(ANTI)SELF-DUAL GAUGE FIELDS IN DIMENSION $d \geq 4$

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The (anti)self-duality equations for gauge fields in dimension $d=4$ and the generalization of these equations for $d>4$ are considered. The results on solutions of the (anti)self-duality equations in $d \geq 4$ are reviewed. Some new classes of solutions of Yang—Mills equations in $d \geq 4$ for arbitrary gauge fields are described.

1. INTRODUCTION

It is well known that the electromagnetic, weak, and strong interactions are described by gauge fields of the group $U(1) \times SU(2) \times SU(3)$. The dynamics of these fields is determined by the Yang—Mills (YM) equations in a space of dimension $d=4$, and therefore the finding of solutions of YM equations is of particular interest.

At the present time there are known in $d=4$ not too many solutions of the YM equations in explicit form (see, for example, [1—3]). The most interesting among them are solutions of instanton, monopole, and vortex type. They can be obtained as solutions of the self-duality equations in Euclidean space $\mathbb{R}^4$ for gauge fields that depend on four, three, and two coordinates.

Yang—Mills equations in a space of dimension $d \geq 4$ appear in multidimensional theories of supergravity, super-Yang—Mills, and in the low-energy limit of superstring theory [4]. The use of solutions of the YM equations in $d \geq 4$ makes it possible to obtain soliton solutions in superstring and supermembrane theories [5,6]. It is also known that the imposition of symmetry conditions on the gauge fields of YM theory in d dimensions leads to Yang—Mills—Higgs (YMH) theory in $k<d$ dimensions [7]. Therefore, solutions of the YMH equations in $d=4$ can be obtained from solutions of YM equations in dimension $d>4$.

In [8—11] linear equations for the components of the field tensor of gauge fields in the Euclidean space $\mathbb{R}^d$, generalizing to $d>4$ the self-duality equations of four-dimensional YM theory, were introduced. Some solutions of these equations were obtained in [9—15]. In this paper, we review the results published in [14—19] on solutions of the (anti)self-duality equations in spaces $\mathbb{R}^d$ with $d \geq 4$.

2. YANG—MILLS EQUATIONS AND SELF-DUALITY IN $\mathbb{R}^d$

2.1. Definitions and Notation. We consider in d-dimensional Euclidean space $\mathbb{R}^d$ with metric δ_{ab} gauge fields A_a of an arbitrary semisimple Lie group G, $a, b, \ldots = 1, \ldots, d$. The fields A_a take values in the Lie algebra $\mathcal{G}$ of the Lie group G. The field tensor F_{ab} of the gauge fields has the form

$$F_{ab} = [D_a, D_b] = [\partial_a + A_a, \partial_b + A_b] = \partial_a A_b - \partial_b A_a + [A_a, A_b]. \tag{1}$$

The YM equations have the form

$$D_a F_{ab} = 0 \Longleftrightarrow \partial_a F_{ab} + [A_a, F_{ab}] = 0, \tag{2}$$

where summation over repeated indices is understood. For an arbitrary tensor F_{ab} of the form (1), the Bianchi identities hold:

$$D_{[a} F_{bc]} = 0 \Longleftrightarrow D_a F_{bc} + D_b F_{ca} + D_c F_{ab} = 0. \tag{3}$$

We use square brackets to denote antisymmetrization with respect to all indices in the brackets, for example,

$$T_{[ab]} = T_{ab} - T_{ba}, \quad M_{a[b} N_{c]d} = M_{ab} N_{cd} - M_{ac} N_{bd}.$$

2.2. Self-Duality. We recall that gauge fields A_μ in $d=4$ are said to be (anti)self-dual if their field tensor $F_{\mu\nu}$ satisfies the equations

$$\epsilon_{\mu\nu\lambda\sigma} F_{\lambda\sigma} = \pm 2 F_{\mu\nu}, \tag{4}$$

Moscow State University; Joint Institute for Nuclear Research, Dubna. Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 94, No. 2, pp. 316-342, February, 1993. Original article submitted April 27, 1992.

where $\mu, \nu, \ldots=1, \ldots, 4$, respectively. By virtue of the Bianchi identities (3), every (anti)self-dual field satisfies the YM equations (2).

In [8], the following generalization of the (anti)self-duality equations (4) to the case $d>4$ was proposed:

$$\mathcal{E}_{abcd}F_{cd} = \lambda F_{ab}, \tag{5}$$

where λ=const, and $\mathcal{E}$ is the completely antisymmetric tensor with constant numerical coefficients. Examples of the tensors $\mathcal{E}$ and the eigenvalues λ corresponding to them will be given below.

We shall call gauge fields A_a whose field tensor F_{ab} satisfies Eqs. (5) self-dual and anti-self-dual, respectively. By virtue of the Bianchi identities, it is obvious that solutions of Eqs. (5) satisfy the YM equations (2). Some solutions of Eqs. (5) were obtained in [9—15].

In contrast to (4), Eqs. (5) are not $SO(d)$-invariant for $d>4$, since $\mathcal{E}_{abcd}$ cannot behave as a (pseudo)scalar under the action of the group $SO(d)$. However, $\mathcal{E}_{abcd}$ can be invariant with respect to a subgroup H of $SO(d)$, and Eqs. (5) are well defined on spaces having H as group of admissible coordinate transformations (see [8—11]).

Remark 1. Equations of the type (5) but with tensor $\mathcal{E}_{abcd}$ that is not always completely antisymmetric arise from the condition of vanishing of the curvature F_{ab} on certain complex m-dimensional planes as a consequence of a geometrical definition of (anti)self-duality (see, for example, [9,13,20—23]) in the generalization of twistor theory to dimension $d=2n>4$ (see [9,20,21,24,25]).

Of course, if the tensor $\mathcal{E}_{abcd}$ is not completely antisymmetric then the solutions of Eqs. (5) will not satisfy the YM equations (2). However, in some cases, on hyper-Kähler spaces, for example, geometrical (anti)self-duality is identical to algebraic (anti)self-duality (5) [9,13,23]. On the other hand, Eqs. (5) cannot always be represented as conditions of integrability on certain m planes in the sense of the twistor approach [8,9,13,20—25].

Remark 2. In a number of studies [26—29] the operator of Hodge * dual conjugation on spaces of even dimension $d=2m$ has been used to introduce (anti)self-duality equations that are nonlinear in the curvature tensor F_{ab} and differ from (5). The solutions of such equations do not satisfy Eqs. (2) but either certain generalized YM equations in $\mathbb{R}^{2n}$ [26—28] or YM equations on symmetric spaces [29].

In this paper, we restrict ourselves to constructing solutions of the algebraic (anti)self-duality equations (5) linear in the curvature F_{ab}.

3. INVARIANT TENSORS $\mathcal{E}_{abcd}$

For spaces $\mathbb{R}^d$ with additional structure we give examples of tensors $\mathcal{E}_{abcd}$ invariant with respect to some subgroup H of the group $SO(d)$.

3.1. Lie Algebras and Self-Duality. We consider a compact Lie algebra $\mathcal{H}$ of the simple Lie group H. We assume that the structure constants f_{abc} are normalized in such a way that the Killing—Cartan metric is $K_{ab}=2\delta_{ab}$, $a, b, \ldots=1, \ldots, \dim\mathcal{H}$. We set $n=1+\dim\mathcal{H}$ and consider the vector space $\mathbb{R}^n=\mathcal{H}\oplus\mathbb{R}$. We choose the metric in this space in the form $\delta_{\mu\nu}=\{\delta_{ab}, \delta_{nn}\}$, $\mu, \nu, \ldots=1, \ldots, n$. We denote the basis in the algebra $\mathcal{H}$ by I_a, and the basis vector of the one-dimensional space $\mathbb{R}$ by I_n. It is obvious that $\mathbb{R}^n=\mathcal{H}\oplus\mathbb{R}$ can be regarded as a Lie algebra with the commutation relations

$$[I_a, I_b] = f_{abc}I_c, \qquad [I_a, I_n] = 0.$$

We introduce in $\mathbb{R}^n$ the following completely antisymmetric 4-index tensor $R_{\mu\nu\lambda\sigma}$:

$$R_{abcd} = 0, \qquad R_{abcn} = f_{abc}. \tag{6}$$

The group $SO(n-1)$ is embedded in the group $SO(n)$, and H can be embedded in $SO(n-1)$. The group H acts on the subspace $\mathcal{H}\subset\mathbb{R}^n$ by the adjoint representation

$$Ad_h = \left\{(Ad_h)^b_a\right\}$$

and preserves the decomposition $\mathbb{R}^n=\mathcal{H}\oplus\mathbb{R}$. The tensor $R_{\mu\nu\lambda\sigma}$ is invariant with respect to the action of the group H. This follows from the invariance of the scalar product $\langle,\rangle$ on $\mathcal{H}(\langle I_a, I_b\rangle=2\delta_{ab})$ with respect to the adjoint action:

$$\begin{aligned}(Ad_h)^d_a(Ad_h)^e_b(Ad_h)^g_c R_{degn} &\equiv (Ad_h)^d_a(Ad_h)^e_b(Ad_h)^g_c f_{deg}\\ &= \langle Ad_h(I_a), [Ad_h(I_b), Ad_h(I_c)]\rangle\\ &= \langle Ad_h(I_a), Ad_h([I_b, I_c])\rangle\\ &= \langle I_a, [I_b, I_c]\rangle = f_{abc} \equiv R_{abcn}.\end{aligned}$$

We give examples of constant (anti)self-dual tensors. We set

$$f^a_{\mu\nu} = \{ f^a_{bc},\ \mu=b,\ \nu=c;\ \delta^a_b,\ \mu=b,\ \nu=n;\ -\delta^a_b,\ \mu=n,\ \nu=b \}, \quad f_{\mu\nu\lambda\sigma} = f^a_{\mu\nu} f^a_{\lambda\sigma} - 2R_{\mu\nu\lambda\sigma}.$$

It is easy to show that

$$R_{\mu\nu\lambda\sigma} f^a_{\lambda\sigma} = 2 f^a_{\mu\nu}, \quad R_{\mu\nu\alpha\beta} f_{\lambda\sigma\alpha\beta} = -2 f_{\mu\nu\lambda\sigma}.$$

3.2. Complex Structure. We consider a space $\mathbb{R}^{2n}$ of even dimension with metric structure δ_{MN}, $M, N, \ldots = 1, \ldots, 2n$. By a complex structure on $\mathbb{R}^{2n}$ we mean a constant tensor $J = (J_{MN})$ whose components satisfy the relations [30]

$$J_{MQ} J_{QN} = -\delta_{MN}. \tag{7}$$

The canonical complex structure J_0 has the form

$$J_0 = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

By means of the tensor J_{MN} we can introduce in $\mathbb{R}^{2n}$ complex coordinates and identify $\mathbb{R}^{2n}$ with $\mathbf{C}^n$.

The group $SO(2n)$ preserves the metric δ_{MN}. A subgroup H in $SO(2n)$ that preserves the complex structure J_{MN} is the group $U(n)$:

$$U(n) = \{ g \in SO(2n):\ gJ = Jg \}.$$

By means of the real antisymmetric 2-index tensor J_{MN} we can specify a completely antisymmetric 4-index tensor:

$$T_{MNPQ} \equiv J_{MN} J_{PQ} - J_{MP} J_{NQ} + J_{MQ} J_{NP}, \tag{8}$$

which is invariant with respect to the group $H = U(n) \subset SO(2n)$.

We introduce the following tensors:

$$K_{PQRS} \equiv \frac{1}{2n} J_{PQ} J_{RS}, \tag{9a}$$

$$\overline{K}_{PQRS} \equiv -\frac{1}{4}\left(T_{PQRS} - \left(1 - \frac{2}{n}\right) J_{PQ} J_{RS} - \delta_{PR}\delta_{QS} + \delta_{PS}\delta_{QR} \right), \tag{9b}$$

$$L_{PQRS} \equiv \frac{1}{4}(T_{PQRS} - J_{PQ} J_{RS} + \delta_{PR}\delta_{QS} - \delta_{PS}\delta_{QR}). \tag{9c}$$

Using (7) and (8), we can readily show that

$$T_{MNPQ} K_{PQRS} = 2(n-1) K_{MNRS}, \tag{10a}$$

$$T_{MNPQ} \overline{K}_{PQRS} = -2 \overline{K}_{MNRS}, \tag{10b}$$

$$T_{MNPQ} L_{PQRS} = 2 L_{MNRS}, \tag{10c}$$

$$K_{MNPQ} + \overline{K}_{MNPQ} + L_{MNPQ} = \frac{1}{2}(\delta_{MP}\delta_{NQ} - \delta_{MQ}\delta_{NP}). \tag{11}$$

Since

$$\begin{aligned} &K_{MNPQ} K_{PQRS} = K_{MNRS}, \quad &&\overline{K}_{MNPQ} \overline{K}_{PQRS} = \overline{K}_{MNRS}, \\ &L_{MNPQ} L_{PQRS} = L_{MNRS}, \quad &&K_{MNPQ} \overline{K}_{PQRS} = 0, \\ &K_{MNPQ} L_{PQRS} = 0, \quad &&\overline{K}_{MNPQ} L_{PQRS} = 0, \end{aligned} \tag{12}$$

it follows that the tensors K_{MNPQ}, $\bar{K}_{MNPQ}$, and L_{PQRS} are projectors onto three orthogonal subspaces in the algebra $so(2n)$. We have

$$so(2n) = u(1) \oplus su(n) \oplus \mathcal{P},$$

where $\mathcal{P}$ is the orthogonal complement of $u(n)$ in $so(2n)$. One can show that K_{MNPQ} projects onto the subalgebra $u(1)$, $\tilde{K}_{MNPQ}$ onto the subalgebra $su(n)$, and L_{MNPQ} onto the subspace $\mathcal{P}$. Note that for $n=2$ $\tilde{K}_{MNPQ}$ projects onto one subalgebra $su(2)$ in $so(4)$, and $K_{MNPQ}+L_{MNPQ}$ projects onto the other subalgebra $su(2)$ in $so(4)$.

3.3. Quaternion Structure. We consider the space $\mathbf{R}^{4n}$ of dimension $4n$ with metric δ_{ab}, $a, b, \ldots=1, \ldots, 4n$. One says that in the space $\mathbf{R}^{4n}$ there is defined a quaternion structure if in $\mathbf{R}^{4n}$ there are defined three complex structures $J^\alpha=(J^\alpha_{ab})$ related by [23]

$$J^\alpha J^\beta = -\delta^{\alpha\beta} - \epsilon^{\alpha\beta\gamma} J^\gamma \Longleftrightarrow J^\alpha_{ac} J^\beta_{cb} = -\delta^{\alpha\beta}\delta_{ab} - \epsilon^{\alpha\beta\gamma} J^\gamma_{ab}, \tag{13}$$

where $\epsilon_{\alpha\beta\gamma}$ are the structure constants of the group $SU(2)$, $\alpha, \beta, \ldots=1, 2, 3$. Using J^α_{ab}, we can identify $\mathbf{R}^{4n}$ with $\mathbf{H}^n$.

It is known (see, for example, [23]) that a subgroup in $SO(4n)$ which preserves the quaternion structure is $Sp(1)Sp(n)\subset SO(4n)$. Accordingly, for the Lie algebra $so(4n)$ we have

$$so(4n) = sp(1) \oplus sp(n) \oplus \mathcal{K}, \tag{14}$$

where $\mathcal{K}$ is the orthogonal complement of $sp(1)\oplus sp(n)$ in the algebra $so(4n)$.

Using the real antisymmetric tensors J^α_{ab}, we can introduce the following completely antisymmetric tensor:

$$Q_{abcd} \equiv \frac{1}{3}\left(J^\alpha_{ab} J^\alpha_{cd} + J^\alpha_{ad} J^\alpha_{bc} - J^\alpha_{ac} J^\alpha_{bd}\right), \tag{15}$$

which is invariant with respect to $Sp(1)Sp(n)\equiv Sp(1)\times Sp(n)/Z_2$.

In accordance with the decomposition (14), we can define the projector N_{abcd} onto the subalgebra $sp(1)$, the projector $\tilde{N}_{abcd}$ onto the subalgebra $sp(n)$, and the projector M_{abcd} onto the subspace $\mathcal{K}$. These projectors were introduced in [13]. They satisfy relations of the type (12). We shall use a definition of the projectors N, $\tilde{N}$, and M in terms of J^α_{ab}:

$$N_{abcd} \equiv \frac{1}{2n} J^\alpha_{ab} J^\alpha_{cd}, \tag{16a}$$

$$\overline{N}_{abcd} \equiv -\frac{1}{4}\left(3Q_{abcd} - J^\alpha_{ab} J^\alpha_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right), \tag{16b}$$

$$M_{abcd} \equiv \frac{1}{4}\left(3Q_{abcd} - \left(1+\frac{2}{n}\right) J^\alpha_{ab} J^\alpha_{cd} + 3\delta_{ac}\delta_{bd} - 3\delta_{ad}\delta_{bc}\right). \tag{16c}$$

By means of (13), we can show that the tensors defined by Eqs. (16) are indeed projectors. We can also show [13] that $M_{abcd}\equiv 0$ for $n=1$. From (13) and the definitions (16) there follow the relations

$$Q_{abpq} N_{cdpq} = \frac{2(1+2n)}{3} N_{abcd}, \tag{17a}$$

$$Q_{abpq} \overline{N}_{cdpq} = -2\overline{N}_{abcd}, \tag{17b}$$

$$Q_{abpq} M_{cdpq} = \frac{2}{3} M_{abcd}. \tag{17c}$$

Therefore, the tensors N_{abcd}, $\tilde{N}_{abcd}$, and M_{abcd} are (anti)self-dual in the sense of the definition (5) with respect to each pair of indices. At the same time, $\lambda_1=2(1+2n)/3$, $\lambda_2=-2$, $\lambda_3=2/3$.

3.4. The Space $\mathbf{R}^7$. Let Ca be the alternative nonassociative algebra of octonions with multiplication law given by the octonion structure constants f_{abc} [31]:

$$e_a e_b = -\delta_{ab} + f_{abc} e_c.$$

Here, e_a and $e_8\equiv 1$ is a basis in the algebra Ca, $a, b, \ldots=1, \ldots, 7$. The octonion structure constants f_{abc} are completely antisymmetric with respect to (abc) and are equal to unity for seven combinations: (123), (246), (435), (367), (651), (572), (714).

We introduce in the space $\mathbf{R}^7$ with metric δ_{ab} the following completely antisymmetric 4-tensor:

$$h_{abcd} = -\frac{1}{3!}\epsilon_{abcdmnk} f_{mnk},$$

where $\epsilon_{abcdmnk}$ is the completely antisymmetric tensor in $d=7$. It is known [11,31] that the tensors f_{abc} and h_{abcd} are invariant with respect to the subgroup G_2 of the group $SO(7)$ of rotations of the 7-dimensional space $\mathbb{R}^7$.

The tensors f_{abc} and h_{abcd} satisfy the identities [11,31]

$$h_{abcd}h_{ijkd} = \delta_{a[i}\delta_{j]b}\delta_{ck} + \delta_{b[i}\delta_{j]c}\delta_{ak} + \delta_{c[i}\delta_{j]a}\delta_{bk} + \frac{1}{2}h_{ij[ab}\delta_{c]k} + \frac{1}{2}h_{jk[ab}\delta_{c]i} + \frac{1}{2}h_{ki[ab}\delta_{c]j} - f_{abc}f_{ijk}, \tag{18a}$$

$$f_{abk}h_{cdek} = \delta_{e[b}f_{a]cd} + \delta_{c[b}f_{a]de} + \delta_{d[b}f_{a]ec}. \tag{18b}$$

The Lie algebra $so(7)$ can be decomposed as follows:

$$so(7) = g_2 \oplus \mathcal{B},$$

where $\mathcal{B}$ is the orthogonal complement of g_2 in $so(7)$. Using h_{abcd} and f_{abc}, we can introduce the projector $\bar{g}_{abcd}$ onto the subalgebra g_2 in $so(7)$ and the projector g_{abcd} onto the subspace $\mathcal{B}$ in the algebra $so(7)$. These projectors have the form

$$g_{abcd} \equiv \frac{1}{6}\delta_{a[c}\delta_{d]b} + \frac{1}{6}h_{abcd} = \frac{1}{6}f_{abk}f_{cdk}, \tag{19a}$$

$$\overline{g}_{abcd} \equiv \frac{1}{3}\delta_{a[c}\delta_{d]b} - \frac{1}{6}h_{abcd} = \frac{1}{2}\delta_{a[c}\delta_{d]b} - \frac{1}{6}f_{abk}f_{cdk}. \tag{19b}$$

It is readily seen that

$$g_{abpq}g_{cdpq} = g_{abcd}, \qquad \overline{g}_{abpq}\overline{g}_{cdpq} = \overline{g}_{abcd}, \qquad g_{abpq}\overline{g}_{cdpq} = 0, \qquad g_{abcd} + \overline{g}_{abcd} = \frac{1}{2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}). \tag{20}$$

From (18)—(20), we obtain

$$h_{abpq}g_{pqcd} = 4g_{abcd}, \tag{21a}$$

$$h_{abpq}\overline{g}_{pqcd} = -2\overline{g}_{abcd}. \tag{21b}$$

Therefore, the tensor g_{abcd} is self-dual, and the tensor $\bar{g}_{abcd}$ is anti-self-dual in the sense of the definition (5).

3.5. Octonion Structure. As a vector space, the algebra of octonions Ca is identical to $\mathbb{R}^8$. In the space $\mathbb{R}^8$ with metric δ_{AB} ($A, B, \ldots = 1, \ldots, 8$) we introduce the following completely antisymmetric 4-index tensor [31]:

$$H_{abcd} = h_{abcd}, \qquad H_{abc8} = f_{abc},$$

where $a, b, \ldots = 1, \ldots, 7$, and h_{abcd} and f_{abc} were introduced in Sec. 3.4. The tensor H_{ABCD} is invariant with respect to the subgroup $Spin(7)$ of the group $SO(8)$ (see [8,10,11,31]) and satisfies the identities

$$H_{ABCD}H_{IJKD} = \delta_{A[I}\delta_{J]B}\delta_{CK} + \delta_{B[I}\delta_{J]C}\delta_{AK} + \delta_{C[I}\delta_{J]A}\delta_{BK}$$
$$+ \frac{1}{2}H_{IJ[AB}\delta_{C]K} + \frac{1}{2}H_{JK[AB}\delta_{C]I} + \frac{1}{2}H_{KI[AB}\delta_{C]J}. \tag{22}$$

The algebra $so(8)$ can be decomposed as follows:

$$so(8) = spin(7) \oplus \mathcal{R},$$

where $\mathcal{R}$ is the orthogonal complement of $spin(7)$ in $so(8)$. We define the projector $\bar{G}_{ABCD}$ onto the subalgebra $spin(7)$ in $so(8)$ and the projector G_{ABCD} onto the subspace $\mathcal{R}$ in $so(8)$ (cf. [31]):

$$G_{ABCD} \equiv \frac{1}{8}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}) + \frac{1}{8}H_{ABCD}, \tag{23a}$$

$$\overline{G}_{ABCD} \equiv \frac{3}{8}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}) - \frac{1}{8}H_{ABCD}. \tag{23b}$$

By means of (22) and (23) we obtain

$$H_{ABIJ}G_{IJCD} = 6G_{ABCD}, \tag{24a}$$

$$H_{ABIJ}\bar{G}_{IJCD} = -2\bar{G}_{ABCD}. \tag{24b}$$

It is easy to show that

$$G_{ABIJ}G_{CDIJ} = G_{ABCD}, \qquad \overline{G}_{ABIJ}\overline{G}_{CDIJ} = \overline{G}_{ABCD},$$

$$G_{ABIJ}\overline{G}_{CDIJ} = 0, \qquad G_{ABCD} + \overline{G}_{ABCD} = \frac{1}{2}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}).$$

4. NAHM AND ROUHANI—WARD EQUATIONS

4.1. Rouhani—Ward Equations. In Sec. 3.1, we introduced the vector space $\mathbf{R}^n = \mathcal{H} \oplus \mathbf{R}$, where $\mathcal{H}$ is a simple compact Lie algebra, and $n = 1 + \dim \mathcal{H}$. On this space, we consider the (anti)self-duality equations

$$R_{\mu\nu\lambda\sigma}F_{\lambda\sigma} = 2F_{\mu\nu}, \tag{25}$$

where the completely antisymmetric tensor $\mathbf{R}_{\mu\nu\lambda\sigma}$ is given by formula (6). Using the explicit form of the tensor $\mathbf{R}_{\mu\nu\lambda\sigma}$, we can rewrite Eqs. (25) in the form

$$f_{abc}F_{cn} = F_{ab}, \tag{26}$$

$$\frac{1}{2}f_{dab}F_{ab} = F_{dn}, \tag{27}$$

where $a, b, \ldots = 1, \ldots, n-1$. We multiply (26) by f_{abd} and use the identity $f_{abc}f_{abd} = 2\delta_{cd}$. We obtain Eqs. (27). Therefore, the (anti)self-dual equations (25) are equivalent to Eqs. (26).

We consider gauge fields A_μ that depend only on $t \equiv x_n$ (cosmological solutions). We make A_n vanish by means of a gauge transformation (gauge fixing). We obtain

$$F_{nc} = \frac{dA_c}{dt} \equiv \dot{A}_c, \qquad F_{ab} = [A_a, A_b]. \tag{28}$$

Substituting (28) in Eqs. (26), we obtain

$$f_{abc}\dot{A}_c = -[A_a, A_b]. \tag{29}$$

Equations (29) were introduced by Rouhani [32] for the case when f_{abc} are the structure constants of the Lie algebra $sl(N, \mathbb{R})$. In the general case, these equations were introduced and studied by Ward [33]. We shall call (29) the Rouhani—Ward (RW) equations.

In (29), f_{abc} are the structure constants of the Lie algebra $\mathcal{H}$, and A_a take values in the Lie algebra $\mathcal{G}$. If $\mathcal{H} = su(2)$, then the RW equations (29) are identical to the well-known Nahm equations [34]. We write them in the form

$$\epsilon_{\alpha\beta\gamma}\dot{T}_\gamma = -[T_\alpha, T_\beta], \tag{30}$$

where $T_\alpha = T_\alpha(\varphi)$, $\dot{T}_\alpha \equiv dT_\alpha/d\varphi$, and $\epsilon_{\alpha\beta\gamma}$ are the structure constants of the group $SU(2)$, $\alpha, \beta, \ldots = 1, 2, 3$. Equations (30) arose in the construction of solutions of YM equations in $\mathbb{R}^4$ [35,16—19]. Equations (30) with φ replaced by a complex parameter ξ are also used in the algorithm for constructing N-monopole solutions of YM equations in $\mathbb{R}^4$ [34,36,37].

The simplest solution of the RW equations (29) has the form

$$A_a = \frac{1}{t}J_a, \tag{31}$$

where J_a are the generators of the algebra $\mathcal{H}$ in an arbitrary representation. This solution gives the simplest cosmological solution of Eqs. (25) for gauge fields of the Lie group H. Solutions of the Nahm equations (30) give a large class of solutions of the RW equations.

4.2. Nahm's Equations. Equations (30) have a representation of Lax type. Indeed, we introduce the matrices

$$L(\zeta) = (1+\zeta^2)T_1 + i(1-\zeta^2)T_2 + 2i\zeta T_3, \quad M(\zeta) = \zeta(T_1 - iT_2) + iT_3.$$

Then (30) can be rewritten in the form of a Lax equation with spectral parameter ζ [36,37,33]:

$$\dot{L}(\zeta) = [L(\zeta), M(\zeta)], \tag{32}$$

where $\dot{L}(\zeta) \equiv dL(\zeta)/d\varphi$. To Eq. (32) we can apply the inverse scattering method (see, for example, [38]), and in terms of theta functions we can write down the general solution of Eq. (32) for any semisimple Lie algebra $\mathcal{G}$.

A special case of this class of solutions can be obtained if we use an observation made in a number of studies [39,33] concerning the possibility of reducing Nahm's equations to the equations of a Toda chain. In the case of a nonperiodic chain, this reduction makes it possible to obtain a solution of Eqs. (30) in elementary functions. As an example, we write down the explicit form of the ansatz for the functions T_α with values in the Lie algebra $\mathcal{G}=su(N)$.

We set

$$T_1 = i\begin{pmatrix} 0 & a_1 & 0 & \dots & \dots & 0 & a_N \\ a_1 & 0 & a_2 & 0 & \dots & \dots & 0 \\ 0 & a_2 & 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 & a_{N-2} & 0 \\ 0 & \dots & \dots & 0 & a_{N-2} & 0 & a_{N-1} \\ a_N & 0 & \dots & \dots & 0 & a_{N-1} & 0 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} 0 & a_1 & 0 & \dots & \dots & 0 & -a_N \\ -a_1 & 0 & a_2 & 0 & \dots & \dots & 0 \\ 0 & -a_2 & 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 & a_{N-2} & 0 \\ 0 & \dots & \dots & 0 & -a_{N-2} & 0 & a_{N-1} \\ a_N & 0 & \dots & \dots & 0 & -a_{N-1} & 0 \end{pmatrix}, \tag{33}$$

$$T_3 = i\begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b_N \end{pmatrix}, \qquad \sum_{J=1}^{N} b_J = 0,$$

where $a_J=a_J(\varphi)$, $b_J=b_J(\varphi)$ are real functions that depend on φ. It is easy to show that for the ansatz (33) the equations $\dot{T}_1=-[T_2, T_3]$ and $\dot{T}_2=-[T_3, T_1]$ are identical. We introduce the matrices

$$L = -T_2 + iT_3, \qquad M = -iT_1,$$

which take values in the algebra $sl(N, \mathbf{R})$. Then the Nahm equations (30) can be rewritten in the form of the Lax equation

$$\dot{L} = [L, M], \tag{34}$$

which is now without a spectral parameter. Rewritten in terms of a_J and b_J, Eqs. (34) are identical to the equations of the ordinary periodic Toda chain (see, for example, [39,40]).

If in (33) we set $a_N=0$, then (34) will be the equations of the (finite) nonperiodic Toda chain associated with the Lie algebra $sl(n, \mathbf{R})$. The equations of the periodic and nonperiodic Toda chains associated with an arbitrary semisimple Lie algebra (see [39,40]) can also be embedded in the Nahm equations.

4.3. Solutions of Nahm's Equations for $G=SU(2)$ **and** $G=SU(3)$. For the algebra $\mathcal{G}=su(2)$, the ansatz (33) reduces Nahm's equations to the equations of the periodic Toda chain with solution that can be expressed in terms of the Jacobi elliptic functions cs, ds, ns [35,16]:

$$a_1 + a_2 = A\,\mathrm{cs}(A\varphi + B \mid m), \qquad a_1 - a_2 = A\,\mathrm{ds}(A\varphi + B \mid m), \qquad b_1 = -b_2 = A\,\mathrm{ns}(A\varphi + B \mid m), \tag{35}$$

where A and B are arbitrary real numbers, $0 \le m \le 1$. For $m=1$, this solution has the form

$$a_1 = \frac{A}{\mathrm{sh}(A\varphi + B)}, \qquad a_2 = 0, \qquad b_1 = -b_2 = A\,\mathrm{cth}(A\varphi + B). \tag{36}$$

If in (36) we set $B/A=C=$const and allow A to tend to zero, then we obtain the solution

$$a_1 = b_1 = -b_2 = \frac{1}{\varphi + C}, \qquad a_2 = 0. \tag{37}$$

We consider the ansatz (33) for $\mathcal{G}=su(3)$ with $a_3=0$ (nonperiodic Toda chain). The solution of Eqs. (34) has the form

[40,17,19]

$$a_1 = A_1 \exp(A_1\varphi + B_1)\frac{\Phi^{1/2}}{\Psi}, \qquad a_2 = A_2 \exp(A_2\varphi + B_2)\frac{\Psi^{1/2}}{\Phi},$$

$$b_1 = \frac{4A_1 + 2A_2}{3} + \frac{A_1 \exp(2A_1\varphi + 2B_1)}{\Psi}\left\{1 - \frac{A_1}{A_1 + A_2}\exp(2A_2\varphi + 2B_2)\right\},$$

$$b_3 = -\frac{2A_1 + 4A_2}{3} - \frac{A_2 \exp(2A_2\varphi + 2B_2)}{\Phi}\left\{1 - \frac{A_2}{A_1 + A_2}\exp(2A_1\varphi + 2B_1)\right\}, \tag{38}$$

$$\Psi = 1 - \exp(2A_1\varphi + 2B_1) + \frac{A_1^2}{(A_1 + A_2)^2}\exp(2A_1\varphi + 2B_1)\exp(2A_2\varphi + 2B_2),$$

$$\Phi = 1 - \exp(2A_2\varphi + 2B_2) + \frac{A_2^2}{(A_1 + A_2)^2}\exp(2A_1\varphi + 2B_1)\exp(2A_2\varphi + 2B_2), \qquad b_2 = -b_1 - b_3,$$

where A_1, A_2, B_1, B_2 are arbitrary constants.

5. ANTI-SELF-DUAL SOLUTIONS OF THE YM EQUATIONS IN $\mathbb{R}^{4n}$

5.1. Ansatz. We consider the space $\mathbb{R}^{4n}$ with metric δ_{ab} and antiself-dual equations of the form (17b) for the gauge fields A_a of the semisimple Lie group G:

$$Q_{abcd}F_{cd} = -2F_{ab}, \tag{39}$$

where Q_{abcd} is given by formula (15), $a, b, \ldots = 1, \ldots, 4n$. For $n=1$, $Q_{abcd}=\epsilon_{abcd}$, and (39) are identical to the standard antiself-duality equations (4).

Equations (39) in $\mathbb{R}^{4n}$ were considered in [9,13,15]. It was shown in [9,13] that the well-known Corrigan—Fairlie—'t Hooft—Wilczek (CFTW) ansatz (see [1—3]) for the gauge fields of the group $SU(2)$ can be generalized to dimension $d=4n$. We generalize this ansatz to gauge fields of an *arbitrary* semisimple Lie group G and describe the corresponding new classes of solutions of YM equations in the space $\mathbb{R}^{4n}$, $n=1, 2, \ldots$ [15].

For the gauge fields A_a, we choose the ansatz

$$A_a = -J^{\alpha}_{ac}T_{\alpha}(\varphi)\partial_c\varphi, \tag{40}$$

where the constant antisymmetric tensors J^{α}_{ab} satisfy (13), φ is an arbitrary function of the coordinates $x^a \in \mathbb{R}^{4n}$, and $\alpha, \beta, \gamma = 1, 2, 3$. The functions T_α depend on φ and take values in the Lie algebra $\mathcal{G}$ of the gauge group G, i.e., $T_\alpha = T^A_\alpha(\varphi)I_A$, where I_A are the generators of the algebra $\mathcal{G}$.

We substitute (40) in the definition of F_{ab}, obtaining

$$F_{ab} = J^{\alpha}_{ac}\left\{T_\alpha\partial_b\partial_c\varphi + \dot{T}_\alpha\partial_b\varphi\partial_c\varphi\right\} - J^{\alpha}_{bc}\left\{T_\alpha\partial_a\partial_c\varphi + \dot{T}_\alpha\partial_a\varphi\partial_c\varphi\right\} + J^{\beta}_{ac}J^{\gamma}_{be}[T_\beta, T_\gamma]\partial_c\varphi\partial_e\varphi, \tag{41}$$

where $\dot{T}_\alpha \equiv dT_\alpha/d\varphi$. Using (13) and (15), we calculate $Q_{abcd}F_{cd}$:

$$Q_{abcd}F_{cd} = -\frac{2}{3}F_{ab} + \frac{4}{3}J^{\beta}_{ac}J^{\gamma}_{be}\epsilon_{\beta\gamma\alpha}\left\{T_\alpha\partial_c\partial_e\varphi + \dot{T}_\alpha\partial_c\varphi\partial_e\varphi\right\}$$

$$-\frac{2}{3}J^{\alpha}_{bc}\epsilon_{\alpha\beta\gamma}[T_\beta, T_\gamma]\partial_c\varphi\partial_a\varphi + \frac{2}{3}J^{\alpha}_{ac}\epsilon_{\alpha\beta\gamma}[T_\beta, T_\gamma]\partial_c\varphi\partial_b\varphi + \frac{2}{3}J^{\alpha}_{ab}\left\{T_\alpha\Box\varphi + (\dot{T}_\alpha + \frac{1}{2}\epsilon_{\alpha\beta\gamma}[T_\beta, T_\gamma])\partial_c\varphi\partial_c\varphi\right\}, \tag{42}$$

where $\Box \equiv \partial_c\partial_c$. It is now easy to show that

$$Q_{abcd}F_{cd} + 2F_{ab} = \frac{2}{3}\left(\dot{T}_\alpha + \frac{1}{2}\epsilon_{\alpha\beta\gamma}[T_\beta, T_\gamma]\right)\{J^{\alpha}_{ab}\partial_c\varphi\partial_c\varphi + 2J^{\alpha}_{ac}\partial_c\varphi\partial_b\varphi - 2J^{\alpha}_{bc}\partial_c\varphi\partial_a\varphi\}$$

$$+\frac{4}{3}\left(\epsilon_{\alpha\beta\gamma}\dot{T}_\gamma + [T_\alpha, T_\beta]\right)J^{\alpha}_{ac}J^{\beta}_{be}\partial_c\varphi\partial_e\varphi + \frac{2}{3}T_\alpha\left(2J^{\alpha}_{ac}\partial_c\partial_b\varphi - 2J^{\alpha}_{bc}\partial_c\partial_a\varphi + 2\epsilon^{\alpha}_{\beta\gamma}J^{\beta}_{ac}J^{\gamma}_{be}\partial_c\partial_e\varphi + J^{\alpha}_{ab}\Box\varphi\right). \tag{43}$$

We assume that

$$\epsilon_{\alpha\beta\gamma}\dot{T}_\gamma = -[T_\alpha, T_\beta], \tag{44}$$

$$2J^\alpha_{ac}\partial_c\partial_b\varphi - 2J^\alpha_{bc}\partial_c\partial_a\varphi + 2\epsilon^\alpha_{\beta\gamma}J^\beta_{ac}J^\gamma_{be}\partial_c\partial_e\varphi + J^\alpha_{ab}\Box\varphi = 0. \tag{45}$$

Equations (44) are equivalent to the equations

$$\dot{T}_\alpha = -\frac{1}{2}\epsilon_{\alpha\beta\gamma}[T_\beta, T_\gamma].$$

Therefore, if Eqs. (44) and (45) are satisfied, the right-hand side of (43) vanishes, and, therefore, the ansatz (40) will give an anti-self-dual gauge field A_a.

Proposition 1 [15]. *With every solution $T_\alpha(\varphi)$ of the Nahm equations* (44), *where the function φ satisfies Eqs.* (45), *one can associate a solution* (40) *of the YM equations for the gauge fields A_a of an arbitrary semisimple Lie group G in spaces $\mathbb{R}^{4n}$ with n=1, 2, ...*

The proof follows from Eqs. (41)—(43).

In accordance with Proposition 1, with every solution of the equations of the Toda chain and Eqs. (45) for φ it is possible to associate a solution of the form (40) of the YM equations in the space $\mathbb{R}^{4n}$. In particular, such solutions result from the substitution of (35)—(37) in (40) for the gauge fields of the group $SU(2)$ and the substitution of (38) in (40) for the gauge fields of the group $SU(3)$. We emphasize that the Nahm equations (44) can be reduced not only to the equations of the Toda chain. They can also be reduced to the equations of the Kowalevski top [41] and to other integrable equations.

5.2. Equations for Scalar Field. To find solutions of Eqs. (45), we replace the indices a, b, ... by double indices (μi), (νj), ..., where μ, ν, ... =1, ..., 4, i, j, ... =1, ..., n. In adopting such notation, we follow [42]. The tensors $J^\alpha_{(\mu i)(\nu j)}$ $(=J^\alpha_{ab})$ can be chosen in the form

$$J^\alpha_{(\mu i)(\nu j)} = \delta_{ij}\eta^\alpha_{\mu\nu}, \tag{46}$$

where $\eta^\alpha_{\mu\nu}$ are the well-known 't Hooft tensors. By definition (see, for example, [1—3])

$$\eta^\alpha_{\beta\gamma} = \epsilon^\alpha_{\beta\gamma}, \qquad \eta^\alpha_{\mu 4} = -\eta^\alpha_{4\mu} = \delta^\alpha_\mu,$$

where α, β, γ=1, 2, 3. It is readily seen that the tensors $J^\alpha_{(\mu i)(\nu j)}$ satisfy (13) by virtue of the following identities for $\eta^\alpha_{\mu\nu}$ [3]:

$$\eta^\alpha_{\mu\lambda}\eta^\beta_{\lambda\nu} = -\delta^{\alpha\beta}\delta_{\mu\nu} - \epsilon^{\alpha\beta\gamma}\eta^\gamma_{\mu\nu}, \qquad \epsilon^\alpha_{\beta\gamma}\eta^\beta_{\mu\lambda}\eta^\gamma_{\nu\sigma} = \delta_{\mu\nu}\eta^\alpha_{\lambda\sigma} - \delta_{\mu\sigma}\eta^\alpha_{\lambda\nu} - \delta_{\lambda\nu}\eta^\alpha_{\mu\sigma} + \delta_{\lambda\sigma}\eta^\alpha_{\mu\nu}. \tag{47}$$

We substitute (46) in Eqs. (45) and use the identities (47). We obtain

$$2\eta^\alpha_{\mu\lambda}(\partial_{\lambda i}\partial_{\nu j}\varphi - \partial_{\lambda j}\partial_{\nu i}\varphi) - 2\eta^\alpha_{\nu\lambda}(\partial_{\lambda j}\partial_{\mu i}\varphi - \partial_{\lambda i}\partial_{\mu j}\varphi)$$

$$+\,\delta_{\mu\nu}\eta^\alpha_{\lambda\sigma}(\partial_{\lambda i}\partial_{\sigma j}\varphi - \partial_{\lambda j}\partial_{\sigma i}\varphi) + \eta^\alpha_{\mu\nu}(2\partial_{\lambda i}\partial_{\lambda j}\varphi + \delta_{ij}\Box\varphi) = 0, \tag{48}$$

where $\partial_{\lambda i} \equiv \partial/\partial x^{\lambda i}$. We recall that, unless stated otherwise, summation over repeated indices is understood always.

It is readily seen that Eqs. (48) are equivalent to

$$\partial_{\mu i}\partial_{\nu j}\varphi = \partial_{\mu j}\partial_{\nu i}\varphi, \tag{49a}$$

$$\partial_{\lambda i}\partial_{\lambda j}\varphi = 0, \tag{49b}$$

where μ and ν can take all values from 1 to 4, and i and j can take all values from 1 to n. Note that from (49b) it follows that

$$\Box\varphi = \textstyle\sum_{j=1}^n \partial_{\lambda j}\partial_{\lambda j}\varphi = 0.$$

5.3. Solutions of Equations for Scalar Field.

Example 1. The simplest solution of Eqs. (49) [and (45)] is the function

$$\varphi = p_a x_a = p_{\mu i} x_{\mu i}, \tag{50}$$

where p_a is a constant vector (momentum) in $\mathbb{R}^{4n}$. Therefore, in accordance with Proposition 1 we can associate with each solution of the Nahm equations (44) a solution of plane-wave type [with φ of the form (50)] of the YM equations in $\mathbb{R}^{4n}$.

Example 2. We set

$$X_\mu \equiv x_{\mu i} p_i, \tag{51}$$

where $p_i \equiv$const. Suppose the function φ depends only on X_μ, i.e., $\varphi = \varphi(X_1, X_2, X_3, X_4)$. It is easy to show that in this case Eqs. (49a) are satisfied identically, while Eqs. (49b) reduce to the Laplace equation with respect to the "collective" coordinates X_μ:

$$\frac{\partial^2 \varphi}{\partial X^\mu \partial X^\mu} = 0. \tag{52}$$

Therefore, taking any solution of Eqs. (52), we obtain a solution of Eqs. (49) [and (45)]. For example, we can choose

$$\varphi = 1 + \sum_{I=1}^{K} \frac{B_I^2}{(X_\mu - C_\mu^I)(X_\mu - C_\mu^I)}, \tag{53}$$

where K is any natural number, and B_I and C^I_μ are arbitrary constants.

Example 3. Suppose the functions φ_i depend only on the coordinates $x_{\mu i}$ with number i, i.e., $\varphi_i(x_{1i}, x_{2i}, x_{3i}, x_{4i})$. We set

$$\varphi = \sum_{i=1}^{n} \varphi_i \tag{54}$$

and substitute (54) in Eqs. (49). It is readily seen that these equations reduce to Laplace equations with respect to the coordinates $x_{\mu i}$ for the functions φ_i:

$$(\partial_{1i}\partial_{1i} + \partial_{2i}\partial_{2i} + \partial_{3i}\partial_{3i} + \partial_{4i}\partial_{4i})\varphi_i = 0,$$

where there is no summation over the index i. Therefore, taking n functions φ_i that each satisfies the Laplace equation with respect to "its" coordinates $x_{\mu i}$, we obtain the solution (54) of Eqs. (49) [and (45)].

We emphasize that the gauge fields corresponding to (54) *are not* a direct sum of n solutions in four-dimensional subspaces, since $T_\alpha(\varphi)$ depend nontrivially on φ and cannot be decomposed into a direct sum of matrices.

Example 4. We introduce K isotropic (complex) constant four-dimensional vectors p^I_μ: $p^I_\mu p^I_\mu = 0$, where $I = 1, \ldots, K$, K is any natural number, and there is no summation over I. It is easy to show that

$$\varphi = \sum_{I=1}^{K} \varphi_I(x_{\mu i} p^I_\mu), \tag{55}$$

where φ_I are arbitrary functions of $X^I_i \equiv x_{\mu i} p^I_\mu$, is a complex solution of Eqs. (49). Therefore

$$\psi = \sum_{I=1}^{K} (\varphi_I(X_i^I) + \overline{\varphi}_I(X_i^I)), \tag{56}$$

where the bar denotes complex conjugation, will be a real solution of Eqs. (49) [and (45)].

5.4. Discussion. Equations (49) arise in the construction of metrics on hyper-Kählerian manifolds of dimension $4n$ [43,42]. At the same time, there exists a correspondence between the space of solutions of Eqs. (49) and the cohomology group $H^1(Z, \mathcal{O}(-2))$ of a certain auxiliary space Z with coefficients in the sheaf $\mathcal{O}(-2)$, whose sections are homogeneous functions of homogeneity degree -2 [9,42,43]. Therefore, the general solution of Eqs. (49) can be expressed as a contour integral with respect to the auxiliary variable $\zeta \in \mathbb{C}P^1$ of an arbitrary holomorphic function of homogeneity degree -2. In particular, for the space $\mathbb{R}^8$ the explicit form of the general solution φ in terms of a contour integral was given by Ward [9]. In [43], a similar solution was given for a function φ that does not depend on x_4^i, $i = 1, \ldots, n$. However, in applications it is more convenient to use the explicit form of solutions of the type (50), (53)–(56) or a superposition of them. Note that $Z \cong \mathbb{C}P^1 \times \mathbb{R}^{4n}$ is the twistor space for $\mathbb{R}^{4n}$ or a certain subspace in $\mathbb{C}P^1 \times \mathbb{R}^{4n}$.

As we have already noted, the general solution of the Nahm equations (44) can also be expressed in terms of theta functions. Therefore, in principle it is possible to give the general solution of Eqs. (44), (45) obtained from the ansatz (40). The behavior at infinity and the topological properties of these solutions must be the subject of a separate investigation.

Note that Eqs. (39), the ansatz (40), and Eqs. (44), (45) admit formal passage to the limit $n \to \infty$. As a result, we obtain gauge fields and anti-self-duality equations on the infinite-dimensional (countable) space $\mathbb{R}^{4\infty}$. At the same time, (50) and (53)–(56) also give solutions of Eq. (45) in the limit $n \to \infty$. Anti-self-dual gauge fields on infinite-dimensional manifolds are used in the approach of geometric quantization of string theory [44].

One can also carry out a different limiting process — with respect to the dimension of the gauge group. Then for the ansatz (33) Eqs. (34) become the equations of an infinite Toda chain, whose solutions are known. For infinite-dimensional Lie algebras $\mathcal{G}$ [of the type $su(\infty)$] one can also obtain [45] a general solution of the Nahm equations (44).

6. (ANTI)SELF-DUAL EQUATIONS IN $\mathbb{R}^4$

6.1. Ansatz and Solutions. We now consider the case of gauge fields in the space $\mathbb{R}^4$ with metric $\delta_{\mu\nu}$ in a little more detail. In accordance with (46), the ansatz (40) in $d=4$ takes the form [18,19]

$$A_\mu = -\eta^\alpha_{\mu\lambda} T_\alpha(\varphi)\partial_\lambda \varphi. \tag{57}$$

It is easy to show that in $\mathbb{R}^4$ (39) are identical to the standard antiself-duality equations (4). Equations (39) were reduced in Sec. 5 to the system of equations (44)—(45). In their turn, Eqs. (45) are equivalent to Eqs. (49), and for $n=1$ Eqs. (49) reduce to the Laplace equation for the function φ:

$$\Box\varphi = 0, \tag{58}$$

where $\Box \equiv \partial_\mu\partial_\mu$. The matrices $T_\alpha(\varphi)$ must still satisfy the Nahm equations (44).

If we choose the simplest solution of the Nahm equations, $T_\alpha=(1/\varphi)I_\alpha$, where I_α are the generators of the group $SU(2)$, the ansatz (57) becomes the CFTW ansatz. If $\varphi=x_\mu x_\mu$, then for $T_\alpha(\varphi)$ of the algebra $su(2)$ (57) becomes the Minkowski ansatz [35], and for $T_\alpha(\varphi)$ belonging to an arbitrary semisimple Lie algebra $\mathcal{G}$ (57) becomes the ansatz of [16,17].

Proposition 2 [16—19]. *Let $T_\alpha(\varphi)$ satisfy the Nahm equations.*

A. *If $\varphi=x_\mu x_\mu$, then the gauge field* (57) *will be self-dual.*

B. *If the function φ satisfies the Laplace equation* (58), *then the gauge field* (57) *will be antiself-dual.*

Proof of A. From (41), (46), and (47) we obtain

$$F_{\mu\nu} = \eta^\alpha_{\mu\lambda}\left\{T_\alpha\partial_\lambda\partial_\nu\varphi + \left(\dot{T}_\alpha - \frac{1}{2}\epsilon_{\alpha\beta\gamma}[T_\beta,T_\gamma]\right)\partial_\lambda\varphi\partial_\nu\varphi\right\}$$

$$-\eta^\alpha_{\nu\lambda}\left\{T_\alpha\partial_\lambda\partial_\mu\varphi + \left(\dot{T}_\alpha - \frac{1}{2}\epsilon_{\alpha\beta\gamma}[T_\beta,T_\gamma]\right)\partial_\lambda\varphi\partial_\mu\varphi\right\} + \eta^\alpha_{\mu\nu}\epsilon_{\alpha\beta\gamma}T_\beta T_\gamma\partial_\lambda\varphi\partial_\lambda\varphi, \tag{59}$$

where $\dot{T}_\alpha \equiv dT_\alpha/d\varphi$. In (59) we substitute $\varphi=x_\sigma x_\sigma$. It is readily seen that if T_α satisfy the equations

$$\epsilon_{\alpha\beta\gamma}\dot{T}_\gamma = [T_\alpha, T_\beta],$$

which are obtained from the Nahm equations (44) by the substitution $T_\alpha \to -T_\alpha$, then the tensor $F_{\mu\nu}$ will be self-dual:

$$F_{\mu\nu} = 4\eta^\alpha_{\mu\nu}(T_\alpha + x_\lambda x_\lambda \epsilon_{\alpha\beta\gamma}T_\beta T_\gamma). \tag{60}$$

Proof of B. This follows from formulas (41)—(45) and (46)—(49).

All the solutions of the Laplace equation in $\mathbb{R}^4$ are known, and for Eq. (59) we can write down the general solution. Therefore, to find solutions of the form (57), it is sufficient to find solutions of Nahm's equations. We discuss these equations and their solutions in Sec. 4. In particular, for the gauge group $SU(2)$ one can take the solutions (35)—(37), and for the gauge group $SU(3)$ the solution (38).

6.2. Monopoles and Vortices. The reduction of the (anti)self-duality equations (4) to three dimensions (for $\partial_4 A_\mu=0$) leads to Bogomol'nyi's equations (see [1—3,34,36,37]). Therefore, every solution of Eqs. (44) and (58) with $\partial_4\varphi=0$ describes a monopole configuration. The ansatz (57) for such monopole solutions takes the form

$$A_\alpha = \epsilon_{\alpha\beta\gamma}T_\beta(\varphi)\partial_\gamma\varphi, \qquad A_4 = T_\alpha(\varphi)\partial_\alpha\varphi,$$

where α, β, $\gamma=1, 2, 3$, $\varphi=\varphi(x^\alpha)$.

It is also known [46] that the reduction of the self-duality equations to two dimensions (with $\partial_3 A_\mu=\partial_4 A_\mu=0$) leads to the equations of the two-dimensional $G^{\mathbb{C}}/G$ σ model, where $G^{\mathbb{C}}$ is the complexification of the Lie group G. Therefore, every solution of Eqs. (44) and (58) with $\partial_3\varphi=\partial_4\varphi=0$ gives a solution of the $d=2$ equations of the $G^{\mathbb{C}}/G$ σ model. The ansatz (57) in this case has the form

$$A_p = -T_3(\varphi)\epsilon_{pq}\partial_q\varphi, \qquad A_3 = \epsilon_{pq}T_p(\varphi)\partial_q\varphi, \qquad A_4 = T_q(\varphi)\partial_q\varphi,$$

where p, $q=1, 2$, $\varphi=\varphi(x^q)$.

Similarly, with every solution of the equation $\partial_q\partial_q\varphi=0$ and the equations obtained from Nahm's equations by the substitution $T_1\to iT_1$, $T_2\to iT_2$, $T_3\to T_3$ we can associate a solution of the equations of the principal chiral model in $\mathbb{R}^2$ [47].

6.3. Plane-Wave Solutions. We have considered (anti)self-dual gauge fields in Euclidean space $\mathbb{R}^{4,0}$. (Anti)self-dual solutions in the Minkowski space $\mathbb{R}^{3,1}$ (see, for example, [48,49]) are also interesting. It is known that self-dual gauge fields of a Lie group G in the space $\mathbb{R}^{3,1}$ are complex. However, as was already shown by Wu and Yang [50] (see also [48,49]), complex gauge fields of the Lie group G can be regarded as real gauge fields of the complexified Lie group $G^{\mathbb{C}}$.

The ansatz (57) is readily continued to Minkowski space $\mathbb{R}^{3,1}$. For this, it is sufficient to set $x^4=-ix^0$. Then Nahm's equations (44) remain unchanged, while in Eq. (58) the Laplacian in Euclidean space is replaced by the Laplacian in Minkowski space:

$$\left(\partial_1^2+\partial_2^2+\partial_3^2-\partial_0^2\right)\varphi=0. \tag{61}$$

As solution of Eq. (61) we take a function of the form

$$\varphi=A_0+\sum_{I=1}^{M}A_I\exp(p_1^Ix^1+p_2^Ix^2+p_3^Ix^3+p_0^Ix^0), \tag{62}$$

where M is any natural number, A_0 and A_I are non-negative constants, and $p^I=(p_1^I,\ p_2^I,\ p_3^I,\ p_0^I)$ are constant isotropic vectors, i.e.,

$$(p_1^I)^2+(p_2^I)^2+(p_3^I)^2-(p_0^I)^2=0, \qquad I=1,\dots,M.$$

As solutions of Nahm's equations (44), we choose solutions of the equations of the finite nonperiodic Toda chain, the reduction to which is described by ansatzes of the type (33). In particular, for the gauge group $SU(N)$ we obtain a reduction to the equations of the ordinary Toda chain associated with the algebra $sl(N,\ \mathbb{R})$. Solutions of these equations are known explicitly for all N [40]. Solutions for $N=2$ and $N=3$ are given in (36) and (38). The solutions that are obtained have the same form as the solutions of De Vega [49], who considered $G=SU(2)$ $[G^{\mathbb{C}}=SL(2,\ \mathbb{C})]$, and generalize them to an arbitrary gauge Lie group G.

Proposition 3. *With every solution of the Nahm equations* (44), *where* φ *has the form* (62), *it is possible to associate a multi-plane-wave solution* (57) *of the YM equations for the gauge fields of an arbitrary semisimple Lie group* $G^{\mathbb{C}}$ *in Minkowski space* $\mathbb{R}^{3,1}$.

6.4. Linearly Rising Potential. As solution of Eq. (61) we take a function of the form

$$\varphi=x_1^2+x_2^2+x_3^2+3x_0^2. \tag{63}$$

Considering the group $SU(N)$, as solutions of the Nahm equations (44) we take, for example, the solution of the nonperiodic Toda chain, the reduction to which is given by the ansatz (33). For φ of the form (63), the free parameters of the solution of Eqs. (34) can be chosen in such a way that $T_\alpha(\varphi)$ will be nonsingular [19]. At the same time, following [40], we can show that for $\varphi\to\infty$ $T_3(\varphi)$ tends to a constant matrix. Therefore, $F_{\mu\nu}$ does not decrease and in the limit $x_\mu\to\infty$ we obtain $F_{\mu\nu}\to F_{\mu\nu}(\infty)=\text{const}\neq 0$ [19].

7. ANTI-SELF-DUAL SOLUTIONS OF THE YM EQUATIONS IN $\mathbb{R}^{4n+2}$

7.1. Anti-self-duality Equations. We consider the space $\mathbb{R}^{2k}$ with metric δ_{MN} and complex structure J_{MN}. We introduce anti-self-duality equations of the form (10b) for the gauge fields A_M of the semisimple Lie group G:

$$T_{MNPQ}F_{PQ}=-2F_{MN}, \tag{64}$$

where T_{MNPQ} is given by formula (8), $M,\ N,\ \dots=1,\ \dots,\ 2k$. For $k=2$, $T_{MNPQ}=\epsilon_{MNPQ}$, and (64) are identical to the standard anti-self-duality equations (4).

If $k=2n$, then $d=2k=4n$. Examples of anti-self-dual fields in $\mathbb{R}^{4n}$ were given in Sec. 5. Therefore, we consider here the case $k=2n+1$, which corresponds to $d=2k=4n+2$, $n=1,\ 2,\ \dots$. Solutions of Eqs. (64) in $d=4n$ will be obtained as a special case of solutions in $d=4n+2$.

The space $\mathbb{R}^{4n+2}$ can be represented as the direct product $\mathbb{R}^{4n}\times\mathbb{R}^2$ of the space $\mathbb{R}^{4n}$ and the space $\mathbb{R}^2$. We shall assume that $a,\ b,\ c,\ \dots=1,\ \dots,\ 4n$; $p,\ q,\ r=1,\ 2$; $\alpha,\ \beta,\ \gamma,\ \delta=1,\ 2,\ 3$. As was shown in Sec. 3.3, in the space $\mathbb{R}^{4n}$ it is always possible to find three constant antisymmetric tensors J^α_{ab}. Each of these tensors can be chosen as a complex structure J_{ab} on $\mathbb{R}^{4n}$ that satisfies in accordance with the definition (7) the relations $J_{ac}J_{cb}=-\delta_{ab}$. We denote the complex structure on $\mathbb{R}^2$ by J_{pq}, $J_{pr}J_{rq}=-\delta_{pq}$. We choose the complex structure J_{MN} on $\mathbb{R}^{4n+2}=\mathbb{R}^{4n}\times\mathbb{R}^2$ in the form

$$J_{MN} = \{ J_{ab} = J^3_{ab},\ M = a,\ N = b;\ J_{aq} = 0,\ M = a,\ N = q;\ J_{pq},\ M = p,\ N = q \}. \tag{65}$$

It is readily seen that the tensor (65) satisfies (7).

Substituting (65) in the definition (8), we find that the only nonvanishing components of the tensor T_{MNPQ} are

$$T_{abcd} = J_{ab}J_{cd} + J_{ad}J_{bc} - J_{ac}J_{bd}, \quad T_{abpq} = -T_{apbq} = T_{pabq} = -T_{paqb} = J_{ab}J_{pq}. \tag{66}$$

In accordance with (66), Eqs. (64) can be rewritten in the form

$$T_{abcd}F_{cd} + 2F_{ab} + J_{ab}J_{pq}F_{pq} = 0, \quad 2F_{pq} + J_{pq}J_{ab}F_{ab} = 0, \quad J_{ab}F_{bp} + J_{pq}F_{aq} = 0. \tag{67}$$

7.2. Ansatz. We shall seek solutions of Eqs. (67) in the form

$$A_a = -J^\alpha_{ac}T_\alpha(\varphi)\partial_c\varphi, \qquad A_q = \varkappa T_q(\varphi). \tag{68}$$

Here, the φ-dependent functions $T_\alpha=\{T_q, T_3\}=\{T_1, T_2, T_3\}$ take values in the Lie algebra $\mathcal{G}$ of the Lie group G, φ is an arbitrary function of the coordinates $x^a\in\mathbb{R}^{4n}$, J^α_{ac} are the constant tensors (13), and $\varkappa=$const. The ansatz (68) generalizes the ansatz (40) for $d=4n$ and goes over into it for $\varkappa=0$.

Substituting (68) in the definition of F_{MN}, we obtain

$$F_{ab} = J^\alpha_{ac}\left\{T_\alpha\partial_b\partial_c\varphi + \dot{T}_\alpha\partial_b\varphi\partial_c\varphi\right\} - J^\alpha_{bc}\left\{T_\alpha\partial_a\partial_c\varphi + \dot{T}_\alpha\partial_a\varphi\partial_c\varphi\right\} + J^\beta_{ac}J^\gamma_{be}[T_\beta, T_\gamma]\partial_c\varphi\partial_e\varphi,$$

$$F_{aq} = \varkappa\left\{\dot{T}_q\partial_a\varphi - [T_\alpha, T_q]J^\alpha_{ac}\partial_c\varphi\right\}, \qquad F_{pq} = \varkappa^2[T_p, T_q], \tag{69}$$

where $\dot{T}_\alpha\equiv dT_\alpha/d\varphi$, $\partial_c\equiv\partial/\partial x^c$.

We substitute (69) in (67). After fairly lengthy calculations, we obtain

$$T_{abcd}F_{cd} + 2F_{ab} + J_{ab}J_{pq}F_{pq} = 2J^3_{ab}\left\{T_3\Box\varphi + (\dot{T}_3 + [T_1, T_2])\partial_c\varphi\partial_c\varphi + \varkappa^2[T_1, T_2]\right\} + 2\left(\dot{T}_q + \frac{1}{2}\epsilon_{q\gamma\delta}[T_\gamma, T_\delta]\right)$$

$$\times\left\{J^q_{ac}\partial_c\varphi\partial_b\varphi - J^q_{bc}\partial_c\varphi\partial_a\varphi + \epsilon^q_{\alpha\beta}J^\alpha_{ac}J^\beta_{be}\partial_c\varphi\partial_e\varphi\right\} + 2T_q\left\{J^q_{ac}\partial_c\partial_b\varphi - J^q_{bc}\partial_c\partial_a\varphi + \epsilon^q_{\alpha\beta}J^\alpha_{ac}J^\beta_{be}\partial_c\partial_e\varphi\right\},$$

$$2F_{pq} + J_{pq}J_{ab}F_{ab} = 2J_{pq}\left\{T_3\Box\varphi + (\dot{T}_3 + [T_1, T_2])\partial_c\varphi\partial_c\varphi + \varkappa^2[T_1, T_2]\right\}, \tag{70}$$

$$J_{ab}F_{bp} + J_{pq}F_{aq} = \varkappa\left(\dot{T}_q + \frac{1}{2}\epsilon_{q\alpha\beta}[T_\alpha, T_\beta]\right)\left\{\delta_{pq}J^3_{ac}\partial_c\varphi + J_{pq}\partial_a\varphi\right\},$$

where $\Box\equiv\partial_c\partial_c$ and we have used the fact that $[T_p, T_q]=J_{pq}[T_1, T_2]$, $\epsilon_{3pq}\equiv J_{pq}$. We recall that $J^3_{ab}=J_{ab}$; $p, q, r=1, 2$; $\alpha, \beta, \gamma, \delta=1, 2, 3$.

It is obvious from (70) that the anti-self-duality equations (67) are satisfied if the following equations hold:

$$\dot{T}_1 + [T_2, T_3] = \dot{T}_2 + [T_3, T_1] = \dot{T}_3 + \left(1 + \frac{\varkappa^2}{\partial_c\varphi\partial_c\varphi}\right)[T_1, T_2] = 0, \tag{71}$$

$$J^q_{ac}\partial_c\partial_b\varphi - J^q_{bc}\partial_c\partial_a\varphi + \epsilon^q_{\alpha\beta}J^\alpha_{ac}J^\beta_{be}\partial_c\partial_e\varphi = 0. \tag{72}$$

Note that we consider real functions $\varphi\not\equiv$const. For such functions, $\partial_c\varphi\partial_c\varphi\not\equiv0$, and, therefore, we can divide by $\partial_c\varphi\partial_c\varphi$. Equations (71) generalize the Nahm equations (44). Equations (72) are equivalent to Eqs. (49), which were considered in Sec. 5, and therefore from (72) the equation $\Box\varphi=0$ follows. The equivalence of (72) and (49) becomes obvious after the substitution of (46) and the use of the identities (47).

Proposition 4. *With every solution of Eqs.* (71) *and* (72) *it is possible to associate an anti-self-dual solution* (68) *of the Yang—Mills equations for the gauge fields A_M of the arbitrary semisimple Lie group G in the space* $\mathbb{R}^{4n+2}$ with $n=1, 2, \ldots$.

The proof follows from formulas (69)—(72).

Remark. All the fields in the ansatz (68) depend only on $x^a\in\mathbb{R}^{4n}$, and for such fields the YM equations in $\mathbb{R}^{4n+2}$ can be regarded as YMH equations in $\mathbb{R}^{4n}$ [4,7,51], the role of the Higgs fields being played by A_q ($q=1, 2$). The YMH Lagrangian in $\mathbb{R}^{4n}$ can be obtained by trivial dimensional reduction of the YM Lagrangian in $\mathbb{R}^{4n+2}$ [7,51]. The

Euler—Lagrange equations for this Lagrangian will be the YMH equations in $\mathbb{R}^{4n}$, and the solutions of Eqs. (67) give solutions of these YMH equations.

7.3. Examples of Solutions. Using the explicit form (46) for J^{α}_{ab}, we can readily show, repeating the calculations of Sec. 5.2, that Eqs. (72) are equivalent to Eqs. (49). In Secs. 5.3 and 5.4 we described in fair detail the solutions of Eqs. (49). It remains to give solutions of Eqs. (71).

Example 1. Note that for $\varkappa=0$ (71) are identical to Nahm's equations, the solutions of which we discussed in Secs. 4.2 and 4.3. For $\varkappa=0$, the solution given by the ansatz (68) is identical to the solution (40)—(41).

Example 2. We choose the simplest solution (50) of Eqs. (72). Then $\partial_c\varphi\partial_c\varphi=p^2=$const and (71) reduces to Nahm's equations (44) after the redefinition

$$T_1 \to \left(\frac{1+\varkappa^2}{p^2}\right)^{-1/2} T_1, \qquad T_2 \to \left(\frac{1+\varkappa^2}{p^2}\right)^{-1/2} T_2, \qquad T_3 \to T_3.$$

Example 3. We assume that the group G is complex. Let $T_1=iT_2$. Then $\dot{T}_3=0 \Rightarrow T_3=$const, and (71) reduce to the equation

$$\dot{T}_1 = [T_1, iT_3]. \tag{73}$$

Let $g\in G$ be a solution of the equation $\dot{g}=igT_3$, where $\dot{g}\equiv dg/d\varphi$. Then the solution of Eq. (73) has the form

$$T_1 = g^{-1}T_3 g, \qquad T_2 = -ig^{-1}T_3 g, \qquad T_3 = -ig^{-1}\dot{g} = \text{const}.$$

Example 4. Let $\mathcal{G}=sl(2N, \mathbb{R})$ or $sl(2n, \mathbb{C})$. We choose T_α in the block form

$$T_1 = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}, \qquad \operatorname{tr} T_3 = O, \tag{74}$$

where each block measures $N\times N$. Then $\dot{T}_3=0 \Rightarrow C$ and D are constant matrices. For the ansatz (74), Eqs. (71) reduce to

$$\dot{A} = BD - CB, \qquad \dot{B} = CA - AD. \tag{75}$$

As a simplification, we shall assume that C and D are diagonal matrices, i.e., $C=\operatorname{diag}(c_1, c_2, \ldots, c_N)$, $D=\operatorname{diag}(d_1, d_2, \ldots, d_N)$. Then for $A=\{a_{ij}\}$, $B=\{b_{ij}\}$ Eqs. (75) in components take the form

$$\dot{a}_{ij} = -(c_i - d_j)b_{ij}, \qquad \dot{b}_{ij} = (c_i - d_j)a_{ij},$$

where there is no summation over $i, j=1, \ldots, N$. The solution of these equations has the form

$$a_{ij} = Q_{ij}\sin(c_i - d_j)\varphi + R_{ij}\cos(c_i - d_j)\varphi, \qquad b_{ij} = R_{ij}\sin(c_i - d_j)\varphi - Q_{ij}\cos(c_i - d_j)\varphi, \tag{76}$$

where Q_{ij} and R_{ij} are arbitrary constants and there is no summation over i and j.

Therefore, taking any solution of Eqs. (72) and the solution (76) of Eqs. (71), we obtain the solution (68) of the YM equations in $\mathbb{R}^{4n+2}$ for $G=SL(2N, \mathbb{R})$ or $G=SL(2N, \mathbb{C})$.

Example 5. We substitute in (71) the ansatz (33) with $a_N=0$ for $\mathcal{G}=su(N)$. Then Eqs. (71) reduce to

$$\frac{da_i}{d\varphi} = (b_{i+1} - b_i)a_i, \qquad \frac{db_i}{d\varphi} = 2\left(1 + \frac{\varkappa^2}{\partial_c\varphi\partial_c\varphi}\right)(a_{i-1}^2 - a_i^2). \tag{77}$$

For $\varkappa=0$, these equations are identical to the equations of the Toda chain (see Sec. 4.2). In the simplest case $G=SU(2)$, Eqs. (77) for $\varphi=1/X_\mu X_\mu$ [cf. (53)] are close in form to the equations that arise in the description of vortex solutions of the YM equations in $\mathbb{R}^4$.

8. ANTI-SELF-DUAL SOLUTIONS OF THE YM EQUATIONS IN $\mathbb{R}^7$

In the space $\mathbb{R}^7$ with metric δ_{ab} we consider the gauge fields A_a of the semisimple Lie group G. We assume that

$$A_a = \frac{3}{2}\bar{g}_{abcd}x_b W_{cd}(u), \tag{78}$$

where the antisymmetric tensor $W_{cd}=-W_{dc}$ depends on $u=\rho^2+x_a x_a$ ($\rho=$const) and takes values in the Lie algebra $\mathcal{G}$ of the Lie group G, and the explicit form of the tensor $\bar{g}_{abcd}$ is given in (19b).

We substitute (78) in the definition of the tensor F_{ab}. We obtain

$$F_{ab} = -3\overline{g}_{abcd}W_{cd} + 3x_a x_k \overline{g}_{bkcd}\dot{W}_{cd} - 3x_b x_k \overline{g}_{akcd}\dot{W}_{cd} + \frac{9}{4}x_m x_n \overline{g}_{amcd}\overline{g}_{bnek}[W_{cd}, W_{ek}], \tag{79}$$

where $\dot{W}_{ce} \equiv dW_{ce}/du$. We substitute (79) in the anti-self-duality equations [cf. (21b)]

$$h_{abcd}F_{cd} = -2F_{ab}, \tag{80}$$

where the tensor h_{abcd} is introduced in Sec. 3.4. Using the identities (18)—(21), we can show that after fairly lengthy manipulations Eqs. (80) for the ansatz (78) reduce to the equations

$$S_{abcdmn}\dot{W}_{mn} = -[W_{ab}, W_{cd}]. \tag{81}$$

Here,

$$S_{abcdmn} = \frac{1}{2}(\delta_{ac}\delta_{b[m}\delta_{n]d} - \delta_{bc}\delta_{a[m}\delta_{n]d} + \delta_{bd}\delta_{a[m}\delta_{n]c} - \delta_{ad}\delta_{b[m}\delta_{n]c})$$

are the structure constants of the group $SO(7)$. Equations (81) are a special case of the RW equations (29).

If $W_{ab}(u)$ satisfy Eqs. (81), then the tensor F_{ab} has the form

$$F_{ab} = -\frac{3}{4}\overline{g}_{abcd}\left\{4W_{cd} + 2(u-\rho^2)\dot{W}_{cd} - 2x_k x_{[c}\dot{W}_{d]k} + 3x_k x_{[c}g_{d]kmn}\dot{W}_{mn}\right\}. \tag{82}$$

The anti-self-duality of the tensor F_{ab} is now obvious [see (21b)].

Proposition 5 [14]. *For the ansatz* (78), *the anti-self-duality equations* (80) *of the $d=7$ YM model with arbitrary semisimple gauge group G reduce to the RW equations* (81). *Conversely, with every solution of the RW equations* (81) *we can associate a solution* (78) *of the $d=7$ anti-self-duality equations* (80).

If we take the solution (31) of the RW equations and replace t by $u=\rho^2+x_a x_a$, then we obtain the simplest anti-self-dual solution of the YM equations in $\mathbb{R}^7$.

9. ANTI-SELF-DUAL SOLUTIONS OF THE YM EQUATIONS IN $\mathbb{R}^8$

In the space $\mathbb{R}^8$ with metric δ_{MN} we consider the gauge fields A_M of the semisimple group G. For A_M we make the ansatz

$$A_M = \frac{4}{3}\overline{G}_{MNCD}x_N W_{CD}(u), \tag{83}$$

where the antisymmetric tensor $W_{CD}=-W_{DC}$, which depends on $u=\rho^2+x_M x_M$ (ρ=const), takes values in the Lie algebra $\mathcal{G}$ of the Lie group G, and the constant tensor $\bar{G}_{MNCD}$ is given by formula (23b).

We substitute (83) in the definition of the tensor F_{MN}. We obtain

$$F_{MN} = -\frac{8}{3}\overline{G}_{MNCD}W_{CD} + \frac{8}{3}x_{[M}\overline{G}_{N]BCD}x_B\dot{W}_{CD} + \frac{16}{9}x_C x_D\overline{G}_{MCPQ}\overline{G}_{NDAB}[W_{PQ}, W_{AB}], \tag{84}$$

where $\dot{W}_{CD} \equiv dW_{CD}/du$. We substitute (84) in the antiself-duality equations [cf. (24b)]

$$H_{CDMN}F_{MN} = -2F_{CD}. \tag{85}$$

After fairly lengthy calculations using the identities (22)—(24), we obtain the equations

$$S_{ABCDMN}\dot{W}_{MN} = -[W_{AB}, W_{CD}], \tag{86}$$

where

$$S_{ABCDMN} = \frac{1}{2}(\delta_{AC}\delta_{B[M}\delta_{N]D} - \delta_{BC}\delta_{A[M}\delta_{N]D} + \delta_{BD}\delta_{A[M}\delta_{N]C} - \delta_{AD}\delta_{B[M}\delta_{N]C})$$

are the structure constants of the group $SO(8)$.

If $W_{AB}(u)$ satisfy Eqs. (86), then the tensor F_{MN} has the form

$$F_{MN} = -\frac{2}{9}\overline{G}_{MNIJ}\left(12W_{IJ} + 6(u-\rho^2)\dot{W}_{IJ} + 8x_B\overline{G}_{CDB[J}x_{I]}\dot{W}_{CD}\right). \tag{87}$$

The anti-self-duality of F_{MN} is obvious on the basis of (87) and (24b).

As in Sec. 8, we obtain the following proposition.

Proposition 6 [14]. *For the ansatz* (83) *the anti-self-duality equations* (85) *of the $d=8$ YM model with arbitrary semisimple gauge group G reduce to the RW equations* (86). *Conversely, with every solution of the RW equations* (86) *we can associate the solution* (83) *of the anti-self-duality equations* (85).

If in the solution (31) of the RW equations we replace t by $u=\rho^2+x_M x_M$ and substitute in (78), then we obtain the simplest anti-self-dual solution of the YM equations in $\mathbb{R}^8$, which was found in [10,11].

10. SOLUTIONS OF YM EQUATIONS IN THE SPACES $\mathbb{R}^{pq}$

In the space $\mathbb{R}^{pq}$ with metric δ_{ab} $(a, b, \ldots=1, \ldots, pq)$ we consider the gauge fields A_a of the semisimple Lie group G. We replace the indices $a, b, \ldots$ by double indices $(\mu i), (\nu j), \ldots$, where $\mu, \nu, \ldots=1, \ldots, p$; $i, j, \ldots=1, \ldots, q$. We take the metric tensor $\delta_{(\mu i)(\nu j)}$ $(=\delta_{ab})$ in the form

$$\delta_{(\mu i)(\nu j)} = \delta_{\mu\nu}\delta_{ij}.$$

We introduce the variables

$$X_\mu = x_{\mu i} p_i, \tag{88}$$

where p_i=const, and, as usual, summation is understood over repeated indices. For the gauge fields $A_{(\mu i)}$ $(=A_b)$ in $\mathbb{R}^{pq}$ we make the ansatz

$$A_{(\mu i)} = A_\mu(X_\nu)p_i, \tag{89}$$

where A_μ depends only on the "composite" coordinates (88).

Substituting the ansatz (89) in the definition of the tensor $F_{(\mu i)(\nu j)}$ $(=F_{ab})$, we obtain

$$F_{(\mu i)(\nu j)} = p_i p_j (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]) \equiv p_i p_j F_{\mu\nu}, \tag{90}$$

where, by definition, $\partial_\mu \equiv \partial/\partial X_\mu$. We substitute (89) and (90) in the YM equations in $\mathbb{R}^{pq}$. We obtain

$$\partial_{\mu i} F_{(\mu i)(\nu j)} + [A_{(\mu i)}, F_{(\mu i)(\nu j)}] = p_j p_i p^i (\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}]) = 0. \tag{91}$$

We assume that the p_i are real constants, and therefore $p_i p_i \neq 0$, and, as a consequence, the ansatz (89) reduces the YM equations in the space $\mathbb{R}^{pq}$ to YM equations in the space $\mathbb{R}^p$ parametrized by the coordinates X_μ:

$$\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0. \tag{92}$$

Proposition 7. *For the ansatz* (89), *the YM equations in $\mathbb{R}^{pq}$ reduce to YM equations in $\mathbb{R}^p$. Conversely, with every solution of the YM equations in the space $\mathbb{R}^p$ one can associate a solution* (89) *of the YM equations in the space $\mathbb{R}^{pq}$, where p, $q=2, 3, \ldots$.*

The proof follows from formulas (88)—(91).

Example 1. Let $p=4$, $q=2, 3, \ldots$. Then for the ansatz (89) the YM equations in $\mathbb{R}^{4q}$ reduce to YM equations in $\mathbb{R}^4$. Therefore, with every anti-self-dual solution of the YM equations in $\mathbb{R}^4$ it is possible to associate an anti-self-dual solution of the YM equations in $\mathbb{R}^{4q}$. Special cases of such an association were considered in [9,15] and in Sec. 5 of this paper (Example 2). Moreover, with every non-self-dual solution of the YM equations in $\mathbb{R}^4$ (for example, a meron solution) one can also associate a solution of the YM equations in $\mathbb{R}^{4q}$. Special cases of this were considered in [52].

Example 2. Let $p=8$, $q=2, 3, \ldots$. In this case, the ansatz (89) makes it possible to associate with every (anti)self-dual solution of the YM equations in $\mathbb{R}^8$ a solution of the YM equations in $\mathbb{R}^{8q}$. We give the explicit form of one such solution:

$$A_{(Mi)} = \frac{4}{3}\bar{G}_{MNCD}\frac{X_N p_i}{(\rho^2 + X_B X_B)} I_{CD},$$

where $X_B=x_{Bi}p_i$, p_i=const, x_{Bi} are coordinates in $\mathbb{R}^{8q}$, and I_{CD} are the generators of the Lie algebra $so(8)$.

11. CONCLUSIONS

In this paper, we have shown that Nahm's equations and the Rouhani—Ward equations, which generalize them, arise in the study of Yang—Mills equations in spaces of dimension $d \geq 4$. By means of the ansatzes considered in this paper one can associate with solutions of the Nahm and Rouhani—Ward equations different types of solution of the Yang—Mills equations in $d \geq 4$.

For Nahm's equations, general solutions of elliptic and trigonometric types are known and can be expressed, respectively, in terms of theta functions and hyperbolic functions. It is desirable to find solutions of such type for the Rouhani—Ward equations too.

We thank A. V. Bolsinov, D. V. Volkov, V. L. Golo, C. Devchand, B. A. Dubrovin, E. A. Ivanov, M. A. Ol'shanetskii, V. I. Tkach, V. V. Trofimov, and A. T. Fomenko for valuable discussions. We also thank V. G. Drinfel'd and R. Ward for comments and valuable remarks.

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