M \leqslant i_1 \leqslant M, -L \leqslant i_\alpha \leqslant L-1, \alpha = 2, \ldots, d \right\}$$

and we shall consider the following b.c.:

(1) $+(-)$, i.e., $\tilde{\sigma}_j = +1(-1)$;
(2) $\pm$, i.e., $\tilde{\sigma}_j = +1$ if $j_1 \geqslant 0$ and $\tilde{\sigma}_j = -1$ if $j_1 < 0$;
(3) 0, i.e., $\tilde{\sigma}_j = 0$;

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(4) $*$, $\sigma_j = +1$ if $j_1 > 0$ or $j_1 = 0$ and $j_2 \geqslant 0$, $\tilde{\sigma}_j = -1$ if $j_1 < 0$ or $j_1 = 0$ and $j_2 < 0$.

The limits $M \to \infty$ and $L \to \infty$ for the Gibbs distributions and expectation values are denoted, respectively, by

$$P_L^b,\, P^b \qquad \text{and} \qquad \langle\ \rangle_L^b, \langle\ \rangle^b$$

The critical temperature in d dimensions, $T_c(d)$, is defined uniquely[1] by $\langle\sigma_i\rangle^+ = -\langle\sigma_i\rangle^- = m^*(T) > 0$ for $T < T_c(d)$ and P^b independent of b for $T > T_c(d)$. The states $\langle\ \rangle^+$ and $\langle\ \rangle^-$ are translation invariant and extremal. It is known that for $d = 2$, $\langle\ \rangle^\pm = (1/2)(\langle\ \rangle^+ + \langle\ \rangle^-)$, $\forall T > 0$, while for $d \geqslant 3$, $\langle\ \rangle^\pm$ is not translation invariant for $T < T_c(d-1)$.[2] Furthermore, for T sufficiently small, the state $\langle\ \rangle^\pm$ in $d \geqslant 3$ has a sharp "Dobrushin interface" located near $i_1 = 0$, which separates the pure $+$ phase at $i_1 \gg 0$ from the pure $-$ phase at $i_1 \ll 0$.[3] This interface is defined, after associating configurations and contours as usual, as the "open" connected contour that extends outside the box. The difference between $d = 2$ and $d \geqslant 3$ at low temperatures is that in the former case, the location of this interface fluctuates with an amplitude proportional $\sqrt{L}$ as $L \to \infty$,[4] while in the latter case, the fluctuations are exponentially small with L. In both cases, however, the "intrinsic width" of this interface is exponentially small at low temperatures, $\beta J \gg 1$.[5]

We define the roughening temperature $T_R(d)$ as the lowest temperature above which $P^\pm$ is translation invariant, i.e., $P^\pm = (1/2)(P^+ + P^-)$, if $T > T_R(d)$. It follows then from Ref. 2 that

$$T_c(d-1) \leqslant T_R(d) \leqslant T_c(d)$$

It is natural to ask whether $T_R(d)$ is strictly less than $T_c(d)$. This is clearly so for $d = 2$ where $T_R = 0$,[4] but there are, however, no rigorous results at present about whether $T_R \leqslant T_c$ for $d \geqslant 3$ with the inequality expected for $d = 3$ and the equality for $d \geqslant 4$.[6] It is further expected that if $T_R < T_c$ then, as in $d = 2$,[7] there are only translation-invariant states above T_R.

Also unknown, even on the heuristic level, is the nature of the changes occurring in the structure of the interface at T_R. We shall describe some inequalities relating to this question in Section 2.

It has also been suggested[8] but not proven that the "step energy" σ, corresponding to the difference (divided by L^{d-2}), between the free energy for $*$ b.c. and $\pm$ b.c., serves as an order parameter for the roughening transition, i.e., defining T^* by the relation $\sigma > 0$ for $T < T^*$, $\sigma = 0$ for $T > T^*$ then $T^* = T_R$. While we cannot prove or disprove this, we show in Section 3 that T^* satisfies the van Beijeren inequalities for T_R, i.e., $T_c(d-1) \leqslant T^*(d) \leqslant T_c(d)$.

Finally we consider the solid-on-solid[9] model, which corresponds to making the nearest-neighbor interaction infinitely strong in the 1-direction.

For this model, $T_c(d) = \infty$ for all d, while it is known that $T_R(2) = 0$,[9] and $0 < T_R(3) < \infty$.[10] We show that for $d \geqslant 4$, $T_R = \infty$ for this model. All our results are thus consistent with expectations.

Proofs are given in Section 6. The proofs make use of a relation between percolation and long-range order. This is discussed in Section 5.

2. FLUCTUATIONS OF THE INTERFACE

In order to analyze the changes which may occur in the structure of the interface at T_R, let us consider first the geometric structure of the pure phases (in zero external field). At low temperatures a typical configuration in the + state consists of a "sea" of + spins in which there are some clusters of − spins, i.e., sites connected by nearest neighbor (n.n.) bonds on which $\sigma_i = -1$. As the temperature is raised, there may appear an infinite cluster of − spins, i.e., there is percolation. $T_p(d)$, the temperature at which this happens, is called the percolation temperature. T_{p^*} is defined similarly but with $*$ clusters of − spins, i.e., sites connected by n.n. *or* next-n.n. bonds (length of a bond is $\leqslant \sqrt{2}$). We now formulate our first result:

Let $P_L^{\pm}(i, j)$ be the probability that a path along the lattice bonds joining i and j crosses the Dobrushin interface an even (possibly zero) number of times, then

$$\lim_{L\to\infty} P_L^{\pm}(i, j) \geqslant \left(\langle\sigma_i\sigma_j\rangle^{\pm} + \langle\sigma_i\sigma_j\rangle^{0}\right)/\left(1 + \langle\sigma_i\sigma_j\rangle^{0}\right) \qquad (1)$$

For $T > T_R$ the right hand side of (1) is equal to $2\langle\sigma_i\sigma_j\rangle^0/(1 + \langle\sigma_i\sigma_j\rangle^0)$. Choosing now i and j such that $i_1 = -j_1 \to \infty$ implies that for $T_R < T < T_c$, when $\langle\sigma_i\sigma_j\rangle^0 \to [m^*(T)]^2 > 0$ for $|i - j| \to \infty$, the interface has some probability to be "at infinity" in the limit $L \to \infty$.

A much stronger result showing that the interface is "at infinity" with probability one holds if $T_R < T_{p^*}$. More precisely let $P_L^{\pm}(\Lambda_0)$ be the probability that the interface intersects a fixed box Λ_0. Then for

$$T_R < T < T_{p^*}, \qquad \lim_{L\to\infty} P_L^{\pm}(\Lambda_0) = 0 \qquad (2)$$

Conversely, if (2) holds then $T > T_R$. (2) is true in $d = 2$ for $T < T_c = T_{p^*}$.[11] We expect $T_R < T_{p^*}$ in three dimensions; numerically $T_p(3) \simeq 0.95\,T_c(3)$.[12]

Remark. The structure of the interface in three dimensions is presumably as follows: At low temperatures the interface is essentially flat (localized) and the state $P^{\pm}$ is not translation invariant. At $T_R \simeq 0.57\,T_c$,[6] $\langle\ \rangle^{\pm}$ becomes translation invariant and if we assume $T_R < T_{p^*}$ then for T just above T_R the interface fluctuates out of sight. Between T_{p^*} and T_c the interface still fluctuates but has a nonzero probability of being "localized"; it is very "thick." Above T_c, we have no results but we expect the interface to "fill" the whole space.

3. THE STEP ENERGY

There is a quasithermodynamic quantity, called the step energy, σ[8] which is expected to characterize the roughening transition in the same way that the surface tension τ characterizes the phase transition: τ is known[13] to be nonzero iff there is a spontaneous magnetization. Similarly one conjectures that σ is nonzero iff the state $P^{\pm}$ is nontranslation invariant. We recall that τ is defined, letting $L_0 = 2L$, as

$$\tau = -\lim_{L\to\infty}\lim_{M\to\infty} L_0^{1-d}\log\left[Z_{L,M}^{\pm}(Z_{L,M}^{+})^{-1}\right] \tag{3}$$

where $Z_\Lambda^b = \sum_{\sigma_i=\pm 1}\exp(-\beta H_\Lambda^b)$ is the partition function. Similarly σ is defined, assuming that the limits exist, by

$$\sigma = \lim_{L\to\infty}\lim_{M\to\infty} L_0^{2-d}\log Z_{L,M}^{\pm}(Z_{L,M}^{*})^{-1} \tag{4}$$

We prove the following bounds:

$$\sigma_{L,M}(d) \geqslant \tau_{L,M}(d-1) \geqslant 0 \tag{5}$$

$$\lim_{M\to\infty}\sigma_{L,M}(d) \leqslant L_0^{2-d} 2\beta J \sum_{\substack{i_2=-1\\ i_1=0}} \langle\sigma_i\rangle_L^+ \tag{6}$$

Here $\tau_{L,M}$ and $\sigma_{L,M}$ are the right sides of (3) and (4) before the taking of limits.

This shows that $\sigma \neq 0$ for $T < T_c(d-1)$ and $\sigma = 0$ for $T > T_c(d)$. This result extends also to the case where we let the interaction J be anisotropic.

4. THE SOLID-ON-SOLID MODEL

If we let the nearest-neighbor coupling go to infinity in the 1-direction while keeping it fixed in the other directions we obtain the solid-on-solid (SOS) model of an interface. In that limit the + b.c. state becomes frozen in the configuration $\sigma_i = 1$ everywhere so that $T_c = \infty$ and, for $\pm$ b.c. the SOS state is characterized by configurations with only one contour, namely, the interface, which crosses once and only once every line in the 1-direction. After taking the limit $M \to \infty$, this state can be equivalently described by a set of integer-valued spins ϕ_x with x running through a $(d-1)$-dimension cube $\mathbf{L}$ with vertices in the hyperplane $i_1 = 0$ and $\phi_x \in \mathbb{Z} + 1/2$ being the height of the interface above x. The interaction becomes

$$-H = 2J \sum_{\langle xy\rangle} |\phi_x - \phi_y| \tag{7}$$

with zero boundary conditions $\phi_x = 0$ for $x \notin \mathbf{L} \subset \mathbb{Z}^{d-1}$. For $d = 2$ (i.e., $d = 1$ for the ϕ_x variables) this is the Temperley model.[9] It can be solved exactly: $T_R = 0$ as in the Ising model; one defines T_R now as the infimum over the temperatures where $\langle\phi_0^2\rangle$ diverges as $L \to \infty$. We can also define T_R as the temperature where $\langle(\phi_0 - \phi_x)^2\rangle_L$ diverges when first L and then x go to infinity. The two definitions are equivalent for $d = 2$ and presumably also in higher dimensions. Using the latter definition, Fröhlich and Spencer proved $0 < T_R < \infty$ for $d = 3$.[10] For $d \geqslant 4$, we consider a box $\Lambda_{L,M}$ with periodic boundary conditions on the lateral sides $i_\alpha = -L$ or $L-1$, $\alpha = 2, \ldots, d$, $+$ b.c. at $i_1 = M$ and $-$ b.c. at $i_1 = -M$. This leads to periodic b.c. for the SOS model. In order to define properly finite volume expectation values in this case, we first add a mass term $-m^2\sum_x \phi_x^2$ to the Hamiltonian (7) that we remove after taking the thermodynamic limit $L \to \infty$. We show for $d \geqslant 4$ that

$$\left\langle(\phi_0 - \phi_x)^2\right\rangle < \infty \tag{8}$$

Uniformly in x, L, and m for all $T < \infty$.

5. PERCOLATION AND LONG-RANGE ORDER

Before proving (1), we state a related result connecting percolation and long-range order whose proof is similar to the proof of (1). Let R_Λ be the probability, in the state with 0 b.c. on the box Λ, that the sites i and j belong to the same cluster of (either $+$ or $-$) spins then

$$R_\Lambda(i,j) \geqslant 2\langle\sigma_i\sigma_j\rangle_\Lambda^0 / \left[1 + \langle\sigma_i\sigma_j\rangle^0\right] \tag{9}$$

To prove (9) we write

$$\langle\sigma_i\sigma_j\rangle_\Lambda^0 = R_\Lambda(i,j) + \langle\sigma_i\sigma_j\rangle'\left[1 - R_\Lambda(i,j)\right]$$

where $\langle\sigma_i\sigma_j\rangle'$ is the conditional expectation value of $\sigma_i\sigma_j$ in the state P_Λ^0 given that i and j do not belong to the same cluster. Of course, we use the fact that if i and j belong to the same cluster, then $\sigma_i\sigma_j = +1$. We claim that $\langle\sigma_i\sigma_j\rangle' \leqslant -\langle\sigma_i\sigma_j\rangle^0$ and this will finish the proof. Indeed, if i and j do not belong to the same cluster there must be a contour separating i and j. We write $\langle\sigma_i\sigma_j\rangle_\Lambda = \sum_\gamma P(\gamma)\langle\sigma_i\sigma_j\rangle_\gamma$ where we condition on the first such countour γ that one crosses while going from i to j. Clearly σ_i is equal to the sign of spins on the side of γ containing i, which is the opposite of the sign of the spins on the j side of γ, and we therefore have

$$\langle\sigma_i\sigma_j\rangle_\gamma = -\langle\sigma_j\rangle_{\Lambda_j(\gamma)}^+$$

where $\langle\ \rangle_{\Lambda_j(\gamma)}^+$ is the expectation in the connected component of the complement of γ that contains j with the spins on the boundary of γ equal to $+1$. By F.K.G. inequalities[14] $\langle\sigma_j\rangle_{\Lambda_j(\gamma)}^+ \geqslant \langle\sigma_j \mid \sigma_i = +1\rangle_\Lambda^0$ where $\langle\sigma_j \mid \sigma_i = +1\rangle_\Lambda^0$ is the expectation value conditioned on σ_i being $+1$. But this equals $\langle\sigma_i\sigma_j\rangle^0$.

6. PROOFS

6.1. Fluctuations of the Interface

In order to prove inequality (1), we follow the same idea as in Section 5. We write

$$\langle\sigma_i\sigma_j\rangle^{\pm} = \langle\sigma_i\sigma_j\rangle''(1 - P_L^{\pm}) + \langle\sigma_i\sigma_j\rangle' P_L^{\pm} \tag{10}$$

where $\langle\ \rangle'$ (respectively, $\langle\ \rangle''$) is conditioned on the fact that a path connecting i and j crosses the interface an even, including zero (respectively, an odd), number of times and $P_L^{\pm}$ is the corresponding probability. Now if the interface crosses at all any path from i to j, it means that i and j lie in different connected regions of the complement of this interface. These regions have pure $+$ or $-$ b.c. and, if the number of crossings is odd the signs of the b.c. on the two regions are opposite. So, if one conditions on a particular interface,

$$\langle\sigma_i\sigma_j\rangle_\lambda = -\langle\sigma_i\rangle^+_{\Lambda_i(\lambda)}\langle\sigma_j\rangle^+_{\Lambda_j(\lambda)}$$

where $\Lambda_i(\lambda)$ is the component containing i with $+$ b.c. By Griffiths' inequalities[15]

$$\langle\sigma_i\rangle^+_{\Lambda_i}\langle\sigma_j\rangle^+_{\Lambda_j} \geqslant \langle\sigma_i\sigma_j\rangle^+_\Lambda \geqslant \langle\sigma_i\sigma_j\rangle^0_\Lambda$$

This combined with $\langle\sigma_i\sigma_j\rangle' \leqslant 1$ and inserted into (10) finishes the proof.

Now we prove (2). Let $T > T_R$. Then

$$\lim_{L\to\infty} P_L^{\pm} = (1/2)(P^+ + P^-)$$

at least for a.a. temperatures.[16] Notice that the boundary of a contour is made of two $*$ clusters one of $+$ and one of $-$ spins. Therefore $P_L^{\pm}(\Lambda_0) \leqslant P_L^{\pm}$ (there exists a $*$ cluster of $-$ spins intersecting Λ_0 and $\mathbb{Z}^d\backslash\Lambda^1$) where Λ^1 is any box in Λ_L containing Λ_0. Since this is a local event its probability converges to the infinite volume probability, which goes to zero (in the P^+ and P^- states) if $T \leqslant T_{P*}$ when $\Lambda^1 \to \infty$.

On the other hand, if $\lim P_L^{\pm}(\Lambda_0) = 0$, take $\langle ij\rangle \subset \Lambda_0$ and write

$$\langle\sigma_i\sigma_j\rangle_L^{\pm} = \langle\sigma_i\sigma_j\rangle''_L P_L^{\pm}(\Lambda_0) + \langle\sigma_i\sigma_j\rangle'_L(1 - P_L^{\pm}(\Lambda_0))$$

where $\langle\ \rangle'$ (respectively, $\langle\ \rangle''$) is the expectation conditioned on the interface not intersecting (respectively, intersecting) Λ_0. $\langle\sigma_i\sigma_j\rangle'_L \geqslant \langle\sigma_i\sigma_j\rangle^+$ because all points in Λ_0 lie in some connected part of the complement of the interface, with pure $+$ or $-$ b.c. So in this case $\lim_{L\to\infty}\langle\sigma_i\sigma_j\rangle_L^{\pm} \geqslant \langle\sigma_i\sigma_j\rangle^+$ and this implies the translation invariance of $\langle\ \rangle^{\pm}$.[16]

6.2. The Step Energy

In order to prove (5) let us define $\sigma_{L,M}(d,h)$ by adding to the Hamiltonian an external field

$$-h \sum_{i_1=1} \sigma_i - \sigma_{\bar{i}} \qquad \text{with} \quad \bar{i} = (-i_1, i_2, \ldots, i_d)$$

in the definition of $Z^{\pm}$ and Z^*, i.e., one puts a field h above the middle layer and a field $-h$ below it. It is easy to see that

$$\lim_{h\to\infty} \sigma_{L,M}(d,h) = \tau_{L,M}(d-1)$$

So the result follows if $(d/dh)\sigma_{L,M}(h) \leqslant 0$;

$$\begin{aligned}\frac{d}{dh}\sigma_{L,M}(h) = \beta \sum_{i_1=1} (\langle\sigma_i\rangle^{\pm} - \langle\sigma_{\bar{i}}\rangle^{\pm}) \\ - \beta \sum_{i_1=1} (\langle\sigma_i\rangle^{*} - \langle\sigma_{\bar{i}}\rangle^{*}) \qquad (11)\end{aligned}$$

Introducing the variables $t_i = \sigma_i - \sigma_{\bar{i}}$, $q_i = \sigma_i + \sigma_{\bar{i}}$, $i_1 \neq 0$, $q_i = \sigma_i$ for $i_1 = 0$ with $\bar{i} = (-i_1, i_2, \ldots, i_d)$, (11) equals

$$\beta \sum_{i_1=1} (\langle t_i\rangle^{\pm} - \langle t_i\rangle^{*})$$

where going from $*$ b.c. to $\pm$ b.c. consists of replacing some boundary external fields acting on q_i with $i_1 = 0$ by their absolute value. Since the q and t variables are negatively correlated,[2] this decreases the t expectation values and the above sum is therefore negative.

Now we prove (6).

Define $\sigma_{L,M}(s)$ by multiplying all bonds $\langle ij\rangle$ with $i_2 = -1$ and $j_2 = 0$ by s. By symmetry $\sigma_{L,M}(0) = 0$ and so we may write

$$\begin{aligned}\sigma_{L,M} &= \int_0^1 \frac{d}{ds}\sigma_{L,M}(s)\,ds \\ &= L_0^{2-d}\int_0^1 \beta J \sum_{\substack{\langle ij\rangle \\ i_2=-1,\, j_2=0}} \left(\langle\sigma_i\sigma_j\rangle_s^{\pm} - \langle\sigma_i\sigma_j\rangle_s^{*}\right) ds \\ &\leqslant L_0^{2-d}\int_0^1 \beta J \left[\sum_{i_2=-1} \left(\langle\sigma_i\rangle_s^{\pm} - \langle\sigma_i\rangle_s^{*}\right) + \sum_{j_2=0} \left(\langle\sigma_j\rangle_s^{\pm} - \langle\sigma_j\rangle_s^{*}\right)\right] ds \qquad (12)\end{aligned}$$

where in the last inequality we use F.K.G.[4] inequalities and the fact that the $\pm$ state dominates the $*$ state in the F.K.G. sense. Now notice that the $*$ state is invariant under the symmetry $\sigma_i \to -\sigma_{i'}$ where $i = (i_1, i_2, \ldots, i_d)$

and $i' = (-i_1, -i_2 - 1, \ldots, i_d)$. This implies

$$\sum_{i_2=-1} \langle \sigma_i \rangle_s^* + \sum_{i_2=0} \langle \sigma_i \rangle_s^* = 0$$

Also by symmetry $\sum_{i_2=-1} \langle \sigma_i \rangle_s^{\pm} = \sum_{i_2=0} \langle \sigma_i \rangle_s^{\pm}$. We can write, using the q variables,

$$\sum_{i_2=-1} \langle \sigma_i \rangle_s^{\pm} = \sum_{i_2=-1, i_1 \geq 0} \langle q_i \rangle_s^{\pm}$$

Since the t and q variables are negatively correlated,[2]

$$\langle q_i \rangle_s^{\pm} \leq \lim_{M \to \infty} \langle q_i \rangle_s^{\pm}$$

and by symmetry

$$\lim_{M \to \infty} \langle \sigma_i \rangle_s^{\pm} = - \lim_{M \to \infty} \langle \sigma_{\bar{i}-1} \rangle_s^{\pm}$$

So,

$$\sum_{\substack{i_2=-1 \\ |i_1| \leq M}} \langle \sigma_i \rangle_s^{\pm} \leq \sum_{\substack{i_2=-1 \\ |i_1| \leq M}} \lim_{M \to \infty} \langle \sigma_i \rangle_s^{\pm}$$

and on the right-hand side only the terms with $i_1 = M$ do not cancel. Now

$$\lim_{M \to \infty} \sum_{\substack{i_2=-1 \\ i_1=M}} \lim_{M \to \infty} \langle \sigma_j \rangle_s^{\pm} = \sum_{\substack{i_2=-1 \\ i_1=0}} \langle \sigma_j \rangle_s^{+} \leq \sum_{\substack{i_2=-1 \\ i_1=0}} \langle \sigma_j \rangle_{s=1}^{+}$$

which completes the proof.

6.3. The Solid-on-Solid Model

In this section we prove infrared bounds for the solid-on-solid model in a more general form defined below. This result implies that in dimensions greater or equal to 3, Eq. (8) holds.

Description of the Model. At each site of a lattice $\mathbb{Z}^d$ we associate a random variable $\phi \in \mathbb{R}$, d here corresponds to $d-1$ for the infinitely anisotropic Ising model discussed in Section 4. Take $\Lambda = [1, L_1] \times [1, L_2] \times \cdots \times [1, L_d]$. We define the Hamiltonian H_Λ with periodic boundary conditions: $H = \sum_{x,y} |\phi_x - \phi_y|$, where the sum runs over all nearest-neighbor pairs of the periodical lattice Λ. The measure at each site $d\nu(\phi)$ is such that finite volume expectations $\langle\ \rangle_\Lambda$ are finite:

$$\langle \cdot \rangle_\Lambda = Z_\Lambda^{-1} \int \cdot \exp(-\beta H_\Lambda) \prod_{x \in \Lambda} d\nu(\phi_x)$$

where

$$Z_\Lambda = \int \exp(-\beta H_\Lambda) \prod_{x \in \Lambda} d\nu(\phi_x)$$

We shall be interested in the two-point function $\langle \phi_0 \phi_x \rangle_\Lambda$. Its Fourier transform is

$$S_\Lambda(p) = \sum_{x \in \Lambda} \langle \phi_0 \phi_x \rangle_\Lambda \exp(ipx)$$

where p belongs to the dual lattice of Λ.

Proposition:

$$S_\Lambda(p) \Big[2 \sum_e (1 - \cos p_e) \Big] \leqslant 2(2d)\beta^{-2} \tag{13}$$

where $p = (p_e)_{e=1}^d$ belongs to the dual lattice of Λ.

Remarks. (1) The theorem implies that

$$\langle (\phi_x - \phi_y)^2 \rangle \leqslant 2(2d)\beta^{-2} \int_{-\pi}^{\pi} d^d p \Big[\sum_e (1 - \cos p_e)^{-1} (1 - \exp ip(x - y)) \Big]$$

which is uniformly bounded in x and y for $d \geqslant 3$. (2) The solid-on-solid model corresponds to $d\nu(\phi) = \sum_{m \in \mathbb{Z}} \delta[\phi - (m + 1/2)]$. (3) The infrared bounds are known to hold when $|\phi_x - \phi_y|$ is replaced by $|\phi_x - \phi_y|^2$ in H, because it is easy to show that the system satisfies reflection positivity.[17,18]

In our case we do not know whether reflection positivity holds. However, we can still establish the result using as a main ingredient the *positivity* of the transfer matrix of the system. Our proof follows essentially the alternate derivation of the infrared bounds given in the appendix of Ref. 17.

Before going to the proof we first recall the *transfer matrix formalism*.[17]

We write $x \in \Lambda$ as $x = (i, \alpha)$, $i = 1, 2, \ldots, L_1$ and

$$\alpha \in \Lambda' \equiv [1, L_2] \times \cdots \times [1, L_d]$$

Assume $d\nu(\phi) = \nu(\phi)\, d\phi$ (the general case can be obtained as a limit). We define

$$F_i(\phi_\alpha) = \prod_{\alpha \in \Lambda'} \nu(\phi_{(i,\alpha)}) \exp\Big[- \sum_{|\alpha - \alpha'| = 1} \beta |\phi_{(i,\alpha)} - \phi_{(i,\alpha')}| \Big]$$

F_i is viewed as a multiplication operator on $\mathbb{R}^M$, $M = L_2, L_3, \ldots, L_d$. We also define an integral operator T_0 on $L^2(R^M)$ with kernel:

$$\prod_{\alpha \in \Lambda'} \exp - \beta |\phi_\alpha - \phi'_\alpha|$$

Obviously

$$Z_\Lambda = \mathrm{Tr}\left(\prod_{i=1}^{L_1} F_i T_0\right) = \mathrm{Tr}(T^{L_1})$$

where $T = F^{1/2}T_0F^{1/2}$ is the transfer matrix associated with the direction 1. We finally introduce an averaged spin variable, $\phi_i(g) = \sum g(\alpha)\phi_{(i,\alpha)}$, where $g(\alpha)$ is a function defined on the lattice sites of an hyperplane. $s(g)$ is the operator on $L^2(R^M)$ given by multiplication by $\phi_0(g)$. The theorem follows from the two following lemmas:

Lemma 1[17]**:**

$$0 \leqslant 2(1-\cos p_1)\sum_j \langle\phi_1(\bar{g})\phi_j(g)\rangle \exp ip_1 j$$

$$\leqslant 2\,\mathrm{Tr}\left\{\left[s(\bar{g}), \left[s(g), T\right] T^{L_1-1}\right\}/Z_\Lambda\right.$$

Lemma 2:

$$\left[s(\bar{g}), \left[s(g), T\right]\right] \leqslant 2\beta^{-2}\|g\|^2 T$$

Proof. As in Ref. 17 we first note that since s is a multiplication operator, it suffices to prove the lemma with T replaced by T_0. The commutator can then be estimated explicitly by going to Fourier transforms.

In momentum space, T_0 is a multiplication operator by $\prod_\alpha 2\beta(k_\alpha + \beta^2)^{-1}$ and $s(g)$ become $-i\sum_\alpha g(\alpha)(\partial/\partial k_\alpha)$. The double commutator is given by

$$\sum_{\alpha,\alpha'} \bar{g}(\alpha)g(\alpha)\frac{\partial^2}{\partial k_\alpha \partial k'_\alpha}\left[\prod_\alpha 2\beta\left(k_\alpha^2+\beta^2\right)^{-1}\right]$$

$$= \left\{-\sum_{\alpha\neq\alpha'} \bar{g}(\alpha)g(\alpha')4k_\alpha k'_\alpha\left(k_\alpha^2+\beta^2\right)^{-1}\left(k_{\alpha'}^2+\beta^2\right)^{-1}\right.$$

$$\left. + \sum_\alpha |g(\alpha)|^2\left[2\left(k_\alpha^2+\beta^2\right)^{-1} - 8k_\alpha^2\left(k_\alpha^2+\beta^2\right)^{-1}\right]\right\}T_0$$

$$= \left\{2\sum_\alpha |g(\alpha)|^2\left(k_\alpha+\beta^2\right)^{-1} - \left|\sum_\alpha g(\alpha)2k_\alpha\left(k_\alpha^2+\beta^2\right)^{-1}\right|^2\right.$$

$$\left. - \sum_\alpha 4k_\alpha^2\left(k_\alpha^2+\beta^2\right)^{-2}|g(\alpha)|^2\right\}T_0$$

$$\leqslant 2\|g\|^2\beta^{-2}T_0$$

In the last line we used the positivity of T_0.

Proof of Proposition. If we choose $g(\alpha) = e^{ip\cdot\alpha}$, Lemmas 1 and 2 imply $2(1 - \cos p_1)S_\Lambda(p) \leqslant 4\beta^{-2}$.

Since the direction 1 is arbitrary we get the result by summing over all d directions.

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