# TIME DISCRETIZATION SCHEMES FOR THE STEFAN PROBLEM IN A CONCENTRATED CAPACITY

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ABSTRACT. We consider two different time discretization algorithms for a nonlinear parabolic PDE arising in heat conduction phenomena with phase changes in two adjoining bodies $\Omega$ and $\Gamma$, where $\Gamma$ can be considered as the boundary of $\Omega$. Stability, convergence and error estimate results are given for both algorithms.

SOMMARIO. Si studiano due algoritmi di discretizzazione nel tempo di un sistema di equazioni a derivate parziali non lineari paraboliche che governa la conduzione del calore, in presenza di cambiamento di fase, in due corpi congiunti $\Omega$ e $\Gamma$, di cui $\Gamma$ possa essere considerato come la frontiera di $\Omega$. Vengono dati risultati di stabilità, convergenza e maggiorazione dell'errore per entrambi gli algoritmi.

KEY WORDS: Free boundary problems, Time discretization, Error estimates.

## 0. INTRODUCTION

The Stefan problems in a *concentrated capacity* arise in heat diffusion phenomena involving phase changes in two adjoining three-dimensional bodies $\Omega$ and $\Gamma$, when assuming that the thermal conductivity along the direction normal to the boundary of $\Omega$ is much greater than in the others. As far as heat diffusion is concerned, the body $\Gamma$ then behaves like a manifold of dimension less than 3. The mathematical formulation of the free boundary problem for the heat conduction on $\Gamma$ has been given in [5] (see also [2], [15] and the references therein). A more general model including a phase change also in $\Omega$ has been studied recently in [11]. The impact of the Stefan problems in a concentrated capacity in a number of physical applications (e.g. the phase change in the bulk $\Omega$ can be used to control the heat conduction on $\Gamma$) motivates their numerical analysis.

Let us introduce the mathematical model studied in [11]. Let $\Omega$ be an open-bounded regular set of $\mathbf{R}^n$; $\Gamma=\partial\Omega$ is the boundary of $\Omega$ and $v$ is the inward normal to $\Gamma$. Let $\theta$ denote the temperature both on $\Gamma$ and in $\Omega$, and let $u$ represent the enthalpy density on $\Gamma$. Let $\theta=\beta(u)$ be the state equation between temperature $\theta$ and enthalpy density $u$ on $\Gamma$, where $\beta\colon \mathbf{R}\to\mathbf{R}$ is a nondecreasing Lipschitz continuous function such that $\beta(0)=0$ and $\beta$ grows at least linearly at $\infty$. Then, the phase change on $\Gamma\times(0,T)$ can be formally described by the nonlinear parabolic equation

$$\text{(I)}\quad \begin{cases} \dfrac{\partial u}{\partial t}-\Delta_g\beta(u)=f+\dfrac{\partial\theta}{\partial v} \quad \text{on } \Sigma=\Gamma\times(0,T) \\ \text{with the initial condition } u(0)=u_0 \quad \text{on } \Gamma, \end{cases}$$

where $t\in(0,T)$ is the time variable, $\Delta_g$ is the Laplace–Beltrami operator on $\Gamma$ with respect to the Riemannian structure $g$ related to the tangential conductivity properties of $\Gamma$, $u_0$ is the initial enthalpy on $\Gamma$, $f$ is a heat source or sink on $\Sigma$, and $\partial\theta/\partial v$ is the *thermal flux* from $\Omega$ to $\Gamma$, which plays the role of an additional heat source on $\Gamma$.

Assuming that a phase change takes place also in $\Omega$, and denoting by $\theta=\gamma(v)$ the state equation for the temperature $\theta$ and the enthalpy density $v$ in $\Omega$, the heat diffusion in $\Omega\times(0,T)$ can be described by the nonlinear parabolic equation

$$\text{(II)}\quad \begin{cases} \dfrac{\partial v}{\partial t}-\Delta\gamma(v)=\varphi \quad \text{in } Q=\Omega\times(0,T) \\ \text{with the boundary condition } \gamma(v)=\beta(u) \text{ on } \Sigma \\ \text{and the initial condition} \qquad v(0)=v_0 \text{ in } \Omega, \end{cases}$$

where $\Delta$ is the Laplace operator in $\mathbf{R}^n$, $v_0$ is the initial enthalpy in $\Omega$, $\varphi$ is a heat source or sink in $Q$, and the constitutive function $\gamma$ bears the same properties of $\beta$.

An existence and uniqueness result for the problem (I)–(II) formulated in suitable Hilbert spaces has been proven in [11].

In order to deal with the numerical approximation of problem (I)–(II), it is useful to study first its time discretization. Here we shall consider two different time discrete algorithms: the first algorithm is the classical implicit Euler finite difference scheme, whereas the second one is a *linear* scheme suggested by the so-called nonlinear Chernoff formula in nonlinear semigroup theory, first introduced in [3] and next studied in [9], [12], [14], [17] for the usual parabolic Stefan-like problems. For both schemes we prove stability and error estimates for temperatures in the natural energy spaces. The full discretization of (I)–(II) is not addressed in this paper and will be the subject of future investigation.

## 1. THE CONTINUOUS PROBLEM

### 1.1. Assumptions and Notation

Let $\Omega$ be an open-bounded set of $\mathbf{R}^n$, $n \geq 2$, whose boundary $\Gamma$ is an oriented connected $C^\infty$ $(n-1)$-manifold, $Q = \Omega \times (0, T)$, $\Sigma = \Gamma \times (0, T)$, $T > 0$. We follow the notation of [7]; in particular we use the $L^2$-Sobolev spaces $H^s(\Omega)$ and $H^s(\Gamma)$, $s$ real, and set $H^0(\Omega) = L^2(\Omega)$, $H^0(\Gamma) = L^2(\Gamma)$. Let us denote by $\nu$ the inward normal to $\Gamma$, by $\nabla$ and $\Delta$ the gradient vector and the Laplace operator in $\mathbf{R}^n$. Moreover we assume that a (proper) $C^\infty$-Riemannian structure $g$ is defined on $\Gamma$ and denote by $\Delta_g$ the Laplace–Beltrami operator on $\Gamma$ with respect to $g$ (see, e.g., [4]) and by $(\eta_1, \eta_2)_g$ the global scalar product with respect to $g$, either for $\eta_1, \eta_2 \in L^2(\Gamma)$ or for $\eta_1 \in H^{-1}(\Gamma)$ and $\eta_2 \in H^1(\Gamma)$. We recall that $\eta \to \Delta_g \eta$ is a linear continuous operator from $H^1(\Gamma)$ into $H^{-1}(\Gamma)$ and that

$$-(\Delta_g \eta_1, \eta_2)_g = (\mathrm{d}\eta_1, \mathrm{d}\eta_2)_g \quad \forall \eta_1, \eta_2 \in H^1(\Gamma),$$

where d is the exterior differential on $\Gamma$. Moreover we shall use also the spaces $H^{r,s}$, $r$ and $s$ nonnegative real numbers, defined in [7, Ch. 4, n. 2.1] as

$$H^{r,s}(Q) = L^2(0, T; H^r(\Omega)) \cap H^s(0, T; L^2(\Omega)),$$

$$H^{r,s}(\Sigma) = L^2(0, T; H^r(\Gamma)) \cap H^s(0, T; L^2(\Gamma)),$$

endowed with their natural norms.

Let $\beta$ and $\gamma$: $\mathbf{R} \to \mathbf{R}$ satisfy $\beta(0) = 0$, $\gamma(0) = 0$ and

$$|\beta(s)| \geq c_1|s| - c_2, \quad |\gamma(s)| \geq c_3|s| - c_4, \quad \forall s \in \mathbf{R},$$

$$c_\beta|\beta(s_1) - \beta(s_2)|^2 \leq (\beta(s_1) - \beta(s_2))(s_1 - s_2),$$

$$c_\gamma|\gamma(s_1) - \gamma(s_2)|^2 \leq (\gamma(s_1) - \gamma(s_2))(s_1 - s_2), \quad \forall s_1, s_2 \in \mathbf{R} \tag{1.1}$$

($c_1$, $c_2$, $c_3$, $c_4$, $c_\beta$, $c_\gamma$ positive numbers).

REMARK 1.1. The functions $\beta$ and $\gamma$ represent the constitutive equations $\theta = \beta(u)$ and $\theta = \gamma(v)$ relating the temperature $\theta$ to the enthalpy densities $u$ on $\Gamma$ and $v$ in $\Omega$, respectively. The simplest physical case corresponding to constant thermal coefficients in each phase is given by

$$\beta(s) = \begin{cases} l_\beta s & \text{if } s \leq 0 \\ 0 & \text{if } 0 \leq s \leq \lambda_0, \\ L_\beta(s - \lambda_0) & \text{if } s \geq \lambda_0 \end{cases}$$

$$\gamma(s) = \begin{cases} l_\gamma s & \text{if } s \leq \lambda_1 \\ l_\gamma \lambda_1 & \text{if } \lambda_1 \leq s \leq \lambda_2, \\ L_\gamma(s - \lambda_2) + l_\gamma \lambda_1 & \text{if } s \geq \lambda_2 \end{cases}$$

where $\lambda_0 \geq 0$ and $\lambda_2 - \lambda_1 \geq 0$ represent the latent heats on $\Gamma$ and $\Omega$, the constants $l_\beta$, $L_\beta$, $l_\gamma$, $L_\gamma$ are the heat capacities, and the temperature of the phase change is $\theta_\Gamma = 0$ on $\Gamma$ and $\theta_\Omega = l_\gamma \lambda_1 \geq 0$ in $\Omega$. □

The hypotheses on the initial data $u_0$, $v_0$ and source terms $f$, $\varphi$ are the following:

$$u_0 \in L^2(\Gamma),\ \beta(u_0) \in H^1(\Gamma), \quad v_0 \in L^2(\Omega),\ \gamma(v_0) \in H^1(\Omega), \tag{1.2}$$

$$\gamma(v_0) = \beta(u_0) \quad \text{on } \Gamma, \tag{1.3}$$

$$f \in L^2(\Sigma), \quad \varphi \in L^2(Q). \tag{1.4}$$

REMARK 1.2. We stress that (1.2) is a proper assumption on the initial data of a phase change problem, because it allows jumps for the enthalpies $u_0$ on $\Gamma$ and $v_0$ in $\Omega$ and for the temperature gradients $\nabla\theta_0$ ($\theta_0 = \beta(u_0)$ on $\Gamma$, $\theta_0 = \gamma(v_0)$ in $\Omega$). The condition (1.3) is an obvious compatibility condition for the temperatures. □

### 1.2. Existence and Uniqueness

Let us introduce the definition of the *weak solution* of problem (I)–(II).

DEFINITION 1.1. Under the assumptions (1.1)–(1.4), $(u, v)$ is a weak solution of problem (I)–(II) if

$$u \in L^\infty(0, T; L^2(\Gamma)),$$

$$\beta(u) \in L^\infty(0, T; H^1(\Gamma)) \cap H^1(0, T; L^2(\Gamma)),$$

$$v \in L^\infty(0, T; L^2(\Omega)),$$

$$\gamma(v) \in L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega)), \tag{1.5}$$

$$\gamma(v) = \beta(u) \quad \text{on } \Sigma, \tag{1.6}$$

and, for all $z \in Z_T = \{z \in H^{1,1}(Q) : z|_\Sigma \in H^{1,1}(\Sigma),\ z(T) = 0\}$, it holds that

$$-\int_0^T \int_\Gamma u \frac{\partial z}{\partial t}\, \mathrm{d}\sigma\, \mathrm{d}t - \int_\Gamma u_0 z(0)\, \mathrm{d}\sigma + \int_0^T (\mathrm{d}\beta(u), \mathrm{d}z)_g\, \mathrm{d}t - \int_0^T \int_\Omega v \frac{\partial z}{\partial t}\, \mathrm{d}x\, \mathrm{d}t - \int_\Omega v_0 z(0)\, \mathrm{d}x + \int_0^T \int_\Omega \nabla\gamma(v) \cdot \nabla z\, \mathrm{d}x\, \mathrm{d}t - \int_0^T \int_\Gamma fz\, \mathrm{d}\sigma\, \mathrm{d}t - \int_0^T \int_\Omega \varphi z\, \mathrm{d}x\, \mathrm{d}t = 0.$$

REMARK 1.3. From (1.5) it follows that $\gamma(v) \in H^{1,1}(Q)$ and, consequently (cf. [7, Ch. 4, n. 2.2]), $\gamma(v)$ has a *trace* on $\Sigma$ which belongs to $H^{1/2,1/2}(\Sigma)$; whence (1.6) is meaningful in $H^{1/2,1/2}(\Sigma)$. □

In [11] the following existence and uniqueness result has been proved.

THEOREM 1.1. *There exists a unique weak solution of problem* (I)–(II).

### 1.3. Properties of the Weak Solution

Let us introduce the space $V$ defined by

$$V = \{\eta \in H^1(\Omega): \eta|_\Gamma \in H^1(\Gamma)\}$$

which, endowed with the norm $\|\eta\|_V = (\|\eta\|^2_{H^1(\Omega)} + \|\eta|_\Gamma\|^2_{H^1(\Gamma)})^{1/2}$, is a Hilbert space. The following property of the weak solution of problem (I)–(II) holds [11].

PROPOSITION 1.1. *If* $(u, v)$ *is the weak solution of* (I)–(II), *there exists a subset* $E$ *of* $(0, T)$ *with* $meas(E) = 0$ *such that, for all* $t \in (0, T) - E$, *we have*

$$\int_\Gamma (u(t)-u_0)\eta \, d\sigma + \int_0^t (d\beta(u(s)), d\eta)_g \, ds + \int_\Omega (v(t)-v_0)\eta \, dx + \int_0^t \int_\Omega \nabla\gamma(v(s))\cdot\nabla\eta \, dx \, ds - \int_0^t \int_\Gamma f(s)\eta \, d\sigma \, ds - \int_0^t \int_\Omega \varphi(s)\eta \, dx \, ds = 0, \quad \forall \eta \in V. \tag{1.7}$$

Further properties of the weak solution $(u, v)$ of (I)–(II) have been proved in [11]. In particular we have that

$$v \in H^1(0, T; H^{-1}(\Omega)) \cap H^{1/2}(0, T; H^{-1/2}(\Omega)) \cap C^0([0, T]; H^{-1/2}(\Omega))$$

and $v$ satisfies the equation

$$\frac{\partial v}{\partial t} - \Delta\gamma(v) = \varphi \tag{1.8}$$

(in the sense of the distributions space $\mathscr{D}'(Q)$) and the initial condition

$$v(0) = v_0 \quad \text{(in the sense of } C^0([0, T]; H^{-1/2}(\Omega))). \tag{1.9}$$

Moreover, the normal derivative of $\gamma(v)$ on $\Sigma$ can be defined in a suitable weak sense, namely,

$$\frac{\partial\gamma(v)}{\partial v} \in \mathscr{G}'(\Sigma), \tag{1.10}$$

where $\mathscr{G}'(\Sigma)$ is a distribution space on $\Sigma$ defined as follows. Let

$$\mathscr{G}(\Sigma) = [H^1_0(0, T; L^2(\Gamma)), L^2(0, T; L^2(\Gamma))]_{1/2} \cap L^2(0, T; H^{1/2}(\Gamma)),$$

where

$$H^1_0(0, T; L^2(\Gamma)) = \{z \in H^1(0, T; L^2(\Gamma)): z(0) = z(T) = 0\}$$

and $[\cdot,\cdot]_\alpha$, $0 \leqslant \alpha \leqslant 1$, denotes the usual Hilbert spaces interpolation method described, e.g., in [7, Ch. 1]. Therefore $\mathscr{G}(\Sigma)$ is a Hilbert space and

$$\mathscr{D}(\Sigma) \subset \mathscr{G}(\Sigma) \subset H^{1/2,1/2}(\Sigma) \subset L^2(\Sigma)$$

with dense and continuous injections. Then $\mathscr{G}'(\Sigma)$ is the dual space of $\mathscr{G}(\Sigma)$ and, denoting by $H^{-1/2,-1/2}(\Sigma)$ the dual space of $H^{1/2,1/2}(\Sigma)$, we have

$$L^2(\Sigma) \subset H^{-1/2,-1/2}(\Sigma) \subset \mathscr{G}'(\Sigma) \subset \mathscr{D}'(\Sigma).$$

Further regularity for $\partial\gamma(v)/\partial v$ can be proved provided $\gamma$ is nondegenerate; e.g. if $\gamma(s) = s$ for all $s \in \mathbf{R}$, then [10]

$$\frac{\partial\gamma(v)}{\partial v} \in L^2(\Sigma).$$

Finally, $u$ verifies the equation

$$\frac{\partial u}{\partial t} - \Delta_g\beta(u) = f + \frac{\partial\gamma(v)}{\partial v} \quad \text{(in the sense of } \mathscr{D}'(\Sigma)), \tag{1.11}$$

and the initial condition

$$u(0) = u_0 \quad \text{(in the sense of} \quad \lim_{t\to 0, t\in(0,T)-E} \int_\Gamma (u(t) - u_0)\eta \, d\sigma = 0, \forall \eta \in V). \tag{1.12}$$

REMARK 1.4. Let us stress that (1.8), (1.9), (1.6), (1.11) and (1.12) give a precise (nonformal) sense to the problem (I)–(II) formulated in the Introduction.

## 2. THE IMPLICIT EULER FINITE DIFFERENCE ALGORITHM (S1)

### 2.1. The Algorithm

In order to guarantee the well-posedness of the implicit Euler finite difference scheme, we need the following preliminary result.

THEOREM 2.1. *Let* $\beta$ *and* $\gamma$ *satisfy* (1.1), *let* $\lambda$ *be a real positive number*, $F \in L^2(\Gamma)$, $\Phi \in L^2(\Omega)$; *then there exists a unique solution* $\{U, V\}$ *to the following problem*:

$$U \in L^2(\Gamma), \quad \beta(U) \in H^1(\Gamma), \quad V \in L^2(\Omega), \quad \gamma(V) \in H^1(\Omega), \tag{2.1}$$

$$\beta(U) = \gamma(V) \quad \text{on } \Gamma, \tag{2.2}$$

$$\int_\Gamma U\eta \, d\sigma + \lambda(d\beta(U), d\eta)_g + \int_\Omega V\eta \, dx + \lambda \int_\Omega \nabla\gamma(V)\cdot\nabla\eta \, dx = \int_\Gamma F\eta \, d\sigma + \int_\Omega \Phi\eta \, dx, \quad \forall \eta \in V. \tag{2.3}$$

REMARK 2.1. The proof of this theorem can be obtained, for example, by the same techniques developed in [11] to prove Theorem 1.1. More precisely, we approximate the functions $\beta$ and $\gamma$ by strictly increasing functions $\beta_\varepsilon$ and $\gamma_\varepsilon$, we solve (using the Leray–Schauder fixed point method) the nondegenerate elliptic problem obtained by replacing $\beta$ and $\gamma$ in (2.1), (2.2), (2.3) by $\beta_\varepsilon$ and $\gamma_\varepsilon$, and finally we pass to the limit as $\varepsilon \to 0$ by standard compactness and monotonicity arguments. □

Now we can introduce the implicit Euler algorithm (S1) which, under the assumptions (1.1)–(1.4), is well posed by virtue of Theorem 2.1.

Let $N$ be a fixed positive integer and let $\tau = T/N$ denote the time step. Let us introduce the notation: $t^0 = 0$, $z^0 = z(0)$, and, for $n = 1, \ldots, N$, $t^n = n\tau$, $I^n = (t^{n-1}, t^n]$, and

$z^n = z(t^n)$, $\bar{z}^n = \tau^{-1}\int_{I^n} z(t)\,dt$, for any continuous or integrable vector-valued function $t \to z(t)$ with values in $L^2(\Gamma)$ or $L^2(\Omega)$. Then, the algorithm (S1) reads as follows.

ALGORITHM (S1). *Set*

$$U^0 = u_0, \quad V^0 = v_0, \tag{2.4}$$

*and, for $n = 1, \dots, N$, let $(U^n, V^n)$ be the solution to the problem*

$$U^n \in L^2(\Gamma),\ \beta(U^n) \in H^1(\Gamma),\ V^n \in L^2(\Omega),\ \gamma(V^n) \in H^1(\Omega), \tag{2.5}$$

$$\beta(U^n) = \gamma(V^n) \quad \text{on } \Gamma, \tag{2.6}$$

$$\int_\Gamma (U^n - U^{n-1})\eta\, d\sigma + \tau(d\beta(U^n), d\eta)_g + \int_\Omega (V^n - V^{n-1})\eta\, dx + \tau \int_\Omega \nabla\gamma(V^n)\cdot\nabla\eta\, dx = \tau \int_\Gamma \bar{f}^n \eta\, d\sigma + \tau \int_\Omega \bar{\varphi}^n \eta\, dx, \quad \forall \eta \in V. \tag{2.7}$$

## 2.2. Stability

The following *stability* result holds.

THEOREM 2.2. *Under the assumptions* (1.1)–(1.4) *there exists a constant $C$ independent of $\tau$ such that*

$$\max_{1 \le n \le N} \|\beta(U^n)\|_{L^2(\Gamma)} + \max_{1 \le n \le N} \|\gamma(V^n)\|_{L^2(\Omega)} + \tau \sum_{n=1}^N \|\beta(U^n)\|^2_{H^1(\Gamma)} + \tau \sum_{n=1}^N \|\gamma(V^n)\|^2_{H^1(\Omega)} \le C. \tag{2.8}$$

*Proof.* By virtue of (2.5) and (2.6), we can take $\eta = \gamma(V^n)$ in (2.7). Adding the resulting expressions over $n$ from 1 to $i$, for any $1 \le i \le N$, we obtain

$$\sum_{n=1}^i \int_\Gamma (U^n - U^{n-1})\beta(U^n)\, d\sigma + \tau \sum_{n=1}^i (d\beta(U^n), d\beta(U^n))_g + \sum_{n=1}^i \int_\Omega (V^n - V^{n-1})\gamma(V^n)\, dx + \tau \sum_{n=1}^i \int_\Omega |\nabla\gamma(V^n)|^2\, dx = \tau \sum_{n=1}^i \int_\Gamma \bar{f}^n \beta(U^n)\, d\sigma + \tau \sum_{n=1}^i \int_\Omega \bar{r}^n \gamma(V^n)\, dx. \tag{2.9}$$

Let us introduce the following notation: if $\lambda: \mathbf{R} \to \mathbf{R}$ is a Lipschitz continuous function so that $\lambda(0) = 0$ and $0 \le \lambda'(s) \le \Lambda$ for a.e. $s \in \mathbf{R}$, we denote by $\Phi_\lambda$ the convex function

$$\Phi_\lambda(s) = \int_0^s \lambda(r)\, dr, \quad \forall s \in \mathbf{R},$$

and note that $\Phi_\lambda$ satisfies

$$\frac{1}{2\Lambda}\lambda^2(s) \le \Phi_\lambda(s) \le \frac{\Lambda}{2}s^2, \quad \forall s \in \mathbf{R}.$$

Then, from (1.1) and (2.4) we readily obtain

$$\sum_{n=1}^i \int_\Gamma (U^n - U^{n-1})\beta(U^n)\, d\sigma \ge \sum_{n=1}^i \int_\Gamma (\Phi_\beta(U^n) - \Phi_\beta(U^{n-1}))\, d\sigma = \int_\Gamma \Phi_\beta(U^i)\, d\sigma - \int_\Gamma \Phi_\beta(u_0)\, d\sigma \ge \frac{c_\beta}{2}\|\beta(U^i)\|^2_{L^2(\Gamma)} - \frac{1}{2c_\beta}\|u_0\|^2_{L^2(\Gamma)}$$

and, similarly,

$$\sum_{n=1}^i \int_\Omega (V^n - V^{n-1})\gamma(V^n)\, dx \ge \frac{c_\gamma}{2}\|\beta(V^i)\|^2_{L^2(\Gamma)} - \frac{1}{2c_\gamma}\|v_0\|^2_{L^2(\Gamma)}.$$

Therefore, from (2.9) we deduce the following estimate:

$$\|\beta(U^i)\|^2_{L^2(\Gamma)} + \|\gamma(V^i)\|^2_{L^2(\Omega)} + \tau \sum_{n=1}^i (d\beta(U^n), d\beta(U^n))_g + \tau \sum_{n=1}^i \|\nabla\gamma(V^n)\|^2_{L^2(\Omega)}$$
$$\le C\Big(\|u_0\|^2_{L^2(\Gamma)} + \|v_0\|^2_{L^2(\Omega)} + \tau \sum_{n=1}^i \|\bar{f}^n\|^2_{L^2(\Gamma)} + \sum_{n=1}^i \|\bar{\varphi}^n\|^2_{L^2(\Omega)} + \tau \sum_{n=1}^i \|\beta(U^n)\|^2_{L^2(\Gamma)} + \tau \sum_{n=1}^i \|\gamma(V^n)\|^2_{L^2(\Omega)}\Big),$$

where $C$ is independent of $\tau$. Finally, using the discrete Gronwall's lemma, we obtain the estimate (2.8) with $C$ depending on $T$, $\|u_0\|_{L^2(\Gamma)}$, $\|v_0\|_{L^2(\Omega)}$, $\|f\|_{L^2(\Sigma)}$, and $\|\varphi\|_{L^2(Q)}$, but independent of $\tau$. □

REMARK 2.2. A straightforward consequence of (2.8) and the linear growth at $\infty$ of $\beta$ and $\gamma$ (see (1.1)) is the stability also for enthalpies, namely,

$$\max_{1 \le n \le N} \|U^n\|_{L^2(\Gamma)} + \max_{1 \le n \le N} \|V^n\|_{L^2(\Omega)} \le C. \tag{2.10}$$ □

## 2.3. Error Estimates

In order to study the order of convergence for the algorithm (S1), let us introduce the temperature and enthalpy errors $e_\theta$ and $e_u$, $e_v$, defined by

$$e_\theta(t) = \gamma(v(t)) - \gamma(V^n), \quad e_v(t) = v(t) - V^n \quad \text{in } \Omega \times I^n,\ n = 1, \dots, N,$$

$$e_\theta(t) = \beta(u(t)) - \beta(U^n), \quad e_u(t) = u(t) - U^n \quad \text{on } \Gamma \times I^n,\ n = 1, \dots, N.$$

Note that, by virtue of (1.6) and (2.6), $e_\theta$ defined on $\Sigma$ is the trace of $e_\theta$ defined in $Q$. We shall prove the following *error estimate.*

THEOREM 2.3. *Under the assumptions* (1.1)–(1.4), *there exists a constant $C$ independent of $\tau$ such that*

$$\|e_\theta\|_{L^2(0,T;L^2(\Gamma))} + \|e_\theta\|_{L^2(0,T;L^2(\Omega))} + \left\|\int_0^t e_\theta(s)\, ds\right\|_{L^\infty(0,T;H^1(\Gamma))} + \left\|\int_0^t e_\theta(s)\, ds\right\|_{L^\infty(0,T;H^1(\Omega))} \le C\tau^{1/2}. \tag{2.11}$$

*Proof.* We add Equation (2.7) over $n$ from 1 to $i$, for any $1 \leqslant i \leqslant N$, and use (2.4) to obtain

$$\int_\Gamma (U^i - u_0)\eta \, d\sigma + \tau \sum_{n=1}^{i} (d\beta(U^n), d\eta)_g + \int_\Omega (V^i - v_0)\eta \, dx + \\ + \tau \sum_{n=1}^{i} \int_\Omega \nabla\gamma(V^n) \cdot \nabla\eta \, dx \\ = \int_0^{t^i} \int_\Gamma f(s)\eta \, d\sigma \, ds + \int_0^{t^i} \int_\Omega \varphi(s)\eta \, dx \, ds, \quad \forall \eta \in V. \tag{2.12}$$

Taking the difference of (2.12) from (1.7), for a.e. $t \in I^i$ and any $1 \leqslant i \leqslant N$, we get

$$\int_\Gamma e_u(t)\eta \, d\sigma + \left( d \int_0^t e_\theta(s) \, ds, d\eta \right)_g + \int_\Omega e_v(t)\eta \, dx + \\ + \int_\Omega \nabla \int_0^t e_\theta(s) \, ds \cdot \nabla\eta \, dx = (t^i - t)(d\beta(U^i), d\eta)_g + \\ + (t^i - t) \int_\Omega \nabla\gamma(V^i) \cdot \nabla\eta \, dx - \int_t^{t^i} \int_\Gamma f(s)\eta \, d\sigma \, ds - \\ - \int_t^{t^i} \int_\Omega \varphi(s)\eta \, dx \, ds, \quad \forall \eta \in V. \tag{2.13}$$

For a.e. $t \in (0, T)$, we can take $\eta = e_\theta(t)$ in (2.13); after integration in time over $(0, \bar{t})$, for any $\bar{t} \in (0, T)$, we get I + II + III + IV = V + VI + VII + VIII. We estimate each term separately. Noting that $\gamma(v(t)) - \gamma(V^i) = \beta(u(t)) - \beta(U^i)$ on $\Gamma$, from (1.1) it readily follows that

$$\text{I} + \text{III} \geqslant c_\beta \|e_\theta\|^2_{L^2(0,\bar{t};L^2(\Gamma))} + c_\gamma \|e_\theta\|^2_{L^2(0,\bar{t};L^2(\Omega))}.$$

On the other hand, we have

$$2(\text{II} + \text{IV}) = \left( d \int_0^{\bar{t}} e_\theta(s) \, ds, d \int_0^{\bar{t}} e_\theta(s) \, ds \right)_g + \\ + \left\| \nabla \int_0^{\bar{t}} e_\theta(s) \, ds \right\|^2_{L^2(\Omega)}.$$

Next we can write

$$|\text{V}| \leqslant \sum_{i=1}^{N} \int_{I^i} |(t^i - t)(d\beta(U^i), de_\theta(t))_g| \, dt \\ \leqslant C\tau \left( \sum_{i=1}^{N} \tau \|\beta(U^i)\|^2_{H^1(\Gamma)} + \|\beta(u)\|^2_{L^2(0,T;H^1(\Gamma))} \right)$$

and, similarly,

$$|\text{VI}| \leqslant C\tau \left( \sum_{i=1}^{N} \tau \|\gamma(V^i)\|^2_{H^1(\Omega)} + \|\gamma(v)\|^2_{L^2(0,T;H^1(\Omega))} \right),$$

$$|\text{VII}| \leqslant C\tau \left( \|f\|^2_{L^2(\Sigma)} + \sum_{i=1}^{N} \tau \|\beta(U^i)\|^2_{L^2(\Gamma)} + \|\beta(u)\|^2_{L^2(\Sigma)} \right),$$

$$|\text{VIII}| \leqslant C\tau \left( \|\varphi\|^2_{L^2(Q)} + \sum_{i=1}^{N} \tau \|\gamma(V^i)\|^2_{L^2(\Omega)} + \|\gamma(v)\|^2_{L^2(Q)} \right).$$

The asserted estimate then follows from (1.4), (1.5) and (2.8). □

REMARK 2.3. The problem of estimating the enthalpy errors $e_u$ on $\Sigma$ and $e_v$ in $Q$ seems to be more difficult. As a simple by-product of the above theorem we obtain a weak error estimates for enthalpies and the convergence of scheme (S1), namely,

$$\|e_v\|_{L^\infty(0,T;H^{-1}(\Omega))} \leqslant C\tau^{1/2}, \\ \|e_u + \mathscr{R}^* e_v\|_{L^\infty(0,T;H^{-1}(\Gamma))} \leqslant C\tau^{1/2}, \tag{2.14}$$

where $\mathscr{R}^* : (H^1(\Omega))' \to H^{-1}(\Gamma)$ is the adjoint operator of the harmonic extension operator $\mathscr{R} : H^1(\Gamma) \to H^1(\Omega)$ defined, for any $\psi \in H^1(\Gamma)$, by $\Delta\mathscr{R}\psi = 0$ in $\Omega$, $\mathscr{R}\psi = \psi$ on $\Gamma$. In fact, from the error equation (2.13), for a.e. $t \in (0, T)$ and all $\eta \in V$, we get

$$\left| \int_\Gamma e_u(t)\eta \, d\sigma + \int_\Omega e_v(t)\eta \, dx \right| \\ \leqslant C \left( \left\| \int_0^t e_\theta(s) \, ds \right\|_{L^\infty(0,T;H^1(\Gamma))} + \right. \\ \left. + \tau^{1/2} \left( \sum_{n=1}^{N} \tau \|\beta(U^n)\|^2_{H^1(\Gamma)} \right)^{1/2} + \tau^{1/2} \|f\|_{L^2(\Sigma)} \right) \|\eta\|_{H^1(\Gamma)} + \\ + C \left( \left\| \int_0^t e_\theta(s) \, ds \right\|_{L^\infty(0,T;H^1(\Omega))} + \right. \\ \left. + \tau^{1/2} \left( \sum_{n=1}^{N} \tau \|\gamma(V^n)\|^2_{H^1(\Omega)} \right)^{1/2} + \tau^{1/2} \|\varphi\|_{L^2(Q)} \right) \|\eta\|_{H^1(\Omega)},$$

whence, by virtue of (2.11), (2.8), and (1.4),

$$\left| \int_\Gamma e_u(t)\eta \, d\sigma + \int_\Omega e_v(t)\eta \, dx \right| \leqslant C\tau^{1/2} \|\eta\|_V,$$

where $C$ is a constant independent of $\tau$, $t$, and $\eta$. The weak convergences $e_v \rightharpoonup 0$ in $L^2(Q)$ and $e_u \rightharpoonup 0$ in $L^2(\Sigma)$ then easily follows from (1.5) and (2.10), whereas the estimates (2.14) follows taking $\eta \in H^1_0(\Omega)$ and $\eta = \mathscr{R}\psi$ ($\psi \in H^1(\Gamma)$), respectively. Note that the estimate on $e_v$ agrees with the regularity $v \in H^1(0, T; H^{-1}(\Omega))$. We conjecture that a more precise estimate of the enthalpy error $e_u$ on $\Sigma$ should be obtained using deep the properties of the weak solution recalled in Section 1.3, in particular (1.10). □

REMARK 2.4. Under the further assumptions $f \in H^1(0, T; L^2(\Gamma))$ and $\varphi \in H^1(0, T; L^2(\Omega))$, the algorithm (S1) can be defined using $f^n$ and $\varphi^n$ in place of $\bar{f}^n$ and $\bar{\varphi}^n$ at the right-hand side of (2.7). The stability and error estimates, Theorems 2.2, 2.3, hold under minor modifications of the proofs. □

## 3. THE ALGORITHM BASED ON THE NONLINEAR CHERNOFF FORMULA (S2)

### 3.1. The Algorithm

The linear scheme suggested by the nonlinear Chernoff formula in semigroup theory reads as follows.

ALGORITHM (S2). *Set*

$$U^0 = u_0, \quad \Xi^0 = \beta(u_0), \quad V^0 = v_0, \quad \Theta^0 = \gamma(v_0), \tag{3.1}$$

*and, for* $n=1,\ldots,N$*, let* $(U^n, \Xi^n, V^n, \Theta^n)$ *be the solution to the problem*

$$U^n \in L^2(\Gamma), \quad \Xi^n \in H^1(\Gamma), \quad V^n \in L^2(\Omega), \quad \Theta^n \in H^1(\Omega), \tag{3.2}$$

$$\Xi^n = \Theta^n \quad \text{on } \Gamma, \tag{3.3}$$

$$\int_\Gamma \Xi^n \eta \, d\sigma + \frac{\tau}{\mu}(d\Xi^n, d\eta)_g + \int_\Omega \Theta^n \eta \, dx + \frac{\tau}{\mu}\int_\Omega \nabla\Theta^n \cdot \nabla\eta \, dx$$
$$= \int_\Gamma \beta(U^{n-1})\eta \, d\sigma + \int_\Omega \gamma(V^{n-1})\eta \, dx + \frac{\tau}{\mu}\int_\Gamma \bar{f}^n \eta \, d\sigma +$$
$$+ \frac{\tau}{\mu}\int_\Omega \bar{\varphi}^n \eta \, dx, \quad \forall \eta \in V, \tag{3.4}$$

$$U^n = U^{n-1} + \mu(\Xi^n - \beta(U^{n-1})) \quad \text{on } \Gamma, \tag{3.5}$$
$$V^n = V^{n-1} + \mu(\Theta^n - \gamma(V^{n-1})) \quad \text{in } \Omega,$$

*where* $\mu$ *is a relaxation parameter which satisfies* $0 < \mu \leqslant \bar{\mu} = \min(c_\beta, c_\gamma)$.

It is easy to see that, under the assumptions (1.1)–(1.4), the algorithm (S2) is well posed. We stress that problem (3.3), (3.4) is linear in the unknowns $\Xi^n$ and $\Theta^n$ whereas Equations (3.5) for $U^n$ and $V^n$ are just pointwise corrections that require the evaluation of given nonlinear functions. Therefore, the algorithm (S2) is expected to be more *efficient* than (S1) from a numerical viewpoint.

### 3.2. Stability

The following *stability* result holds.

THEOREM 3.1. *Under the assumptions* (1.1)–(1.4) *and* $0 < \mu \leqslant \bar{\mu}$ *fixed, there exists a constant* $C$ *independent of* $\tau$ *such that*

$$\max_{1\leqslant n\leqslant N} \|\beta(U^n)\|_{L^2(\Gamma)} + \max_{1\leqslant n\leqslant N} \|\gamma(V^n)\|_{L^2(\Omega)} +$$
$$+ \sum_{n=1}^{N} \|U^n - U^{n-1}\|^2_{L^2(\Gamma)} + \sum_{n=1}^{N} \|V^n - V^{n-1}\|^2_{L^2(\Omega)} +$$
$$+ \tau \sum_{n=1}^{N} (d\Xi^n, d\Xi^n)_g + \tau \sum_{n=1}^{N} \|\nabla\Theta^n\|^2_{L^2(\Omega)} \leqslant C. \tag{3.6}$$

*Proof.* Let us introduce the functions $\alpha$ and $\delta$ defined by

$$\alpha(s) = s - \mu\beta(s), \quad \delta(s) = s - \mu\gamma(s), \quad \forall s \in \mathbf{R}, \tag{3.7}$$

which, in view of the stability constraint $0 < \mu \leqslant \bar{\mu}$, satisfy

$$0 \leqslant \alpha'(s) \leqslant 1, \quad 0 \leqslant \delta'(s) \leqslant 1 \quad \text{for a.e. } s \in \mathbf{R}. \tag{3.8}$$

Using (3.5) and (3.7), we can split $\Xi^n$ and $\Theta^n$ as follows:

$$\Xi^n = \frac{1}{2}\beta(U^n) + \frac{1}{2\mu}(\alpha(U^n) - \alpha(U^{n-1})) - \frac{1}{2\mu}\alpha(U^{n-1}) + \frac{1}{2\mu}U^n,$$
$$\Theta^n = \frac{1}{2}\gamma(V^n) + \frac{1}{2\mu}(\delta(V^n) - \delta(V^{n-1})) - \frac{1}{2\mu}\delta(V^{n-1}) + \frac{1}{2\mu}V^n. \tag{3.9}$$

In addition, still using (3.5), we can reformulate the discrete equation (3.4) as follows:

$$\int_\Gamma (U^n - U^{n-1})\eta \, d\sigma + \int_\Omega (V^n - V^{n-1})\eta \, dx + \tau(d\Xi^n, d\eta)_g +$$
$$+ \tau\int_\Omega \nabla\Theta^n \cdot \nabla\eta \, dx = \tau\int_\Gamma \bar{f}^n \eta \, d\sigma + \tau\int_\Omega \bar{\varphi}^n \eta \, dx, \quad \forall \eta \in V. \tag{3.10}$$

By virtue of (3.2) and (3.3) we can take $\eta = \Theta^n$ in (3.10). We add the resulting expressions over $n$ from 1 to $i$, for any $1 \leqslant i \leqslant N$, and proceed to estimate each term separately. Using (3.9), (3.1), (1.1), (3.8), the convexity of $\Phi_\beta, \Phi_\alpha, \Phi_\gamma, \Phi_\delta$, and the elementary identity $2a(a-b) = a^2 - b^2 + (a-b)^2$ for $a, b \in \mathbf{R}$, we obtain first

$$2\sum_{n=1}^{i}\int_\Gamma (U^n - U^{n-1})\Xi^n \, d\sigma \geqslant \int_\Gamma (\Phi_\beta(U^i) - \Phi_\beta(u_0)) \, d\sigma +$$
$$+ \frac{1}{\mu}\int_\Gamma (\Phi_\alpha(u_0) - \Phi_\alpha(U^i)) \, d\sigma +$$
$$+ \frac{1}{2\mu}\left(\|U^i\|^2_{L^2(\Gamma)} - \|u_0\|^2_{L^2(\Gamma)} + \sum_{n=1}^{i} \|U^n - U^{n-1}\|^2_{L^2(\Gamma)}\right)$$
$$\geqslant -\frac{1}{\mu}\|u_0\|^2_{L^2(\Gamma)} + \frac{c_\beta}{2}\|\beta(U^i)\|^2_{L^2(\Gamma)} +$$
$$+ \frac{1}{2\mu}\sum_{n=1}^{i} \|U^n - U^{n-1}\|^2_{L^2(\Gamma)}$$

and, similarly,

$$2\sum_{n=1}^{i}\int_\Omega (V^n - V^{n-1})\Theta^n \, dx$$
$$\geqslant -\frac{1}{\mu}\|v_0\|^2_{L^2(\Omega)} + \frac{c_\gamma}{2}\|\gamma(V^i)\|^2_{L^2(\Omega)} +$$
$$+ \frac{1}{2\mu}\sum_{n=1}^{i} \|V^n - V^{n-1}\|^2_{L^2(\Omega)}.$$

In addition, using (3.5), we have

$$\left|\sum_{n=1}^{i} \tau\int_\Omega \bar{f}^n \Xi^n \, d\sigma\right| \leqslant C\|f\|^2_{L^2(\Sigma)} +$$
$$+ C\tau\sum_{n=1}^{i} \|\beta(U^{n-1})\|^2_{L^2(\Gamma)} + \frac{1}{8\mu}\sum_{n=1}^{i} \|U^n - U^{n-1}\|^2_{L^2(\Gamma)},$$

$$\left|\sum_{n=1}^{i} \tau\int_\Omega \bar{\varphi}^n \Theta^n \, dx\right| \leqslant C\|\varphi\|^2_{L^2(Q)} +$$
$$+ C\tau\sum_{n=1}^{i} \|\gamma(V^{n-1})\|^2_{L^2(\Omega)} + \frac{1}{8\mu}\sum_{n=1}^{i} \|V^n - V^{n-1}\|^2_{L^2(\Omega)}.$$

Noting that the remaining terms are nothing but

$$\tau\sum_{n=1}^{i} (d\Xi^n, d\Xi^n)_g + \tau\sum_{n=1}^{i} \|\nabla\Theta^n\|^2_{L^2(\Omega)},$$

using the discrete Gronwall's lemma, we obtain the stability estimate (3.6). □

REMARK 3.1. A straightforward consequence of (3.6) and (3.5) is the estimate

$$\max_{1\leqslant n\leqslant N}\|\Xi^n\|_{L^2(\Gamma)}+\max_{1\leqslant n\leqslant N}\|\Theta^n\|_{L^2(\Omega)}\leqslant C, \tag{3.11}$$

whence, using again (3.6), it follows that

$$\tau\sum_{n=1}^{N}\|\Xi^n\|^2_{H^1(\Gamma)}+\tau\sum_{n=1}^{N}\|\Theta^n\|^2_{H^1(\Omega)}\leqslant C. \tag{3.12}$$

□

### 3.3. Error Estimates

Let us define the errors $e_\theta$, $e_u$ and $e_v$ by

$$e_\theta(t)=\gamma(v(t))-\Theta^n,\quad e_v(t)=v(t)-V^n$$
in $\Omega\times I^n$, $n=1,\ldots,N$,
$$e_\theta(t)=\beta(u(t))-\Xi^n,\quad e_u(t)=u(t)-U^n$$
on $\Gamma\times I^n$, $n=1,\ldots,N$.

Note that, by virtue of (1.6) and (3.3), $e_\theta$ defined on $\Sigma$ is the trace of $e_\theta$ defined in $Q$. We shall prove the following *error estimate.*

THEOREM 3.2. *Under the assumptions* (1.1)–(1.4) *and* $0<\mu\leqslant\bar\mu$ *fixed, there exists a constant C independent of* $\tau$ *such that*

$$\|e_\theta\|_{L^2(0,T;L^2(\Gamma))}+\|e_\theta\|_{L^2(0,T;L^2(\Omega))}+\left\|\int_0^t e_\theta(s)\,\mathrm{d}s\right\|_{L^\infty(0,T;H^1(\Gamma))}+$$
$$+\left\|\int_0^t e_\theta(s)\,\mathrm{d}s\right\|_{L^\infty(0,T;H^1(\Omega))}\leqslant C\tau^{1/4}. \tag{3.13}$$

*Proof.* We closely follow the proof of Theorem 2.3. We add Equation (3.10) over $n$ from 1 to $i$, for any $1\leqslant i\leqslant N$, and use (3.1) to obtain

$$\int_\Gamma (U^i-u_0)\eta\,\mathrm{d}\sigma+\tau\sum_{n=1}^{i}(\mathrm{d}\Xi^n,\mathrm{d}\eta)_g+\int_\Omega (V^i-v_0)\eta\,\mathrm{d}x+$$
$$+\tau\sum_{n=1}^{i}\int_\Omega \nabla\Theta^n\cdot\nabla\eta\,\mathrm{d}x$$
$$=\int_0^{t^i}\int_\Gamma f(s)\eta\,\mathrm{d}\sigma\,\mathrm{d}s+\int_0^{t^i}\int_\Omega \varphi(s)\eta\,\mathrm{d}x\,\mathrm{d}s,\quad\forall\eta\in V. \tag{3.14}$$

Taking the difference of (3.14) from (1.7), for a.e. $t\in I^i$ and any $1\leqslant i\leqslant N$, we get

$$\int_\Gamma e_u(t)\eta\,\mathrm{d}\sigma+\left(\mathrm{d}\int_0^t e_\theta(s)\,\mathrm{d}s,\mathrm{d}\eta\right)_g+\int_\Omega e_v(t)\eta\,\mathrm{d}x+$$
$$+\int_\Omega\nabla\int_0^t e_\theta(s)\,\mathrm{d}s\cdot\nabla\eta\,\mathrm{d}x=(t^i-t)(\mathrm{d}\Xi^i,\mathrm{d}\eta)_g+$$
$$+(t^i-t)\int_\Omega\nabla\Theta^i\cdot\nabla\eta\,\mathrm{d}x-\int_t^{t^i}\int_\Gamma f(s)\eta\,\mathrm{d}\sigma\,\mathrm{d}s-$$
$$-\int_t^{t^i}\int_\Omega\varphi(s)\eta\,\mathrm{d}x\,\mathrm{d}s,\quad\forall\eta\in V. \tag{3.15}$$

For a.e. $t\in(0,T)$, we can take $\eta=e_\theta(t)$ in (3.15); after integration in time over $(0,\bar t)$, for any $\bar t\in(0,T)$, we get I + II + III + IV = V + VI + VII + VIII. Noting that $\gamma(v(t))-\Theta^i=\beta(u(t))-\Xi^i$ on $\Gamma$, the estimate of terms II, IV, V, VI, VII, VIII proceeds along the same lines of the corresponding terms in the proof of Theorem 2.3, thus leading to

$$2(\mathrm{II}+\mathrm{IV})=\left(\mathrm{d}\int_0^{\bar t}e_\theta(s)\,\mathrm{d}s,\mathrm{d}\int_0^{\bar t}e_\theta(s)\,\mathrm{d}s\right)_g+$$
$$+\left\|\nabla\int_0^{\bar t}e_\theta(s)\,\mathrm{d}s\right\|^2_{L^2(\Omega)},$$

$$|\mathrm{V}+\mathrm{VI}+\mathrm{VII}+\mathrm{VIII}|\leqslant C\tau\left(\sum_{i=1}^{N}\tau\|\Xi^i\|^2_{H^1(\Gamma)}+\right.$$
$$+\sum_{i=1}^{N}\tau\|\Theta^i\|^2_{H^1(\Omega)}+\|\beta(u)\|^2_{L^2(0,T;H^1(\Gamma))}+$$
$$\left.+\|\gamma(v)\|^2_{L^2(0,T;H^1(\Omega))}+\|f\|^2_{L^2(\Sigma)}+\|\varphi\|^2_{L^2(Q)}\right).$$

It remains to bound from below terms I and III. We first decompose I as

$$\mathrm{I}=\mu\|e_\theta\|^2_{L^2(0,\bar t;L^2(\Gamma))}+$$
$$+\int_0^{\bar t}\int_\Gamma (e_u(t)-\mu e_\theta(t))e_\theta(t)\,\mathrm{d}\sigma\,\mathrm{d}t=\mathrm{I}_1+\mathrm{I}_2.$$

Next, we use (3.5) and (3.7) to split $e_\theta$ and $e_u$ on $\Gamma\times\mathrm{I}^i$, for any $1\leqslant i\leqslant N$, as follows:

$$e_\theta(t)=\beta(u(t))-\beta(U^{i-1})-\frac{1}{\mu}(U^i-U^{i-1}),$$

$$e_u(t)-\mu e_\theta(t)=\alpha(u(t))-\alpha(U^{i-1}),$$

whence, in view of (1.1) and (3.8), we have

$$(e_u(t)-\mu e_\theta(t))e_\theta(t)$$
$$=(\alpha(u(t))-\alpha(U^{i-1}))(\beta(u(t))-\beta(U^{i-1}))-$$
$$-\frac{1}{\mu}(U^i-U^{i-1})(\alpha(u(t))-\alpha(U^{i-1}))$$
$$\geqslant-\frac{1}{\mu}(U^i-U^{i-1})(\alpha(u(t))-\alpha(U^{i-1})).$$

Therefore, using again (3.8), we can estimate $\mathrm{I}_2$ as follows:

$$\mathrm{I}_2\geqslant-C\tau^{1/2}\left(\sum_{i=1}^{N}\|U^i-U^{i-1}\|^2_{L^2(\Gamma)}\right)^{1/2}\times$$
$$\times\left(\|u\|_{L^2(0,T;L^2(\Gamma))}+\left(\sum_{i=1}^{N}\tau\|U^{i-1}\|^2_{L^2(\Gamma)}\right)^{1/2}\right).$$

Similarly, we get

$$\mathrm{III}\geqslant\mu\|e_\theta\|^2_{L^2(0,\bar t;L^2(\Omega))}-$$
$$-C\tau^{1/2}\left(\sum_{i=1}^{N}\|V^i-V^{i-1}\|^2_{L^2(\Omega)}\right)^{1/2}\times$$
$$\times\left(\|v\|_{L^2(0,T;L^2(\Omega))}+\left(\sum_{i=1}^{N}\tau\|V^{i-1}\|^2_{L^2(\Gamma)}\right)^{1/2}\right).$$

Collecting all previous estimates and using (1.5) and (3.6) leads to the asserted error bound (3.13). □

REMARK 3.2. From equation (3.15), one can proceed as in Remark 2.3 and obtain the convergence of scheme (S2) as well as error estimates for enthalpy, namely, $e_v \rightharpoonup 0$ in $L^2(Q)$, $e_u \rightharpoonup 0$ in $L^2(\Sigma)$, and

$$\|e_v\|_{L^\infty(0,T;H^{-1}(\Omega))} \leqslant C\tau^{1/4},$$

$$\|e_u + \mathscr{R}^* e_v\|_{L^\infty(0,T;H^{-1}(\Gamma))} \leqslant C\tau^{1/4}. \qquad \square$$

REMARK 3.3. Remark 2.4 applies also for algorithm (S2). □

REMARK 3.4. A number of linear approximation schemes for Stefan-like problems has been proposed during recent years; see, e.g. [1], [6], [8], [13], [16]. The application of these methods to the Stefan problems in a concentrated capacity should be investigated. □

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