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EXACT SOLUTIONS OF THE NONLINEAR BOLTZMANN EQUATION AND THE THEORY OF RELAXATION OF A MAXWELLIAN GAS

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Results obtained in recent years in the theory of the nonlinear Boltzmann equation for Maxwellian molecules are reviewed. The general theory of spatially homogeneous relaxation based on Fourier transformation with respect to the velocity is presented. The behavior of the distribution function $f(\mathbf{v}, t)$ is studied in the limit $|\mathbf{v}| \to \infty$ (the formation of the Maxwellian tails) and $t \to \infty$ (relaxation rate). An analytic transformation relating the nonlinear and linearized equations is constructed. It is shown that the nonlinear equation has a countable set of invariants, families of particular solutions of special form are constructed, and an analogy with equations of Korteweg–de Vries type is noted.

1. Introduction

Derived for the first time on the basis of phenomenological arguments in 1872, the Boltzmann equation [1] immediately became the source of many problems of both a fundamental nature (reconciliation of time reversibility of the equations of classical mechanics with the irreversible behavior of the solution of the Boltzmann equation) and practical nature (solution of the equation). In the question of the foundation of the Boltzmann equation the monograph [2] of Bogolyubov has played a pioneering role; in it, he also described systematic methods for deriving generalized kinetic equations from the Liouville equation. The development of Bogolyubov's ideas and methods led subsequently to the construction of new, more complicated kinetic equations, but the solution of even the "simplest" of them – the Boltzmann equation – still remains a rather difficult problem. The comparatively restricted group of questions with which the present paper is concerned relates to this problem. For brevity, it is convenient here simply to postulate the Boltzmann equation, regard it as a mathematical model of a rarefied gas, and not return to its systematic derivation.

In the classical kinetic theory of rarefied monatomic gases, the state of the gas at the time $t \geq 0$ is characterized by the (single-particle) distribution function $f(\mathbf{x}, \mathbf{v}, t)$ of its molecules with respect to the spatial coordinates $\mathbf{x} \in R^3$ and the velocities $\mathbf{v} \in R^3$, where R^3 denotes the real three-dimensional Euclidean space. The function $f(\mathbf{x}, \mathbf{v}, t)$ is, roughly speaking, the number of particles in unit volume of the phase space $R^3 \times R^3$ at the time t, and its time evolution is described by the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} = I[f, f], \tag{1.1}$$

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on the right of which we have the so-called collision integral – a nonlinear integral operator that acts on $f(\mathbf{x}, \mathbf{v}, t)$ only with respect to the variable $\mathbf{v}$. Omitting here and in what follows the unimportant arguments $\mathbf{x}$ and t, we can represent the right-hand side of (1.1) in the form

$$I[f,f]=\int d\mathbf{w}\, d\mathbf{n}\, g\left(u, \frac{\mathbf{un}}{u}\right) \{f(\mathbf{v}')f(\mathbf{w}')-f(\mathbf{v})f(\mathbf{w})\}, \tag{1.2}$$

where $\mathbf{w} \in R^3$, $d\mathbf{w}$ is the volume element of R^3, $\mathbf{n} \in R^3$ is a unit vector, i.e., $|\mathbf{n}| = 1$, $d\mathbf{n}$ is the element of area of the surface of the unit sphere Ω in R^3; the integration is over the complete five-dimensional space $R^3 \times \Omega$. We also use the notation

$$\mathbf{u}=\mathbf{v}-\mathbf{w}, \quad u=|\mathbf{u}|, \quad g(u,\mu)=u\sigma(u,\mu), \quad \mathbf{v}'={}^1/_2(\mathbf{v}+\mathbf{w}+u\mathbf{n}), \quad \mathbf{w}'={}^1/_2(\mathbf{v}+\mathbf{w}-u\mathbf{n}). \tag{1.3}$$

It is assumed that the collisions of the molecules take place in accordance with the laws of classical mechanics of particles interacting with two-body potential $U(r)$, where r is the distance between the particles. The function $\sigma(u, \mu)$ in (1.3) is the differential cross section of scattering through angle $0 < \theta \leq \pi$ in the center-of-mass system of the colliding particles expressed as a function of the arguments $u > 0$ and $\mu = \cos\theta$. In the theory of the Boltzmann equation, $g(u, \mu)$ in (1.2) is usually assumed simply to be a given non-negative function subject to restrictions dictated by physical considerations. For example, for hard-sphere molecules of radius r_0 we obtain $g(u, \mu) = ur_0^2$ and for point-particle molecules interacting in accordance with the power law $U(r)=\alpha/r^n$ $(\alpha>0,\ n\geqslant 2)$, $g(u,\mu)=u^{1-4/n}g_n(\mu)$, where $g_n(\mu)(1-\mu)^{3/2}$ is a bounded function.

It is the nonlinearity and complicated structure of the collision integral (1.2) that are the main obstacles in the attempt to solve the Boltzmann equation. For this equation, one can consider problems with both initial conditions and boundary conditions. The simplest problem, which clearly reveals all the difficulties associated with the collision integral, is the spatially homogeneous Cauchy problem

$$\frac{\partial f}{\partial t}=I[f,f], \quad f|_{t=0}=f_0(\mathbf{v}), \tag{1.4}$$

or the problem of relaxation (approach to equilibrium) of a spatially homogeneous gas. This problem is of independent interest and, in addition, its solution is a necessary intermediate step in the solution of the complete, i.e., spatially inhomogeneous, equation (1.1).

Many but by no means all of the general mathematical problems relating to the existence and uniqueness of solutions of the Cauchy problem and boundary-value problems have been solved for the Boltzmann equation, and approximate approaches generalizing the well-known Hilbert, Chapman–Enskog, and Grad methods [3], have been developed. However, even comparatively recently studies on the nonlinear Boltzmann equation could be characterized by the almost complete absence of exact analytic results; for example, the first nontrivial exact solution of this equation was constructed only in 1975 [4, 5] (see also [6, 7]). In this connection, the results reviewed in the present paper may have some interest. The results apply mainly to a special case of the Boltzmann equation – the so-called model of Maxwellian molecules – but in the framework of this model, which is fairly typical from the physical point of view, one can obtain detailed analytic information about the behavior of the solutions of the spatially homogeneous equation and essentially construct an exact theory of the relaxation of such a gas.

Maxwellian molecules are particles that interact with a repulsive potential $U(r) = \alpha/r^4$. In this case, the cross section $\sigma(u, \mu)$ is inversely proportional to the modulus of the relative velocity $\mathbf{u}$, and the function $g(u, \mu)$ in (1.2) does not depend on u, which leads to some simplifications of the calculations related to the collision integral. These simplifications have long been known (see, for example, Chap. 3 of Boltzmann's book [1]) and were used by practically everyone who worked with the Boltzmann equation. However, it was only in 1975 that it was found [4, 8] that one can here achieve a much more significant simplification of the nonlinear operator (1.2) than was assumed earlier; this simplification is achieved by means of an ordinary Fourier transformation with respect to the velocity. It was Fourier transformation that made it possible to construct an exact solution for the first time [4] and served as the key method for obtaining the majority of the results described below.

The idea of the approach is as follows. In (1.1), we go over to the Fourier representation, setting

$$\varphi(\mathbf{x},\mathbf{k},t)=\int d\mathbf{v}\, e^{-i\mathbf{k}\mathbf{v}} f(\mathbf{x},\mathbf{v},t), \tag{1.5}$$

and as a result we obtain an equation for $\varphi(\mathbf{x}, \mathbf{k}, t)$:

$$\frac{\partial \varphi}{\partial t} + i \frac{\partial^2 \varphi}{\partial \mathbf{k} \, \partial \mathbf{x}} = J[\varphi, \varphi] = \int d\mathbf{v} e^{-i\mathbf{k}\mathbf{v}} I[f, f]. \quad (1.6)$$

Now in the exceptional case of Maxwellian molecules (or rather, for any u-independent function $g(u, \mu)$ in (1.2)) the operator $J[\varphi, \varphi]$ simplifies strongly compared with the operator $I[f, f]$ (see Sec.2), as a result of which it becomes much easier to work with the transformed equation. Unfortunately, the appearance of the mixed derivative in (1.6) makes it impossible to use this property effectively to solve spatially inhomogeneous problems. However, for the relaxation problem (1.4), which has in the Fourier representation the form

$$\frac{\partial \varphi}{\partial t} = J[\varphi, \varphi], \quad \varphi|_{t=0} = \varphi_0(\mathbf{k}), \quad (1.7)$$

this difficulty is absent, and the simplification is decisive. These are the reasons why we shall in what follows basically restrict ourselves to Maxwellian molecules and the spatially inhomogeneous problem.

Fourier transformation made it possible to obtain a number of new results [4, 5, 8-18] and ultimately construct a comparatively complete theory of the spatially homogeneous Boltzmann equation for Maxwellian molecules, including generalizations of previously known facts. The main aim of this paper is to give a brief exposition of this theory. After the first papers [4-9] published in 1975-1977 there followed rapidly quite a large number of publications on the theme that one may call "exact solutions of nonlinear models of the Boltzmann equation." Many of them are very interesting but do not directly relate to the aims of the present paper. Therefore, we quote only individual papers, especially since there are fairly complete reviews devoted basically to model equations [16, 17].

The material is arranged as follows. In Sec.2, which can be regarded as the "elementary theory" of the Boltzmann equation for Maxwellian molecules, we perform the Fourier transformation and describe new facts (symmetry, properties of the linearized equation, moment system, self-similar solutions) obtained comparatively easily after the transition to the Fourier representation. These results can be expressed by explicit expressions, but they have a somewhat special nature. More general questions such as the solvability of the Cauchy problem for the nonlinear Boltzmann equation and the asymptotic behaviors of the solution as $|\mathbf{v}| \to \infty$ and as $t \to \infty$ are considered in Sec.3. In Sec.4, we construct an analytic transformation relating the nonlinear and linearized equations in the Fourier representation, we establish the equivalence of these equations in a certain class of functions, and we describe the consequences of this equivalence.

2. Fourier Transformation and Consequences

Fourier Transformation. In accordance with its physical meaning, the distribution function $f(\mathbf{v})$ must be non-negative and possess finite moments up to the second order:

$$\int d\mathbf{v} f(\mathbf{v}) (1 + v^2) < \infty. \quad (2.1)$$

Assuming at the start for brevity that all the additional conditions needed for convergence of integrals, etc., are satisfied, we make a Fourier transformation of a collision integral of general form, i.e., we simplify the right-hand side of Eq. (1.6). For this, we use the standard identity

$$\int d\mathbf{v} I[f, f] h(\mathbf{v}) = \int d\mathbf{v}\, d\mathbf{w} f(\mathbf{v}) f(\mathbf{w}) \int d\mathbf{n} g\left(u, \frac{\mathbf{u}\mathbf{n}}{u}\right) [h(\mathbf{v}') - h(\mathbf{v})], \quad (2.2)$$

in which the notation (1.3) is used. For $h(\mathbf{v}) = \exp(-i\mathbf{k}\mathbf{v})$, the right-hand side has the form

$$\int d\mathbf{v}\, d\mathbf{w} f(\mathbf{v}) f(\mathbf{w}) e^{-i\mathbf{k}\frac{\mathbf{v}+\mathbf{w}}{2}} \int d\mathbf{n} g\left(u, \frac{\mathbf{u}\mathbf{n}}{u}\right) [e^{-i\mathbf{k}\frac{u\mathbf{n}}{2}} - e^{-i\mathbf{k}\frac{\mathbf{u}}{2}}]. \quad (2.3)$$

We consider here the inner integral and show that

$$\int d\mathbf{n} g\left(u, \frac{\mathbf{u}\mathbf{n}}{u}\right) [e^{-i\mathbf{k}\frac{u\mathbf{n}}{2}} - e^{-i\mathbf{k}\frac{\mathbf{u}}{2}}] = \int d\mathbf{n} g\left(u, \frac{\mathbf{k}\mathbf{n}}{k}\right) [e^{-ik\frac{\mathbf{u}\mathbf{n}}{2}} - e^{-i\mathbf{k}\frac{\mathbf{u}}{2}}]. \quad (2.4)$$

Indeed, the left-hand side of (2.4) is an isotropic scalar function of the two vectors $\mathbf{k}$ and $\mathbf{u}$ and, therefore, depends only on their absolute magnitudes $k = |\mathbf{k}|$ and $u = |\mathbf{u}|$ and the scalar product $\mathbf{k} \cdot \mathbf{u}$. Such a function is obviously unaffected by interchanging the directions of the vectors $\mathbf{k}$ and $\mathbf{u}$ (but not their absolute magnitudes), and the right-hand side of (2.4) is the result of such an interchange.

Substituting (2.4) in (2.3) and changing the order of integration, we obtain

$$\int d\mathbf{n}\int d\mathbf{v}\,d\mathbf{w}f(\mathbf{v})f(\mathbf{w})g\left(u,\frac{\mathbf{kn}}{k}\right)\left[\exp\left(-i\mathbf{v}\frac{\mathbf{k}+k\mathbf{n}}{2}-i\mathbf{w}\frac{\mathbf{k}-k\mathbf{n}}{2}\right)-\exp(-i\mathbf{kv})\right]. \tag{2.5}$$

It is clear from this that Fourier transformation leads to a definite simplification of the collision integral for molecules interacting in accordance with a power law and for hard spheres (see, for example, [19]), the greatest simplification being achieved for Maxwellian molecules, for which $g(u,\mu)\equiv g(\mu)$. In this case, the inner (six-fold) integral in (2.5) reduces simply to the difference, multiplied by $g(\mathbf{kn}/k)$, of products of the Fourier transforms of the distribution function, i.e., the final result – the right-hand side of (1.6) – has the form

$$J[\varphi,\varphi]=\int d\mathbf{v}I[f,f]e^{-i\mathbf{kv}}=\int d\mathbf{n}g\left(\frac{\mathbf{kn}}{k}\right)\left\{\varphi\left(\frac{\mathbf{k}+k\mathbf{n}}{2}\right)\varphi\left(\frac{\mathbf{k}-k\mathbf{n}}{2}\right)-\varphi(0)\varphi(\mathbf{k})\right\}, \tag{2.6}$$

where $f(\mathbf{v})$ and $\varphi(\mathbf{k})$ are related by the Fourier transformation (1.5) (the arguments $\mathbf{x}$ and t are not important).

Thus, for the considered model the transition to the Fourier representation has led to two important simplifications: 1) instead of the five-fold integral (1.2) the two-fold integral (2.6) has been obtained and 2) the integrand has also been significantly simplified. These advantages are most strongly manifested for distribution functions isotropic with respect to $\mathbf{v}$: $f\equiv f(|\mathbf{v}|)$. Then in (2.6) we can set $\varphi=\varphi(k^2/2)$ and, introducing the notation $x=k^2/2$, reduce (2.6) to the very simple form

$$J[\varphi,\varphi]=\int_0^1 ds\rho(s)\{\varphi(sx)\varphi[(1-s)x]-\varphi(0)\varphi(x)\},\quad x\geqslant 0,\quad \rho(s)=4\pi g(1-2s), \tag{2.7}$$

from which one can clearly see the advance achieved compared with (1.2).

It is readily verified that for $g\equiv g(\mu)$ the condition (2.1) and the inequality

$$0\leqslant g(\mu)\leqslant \text{const}\,(1-\mu)^{-3/2+\varepsilon},\quad \varepsilon>0, \tag{2.8}$$

guarantee validity of all the transformations (change in the order of integration, etc.) made above in the transition from (2.2) to (2.6). In all that follows we shall, when special requirements are not stipulated, assume that the conditions (2.1) and (2.8) are satisfied. The inequality (2.8) is valid, in particular, for true Maxwellian molecules, i.e., for the potential $U(r)\sim r^{-4}$, for all $0<\varepsilon<1/4$.

Relaxation Problem. We shall consider the Cauchy problem for the spatially homogeneous Boltzmann equation in the notation (1.3):

$$f_t=I[f,f]=\int d\mathbf{w}\,d\mathbf{n}g\left(\frac{\mathbf{un}}{u}\right)\{f(\mathbf{v}')f(\mathbf{w}')-f(\mathbf{v})f(\mathbf{w})\}, \tag{2.9}$$

where the subscript denotes the derivative with respect to t; without loss of generality, we can assume that the initial condition satisfies the normalization conditions

$$f|_{t=0}=f_0(\mathbf{v}):\quad \int d\mathbf{v}f_0(\mathbf{v})=1,\quad \int d\mathbf{v}\mathbf{v}f_0(\mathbf{v})=0,\quad \int d\mathbf{v}\mathbf{v}^2f_0(\mathbf{v})=3. \tag{2.10}$$

By virtue of the conservation laws for the particle number, the momentum, and the energy, the solution $f(\mathbf{v},t)$ of the problem (2.9)–(2.10) will satisfy the same requirements for all $t>0$:

$$\int d\mathbf{v}f(\mathbf{v},t)=1,\quad \int d\mathbf{v}\mathbf{v}f(\mathbf{v},t)=0,\quad \int d\mathbf{v}\mathbf{v}^2f(\mathbf{v},t)=3, \tag{2.11}$$

and the corresponding Maxwellian distribution has the form

$$f_M(\mathbf{v})=(2\pi)^{-3/2}\exp(-v^2/2). \tag{2.12}$$

The formal approach to the solution of the problem, and also to any other problem associated with Eq. (2.9), is rather obvious. Going over to the Fourier representation

$$\varphi(\mathbf{k},t)=\int d\mathbf{v}f(\mathbf{v},t)\exp(-i\mathbf{kv}), \tag{2.13}$$

we obtain instead of (2.9) the simpler equation

$$\varphi_t = J[\varphi, \varphi] = \int d\mathbf{n} g\left(\frac{\mathbf{kn}}{k}\right)\left\{\varphi\left(\frac{\mathbf{k}+k\mathbf{n}}{2}\right)\varphi\left(\frac{\mathbf{k}-k\mathbf{n}}{2}\right) - \varphi(0)\varphi(\mathbf{k})\right\} \tag{2.14}$$

with initial condition

$$\varphi|_{t=0} = \varphi_0(\mathbf{k}) = \int d\mathbf{v} f_0(\mathbf{v}) \exp(-i\mathbf{kv}): \quad \varphi_0|_{\mathbf{k}=0} = 1, \quad \left.\frac{\partial \varphi_0}{\partial \mathbf{k}}\right|_{\mathbf{k}=0} = 0, \quad \left.\frac{\partial^2 \varphi_0}{\partial \mathbf{k}^2}\right|_{\mathbf{k}=0} = -3, \tag{2.15}$$

after which we study the solution $\varphi(\mathbf{k}, t)$ of the problem (2.14)-(2.15) and, finally, formulate the final results for the distribution function $f(\mathbf{v}, t)$, using, for example, the inversion formula

$$f(\mathbf{v}, t) = (2\pi)^{-3} \int d\mathbf{v} \varphi(\mathbf{k}, t) \exp(i\mathbf{kv}) \tag{2.16}$$

under the assumption that this integral converges. For the purposes of this section this formal scheme is entirely adequate, and rigorous definitions of the concepts of the distribution function and the solution of the Cauchy problem (2.9)-(2.10) will be given in Sec. 3.

The Fourier analogs of Eqs. (2.11)-(2.12) have the form

$$\varphi(0, t) = 1, \quad \left.\frac{\partial \varphi(\mathbf{k}, t)}{\partial \mathbf{k}}\right|_{\mathbf{k}=0} = 0, \quad \left.\frac{\partial^2 \varphi(\mathbf{k}, t)}{\partial \mathbf{k}^2}\right|_{\mathbf{k}=0} = -3; \quad \varphi_M(\mathbf{k}) = \exp\left(-\frac{k^2}{2}\right). \tag{2.17}$$

Remark. For brevity, we shall not dwell here on the solution to the relaxation problem for small deviations from equilibrium, for which the linearized equation can be used. We merely mention that Fourier transformation made it possible to generalize significantly the classical results [3] for this equation, i.e., to give a complete solution to the eigenvalue problem for the linearized collision operator [4, 8] and construct in explicit form a solution to the linear problem of relaxation in a maximally large (from the physical point of view) class of initial conditions [11].

We now describe some properties of the nonlinear equation (2.9); their proof in the Fourier representation is very simple.

Symmetry Property. Equation (2.9) is invariant with respect to the one-parameter semigroup of transformations

$$T_\theta f = \exp\left[\frac{\theta}{2}\frac{\partial^2}{\partial \mathbf{v}^2}\right] f = (2\pi\theta)^{-3/2} \int d\mathbf{w}\, f(\mathbf{w}, t) \exp\left[-\frac{(\mathbf{v}-\mathbf{w})^2}{2\theta}\right], \quad \theta \geqslant 0, \tag{2.18}$$

which leave the distribution function non-negative.

For the proof, it is sufficient to note that the corresponding transformations $\varphi(\mathbf{k}, t)$ have the form

$$\tilde{T}_\theta \varphi = \exp(-\theta k^2/2)\varphi(\mathbf{k}, t), \quad \theta \geqslant 0, \tag{2.19}$$

and, obviously, do not change Eqs. (2.14).

This property, which is peculiar to Maxwellian molecules, has a number of interesting consequences. The simplest of them is that besides the classical H function of Boltzmann one can find a one-parameter family of functionals $H_\theta[f]$ on the solution $f(\mathbf{v}, t)$ of Eq. (2.9),

$$H_\theta[f] = \int d\mathbf{v}\, f_\theta(\mathbf{v}, t) \ln f_\theta(\mathbf{v}, t), \quad f_\theta = T_\theta f, \quad \theta > 0, \tag{2.20}$$

which have the same property, i.e., they do not increase with increasing t. An advantage of these functionals is that, in contrast to the usual H function, they are also defined on generalized solutions of the Boltzmann equation (see Sec. 3). This result cannot be generalized to the spatially inhomogeneous equation (1.1), for which the classical H function is the only nonincreasing functional [20].

Another trivial consequence of this symmetry, on which we shall dwell below, is that the equation for the moments (normalized in a special manner) of the function $f(\mathbf{v}, t)$ is identical to the equation for the coefficients of its expansion in Hermite polynomials (Laguerre polynomials in the case isotropic with respect to $\mathbf{v}$).

Finally, combining (2.19) with the scale transformation $\mathbf{k} \to \alpha\mathbf{k}$ and the shift $t \to t + \tau$, which also leave Eq. (2.14) unchanged, we can readily construct a two-parameter family of transformations that leave not only Eq. (2.14) itself but also the normalization conditions (2.17) invariant. In exponential form, these transformations are

$$\tilde{U}_{\tau,\mu}\varphi = \exp\left\{\tau\left[-\frac{\partial}{\partial t} - \mu\left(\mathbf{k}^2 + \mathbf{k}\frac{\partial}{\partial \mathbf{k}}\right)\right]\right\}\varphi(\mathbf{k}, t), \tag{2.21}$$

and for $f(\mathbf{v}, t)$ they correspond to the operators

$$U_{\tau,\mu} f = \exp\left\{\tau\left[-\frac{\partial}{\partial t} + \mu\left(\frac{\partial^2}{\partial \mathbf{v}^2} + \frac{\partial}{\partial \mathbf{v}}\mathbf{v}\right)\right]\right\} f(\mathbf{v}, t). \tag{2.22}$$

We describe in this section invariant solutions of the Boltzmann equation, these being fixed points of the transformations (2.22).

The invariance of the nonlinear equation (2.9) with respect to the transformations (2.18) was established for the first time in [4, 8], but the analog of this property for the linear equation was known much earlier [21].

<u>Moment Equations.</u> In what follows, we shall mainly consider isotropic solutions $f(|\mathbf{v}|, t)$ of the Cauchy problem (2.9)-(2.10).

In the Fourier representation, it is convenient to introduce the variable $x = k^2/2$, set $\varphi \equiv \varphi(x, t)$, and rewrite (2.14)-(2.15) in the form

$$\varphi_t = \int_0^1 ds\, \rho(s)\, \{\varphi(sx)\varphi[(1-s)x] - \varphi(0)\varphi(x)\}, \quad \rho(s) = 4\pi g(1-2s); \tag{2.23}$$

$$\varphi|_{t=0} = \varphi_0(x): \quad \varphi_0(0) = 1, \quad \varphi_0'(0) = -1, \tag{2.24}$$

where the prime denotes the derivative with respect to x.

We now formulate two properties of isotropic solutions of the Boltzmann equation (2.9); the Fourier representation (2.23) makes these properties obvious.

A. We set

$$z_n(t) = \frac{1}{(2n+1)!!}\int d\mathbf{v}\, f(|\mathbf{v}|, t)\,\mathbf{v}^{2n}, \quad n = 0, 1, \ldots; \tag{2.25}$$

then the equations for $z_n(t)$ have the form

$$\dot{z}_0 = \dot{z}_1 = 0, \quad \dot{z}_n = \sum_{k=1}^{n-1} H_{k,n-k}(z_k z_{n-k} - z_0 z_n), \quad n = 2, 3, \ldots, \tag{2.26}$$

where the dot denotes the derivative with respect to t,

$$H_{k,l} = H(k, l) = \binom{k+l}{k}\int_0^1 ds\, \rho(s)\, s^k (1-s)^l; \quad k, l = 1, 2, \ldots. \tag{2.27}$$

B. Let $f(|\mathbf{v}|, t)$ be represented by a Fourier series in Laguerre polynomials:

$$f(|\mathbf{v}|, t) = (2\pi)^{-3/2} e^{-v^2/2} \sum_{n=0}^{\infty} u_n(t)\, \mathcal{L}_n^{1/2}(v^2/2), \tag{2.28}$$

this being convergent with respect to the metric of the Hilbert space L_2 with norm

$$\|f\|^2 = (2\pi)^{3/2}\int d\mathbf{v}\, e^{v^2/2} |f(\mathbf{v})|^2 = \sum_{n=0}^{\infty} \frac{(2n+1)!!}{(2n)!!} |u_n|^2 < \infty; \tag{2.29}$$

then the system of equations for the coefficients $u_n(t)$ can be obtained from the moment system (2.26) by the simple replacement in (2.26) of $z_n(t)$ by $u_n(t)$, $n = 0, 1, \ldots$.

To prove properties **A** and **B**, it is sufficient to note that the corresponding solution $\varphi(x, t)$ of Eq. (2.23) can be represented in the form of the power series

$$\varphi(x, t) = \sum_{n=0}^{\infty} (-1)^n z_n(t) \frac{x^n}{n!} = e^{-x} \sum_{n=0}^{\infty} u_n(t) \frac{x^n}{n!}, \tag{2.30}$$

substitution of which in (2.23) immediately leads to equations of the form (2.26). The identity of the equations for $z_n(t)$ and $u_n(t)$ is an obvious consequence of the invariance of (2.23) with respect to multiplication of $\varphi(x, t)$ by $\exp(-x)$ and replacement of x by $-x$.

It follows from the conditions (2.24) and the first equations in (2.26) that $z_0 = z_1 = 1$, $u_0 = 0$, $u_1 = 0$. Therefore, the equations for $u_n(t)$ when $n = 2, \ldots$ reduce to the form

$$\dot{u}_{2,3}+\lambda_{2,3}u_{2,3}=0, \quad \dot{u}_n+\lambda_n u_n=\sum_{k=2}^{n-2} H_{k,n-k}u_k u_{n-k}, \quad n=4,\ldots, \tag{2.31}$$

where we have used the notation (2.27) and

$$\lambda_n=\lambda(n)=\sum_{k=1}^{n-1} H_{k,n-k}=\int_0^1 ds\,\rho(s)\,[1-s^n-(1-s)^n]. \tag{2.27a}$$

The solution to the Cauchy problem for the system (2.31) can obviously be given in the recursive form

$$u_{2,3}(t)=u_{2,3}(0)\exp(-\lambda_{2,3}t), \quad u_n(t)=u_n(0)\exp(-\lambda_n t)+$$

$$\sum_{k=2}^{n-2} H_{k,n-k}\int_0^t d\tau\, u_k(\tau)\,u_{n-k}(\tau)\exp[-\lambda_n(t-\tau)], \quad n=4,\ldots. \tag{2.32}$$

Similar expressions can be written down for $z_n(t)$, $n = 2, \ldots$.

Thus, we have at our disposal now simple expressions making it possible in principle to write down in the form of a finite sum of exponentials an expression for the moment $z_n(t)$ (or the coefficient $u_n(t)$ of the series (2.28)) of arbitrary order $n = 0, 1, \ldots$. In reality, such a procedure is effective only for comparatively small $n \leqslant 10$, since the number of terms in these sums increases very rapidly with increasing n. Therefore, in Sec.4 we will describe a different approach to the solution of the system (2.31).

The method of solving the nonlinear Boltzmann equation by expansion in series of the type (2.28), which in the general case contain tensor Hermite polynomials

$$H_{(n)}(\mathbf{v})=(-1)^n\exp(v^2/2)\frac{\partial^n}{\partial\mathbf{v}^n}\exp(-v^2/2)$$

or their irreducible representations

$$F_{n,l,m}(\mathbf{v})=v^l\mathscr{L}_n^{l+1/2}(v^2/2)\,Y_{l,m}(\mathbf{v}/v),$$

was first proposed in 1949 by Grad [22]. In particular, for Maxwellian molecules, for which the last expression determines the eigenfunctions of the linearized collision operator, Grad [22] established the structure of the equations for the expansion coefficients and showed that the spatially homogeneous problem can be solved by a recursive scheme analogous to (2.32). The corresponding expressions for the solution $f(\mathbf{v}, t)$ of the Cauchy problem (2.9)-(2.10) were apparently given explicitly for the first time in [23] (see also the earlier paper [24] of Kac on the one-dimensional model that he proposed). However, investigations of this kind were long formal in nature, since the constant coefficients in moment systems of the type (2.21) were expressed in terms of cumbersome multiple integrals difficult to estimate. In this respect, the use of Fourier transformation changed the situation greatly and made it possible to obtain the final, i.e., not admitting further simplifications, form of the equations for the coefficients of expansion in series of the type (2.28). For the isotropic case, the result – the system (2.31) – is obvious and was given for the first time in [4] (see also [10]); for the anisotropic case, the analogous problem was solved in [25] (see also [26]). Equations (2.26) for the normalized moments (2.25) in the special case $\rho(s) = 1$ were solved explicitly for the first time in [6, 7], in which Fourier transformation was not used. The obvious derivation of Eqs. (2.26) described here was given in the general case in [10], in which the identity of the systems for $z_n(t)$ and $u_n(t)$ due to the symmetry properties was noted.

It will be shown further that for many practically interesting initial conditions representable in the form of the series (2.28) for $t = 0$ the corresponding series for the function $f(|\mathbf{v}|, t)$ converges only over a very short time interval $0 \le t < t_0 \ll 1$. For this reason, the use of the space L_2 with the norm (2.29) is inconvenient for solving the nonlinear problem (2.9)-(2.10), and we replace it in Sec.3 by a larger class of functions.

Thus, Fourier transformation significantly simplified the procedure for calculating any finite number of moments of the solution $f(\mathbf{v}, t)$ of the Boltzmann equation (2.9) but did not make trivial the construction of the actual function $f(\mathbf{v}, t)$, which depends on an infinite number of moments. To investigate the properties of this function, it is necessary (in the case isotropic with respect to $\mathbf{v}$) to be able to describe simultaneously the properties of the complete set of quantities $u_n(t)$ or $z_n(t)$ for $n = 0, 1, \ldots$. We do this now in a special case associated with symmetry properties.

Self-Similar Solutions. The simplest scheme for constructing such solutions of Eq. (2.14) is as follows. It is natural to seek a solution in the form $\varphi(\mathbf{k}, t) = \psi[\mathbf{k}\theta(t)]$, from which after substitution in (2.14) we find that $\theta(t) = \exp(-\mu t)$, where μ is a constant. We are interested only in solutions that have physical meaning and describe the relaxation process, i.e., tend in the limit $t \to \infty$ to the equilibrium function $\varphi_M(\mathbf{k})$ and satisfy the conservation laws (2.17). These conditions are not explicitly satisfied by the self-similar function $\psi[\mathbf{k}\exp(-\mu t)]$, but this defect can be readily corrected by using the symmetry property (2.19). The final form in which we shall seek the solution of (2.14) can be conveniently expressed in terms of the corresponding initial condition $\varphi_0(\mathbf{k})$:

$$\varphi(\mathbf{k}, t) = \varphi_0(\mathbf{k}e^{-\mu t}) \exp\left[-\frac{k^2}{2}(1 - e^{-2\mu t})\right], \tag{2.33}$$

where the function $\varphi_0(\mathbf{k})$ and the constant μ are as yet unknown. Substituting (2.33) in (2.14), we obtain an equation for their determination:

$$\mu\left(\mathbf{k}^2 + \mathbf{k}\frac{\partial}{\partial \mathbf{k}}\right)\varphi_0 + J[\varphi_0, \varphi_0] = 0. \tag{2.34}$$

The functions (2.33) are obviously fixed points of the group transformations (2.21).

We restrict ourselves to the case isotropic with respect to $\mathbf{k}$, i.e., we consider Eq. (2.23) and describe its solutions of the form (2.33)

$$\varphi(x, t) = y(xe^{-\beta t}) \exp[-x(1 - e^{-\beta t})], \quad x = k^2/2, \quad \beta = 2\mu. \tag{2.35}$$

For $y(x)$, we obtain from (2.23) the equation

$$\beta x[y'(x) + y(x)] + \int_0^1 ds\, \rho(s)\{y(sx)y[(1-s)x] - y(0)y(x)\} = 0. \tag{2.36}$$

It is natural to seek a solution in the form of the series

$$y(x) = \sum_{n=0}^{\infty} (-1)^n y_n \frac{x^n}{n!}, \quad y_0 = y_1 = 1, \tag{2.37}$$

it being assumed for the sake of definiteness that the conditions (2.24) are satisfied.

Substitution of (2.37) in (2.36) leads to the simple algebraic system

$$y_n(\beta n - \lambda_n) = \beta n y_{n-1} - \sum_{k=1}^{n-1} H_{k,n-k} y_k y_{n-k}, \quad y_1 = 1, \tag{2.38}$$

where $n = 2, 3, \ldots$ and we have used the notation of the system (2.31). The trivial solution of the system (2.38) has the form $y_n = 1$ for all $n = 2, 3, \ldots$. It is readily concluded after an elementary investigation of (2.38) that nontrivial solutions can exist only for definite values of the parameter β:

$$\beta = \beta_p = \frac{\lambda_p}{p} = \frac{1}{p}\int_0^1 ds\, \rho(s)[1 - s^p - (1-s)^p], \quad p = 2, 3, \ldots, \tag{2.39}$$

which are associated with eigenvalues of the linearized operator.

For $p \geq 4$, the solution of the system (2.38), in which $\beta = \beta_p$, is constructed as follows: $y_n \equiv 1$ for $2 \leq n \leq p-1$, y_p can be chosen arbitrarily, and y_n for $n \geq p+1$ are determined in accordance with recursion relations obtained from (2.38) by dividing this equation term by term by the factor $n(\beta_p - \beta_n)$, which occurs on the left-hand side of (2.38) and is positive for $n \geq p+1$.

The case $p = 2, 3$ requires special study, since $\beta_2 = \beta_3$. In this case, not one but two quantities can be chosen arbitrarily, y_2 and y_3 in (2.38), and y_n for $n \geq 4$ is calculated in the same way as described above. In particular, one can set $y_2 = y_3 = 0$ and conclude that then $y_n \equiv 0$ for all $n = 2, 3, \ldots$. This is the only case for which the series (2.37) terminates, and we obtain the simplest solution of Eq. (2.23), represented in the form (2.35):

$$\varphi(x,t)=(1-\theta xe^{-\lambda t})\exp[-x(1-\theta e^{-\lambda t})],\quad \lambda=\beta_{2,3}=\int_0^1 ds\,\rho(s)s(1-s), \tag{2.40}$$

where the constant $\theta=\exp(-\lambda t_0)$ takes into account the possibility of arbitrary choice of the origin of time. For $x=k^2/2$, $\rho(s)=4\pi g(1-2s)$, Eq. (2.40) determines the solution of Eq. (2.14), and inverting the Fourier transformation in accordance with (2.16) we obtain the solution of the original Boltzmann equation (2.9):

$$f(\mathbf{v},t)=(2\pi\tau)^{-3/2}\exp\left(-\frac{v^2}{2\tau}\right)\left[1+\frac{1-\tau}{\tau}\left(\frac{v^2}{2\tau}-\frac{3}{2}\right)\right],$$
$$\tau=\tau(t)=1-\theta e^{-\lambda t},\quad \lambda=(\pi/2)\int_{-1}^{1}d\mu\,g(\mu)(1-\mu^2),\quad t\geqslant 0, \tag{2.41}$$

where $f(v,t)\geqslant 0$ for $0\leqslant\theta\leqslant{}^2/_5$, but in what follows nonpositive solutions corresponding to ${}^2/_5<\theta<1$ will also be helpful.

This is the first and as yet unique nontrivial (i.e., non-Maxwellian) solution of the nonlinear Boltzmann equation that can be expressed in closed form in terms of elementary and special functions. We give here also exact solutions of the moment equations (2.26) and (2.31) corresponding to the series expansion (2.30) of the function (2.40):

$$z_0=z_1=1,\quad z_n(t)=(1-\theta e^{-\lambda t})^{n-1}[1+(n-1)\theta e^{-\lambda t}];\quad u_n(t)=(1-n)\theta^n e^{-n\lambda t};\quad \lambda=\lambda_2/2=\lambda_3/3,\quad n=2,3,\ldots. \tag{2.42}$$

The solution (2.41) was constructed for the first time in [4] by the method described here, and then in [5] an entirely elementary derivation of (2.41) without use of Fourier transformation was given. Somewhat later, the same solution for the special case $\rho(s)\equiv 1$ was constructed independently in [6] on the basis of the moment system (2.26) obtained in [6] for $\rho(s)=1$; it was then generalized in [7] to arbitrary functions $\rho(s)$. Self-similar solutions were also first constructed in [4], and then in [9] the group properties of the corresponding solutions $f(|\mathbf{v}|,t)$ of the Boltzmann equation were studied in more detail. These results were repeated later in [27,28]. The exact solution in elementary functions and the self-similar solutions have been much discussed in the literature [16], and therefore we shall not dwell on them but turn to more general questions.

3. General Relaxation Theory

We make more precise the concepts of the distribution function and solution of the Cauchy problem (2.9)-(2.10). Note that from Eq. (2.9), using Eq. (2.2), we can derive the well-known equation for the evolution of the mean value of the function $h(\mathbf{v})$,

$$\frac{d}{dt}\langle h(\mathbf{v})\rangle=\left\langle\!\!\left\langle\int d\mathbf{n}\,g\left(\frac{\mathbf{un}}{u}\right)[h(\mathbf{v}')-h(\mathbf{v})]\right\rangle\!\!\right\rangle, \tag{3.1}$$

where

$$\langle h(\mathbf{v})\rangle=\int d\mathbf{v}f(\mathbf{v},t)h(\mathbf{v}),\qquad \langle\!\langle H(\mathbf{v},\mathbf{w})\rangle\!\rangle=\int d\mathbf{v}\,d\mathbf{w}\,f(\mathbf{v},t)f(\mathbf{w},t)H(\mathbf{v},\mathbf{w}).$$

Equation (3.1) has a clear physical (probability) interpretation: The change in the mean value of $h(\mathbf{v})$ in unit time is equal to the mean change in $h(\mathbf{v})$ in one collision, averaged then over the number of collisions. On the basis of this interpretation, it is possible to give a phenomenological derivation of (3.1) without recourse to the Boltzmann equation (2.9), using however in the calculation of the number of collisions the same hypotheses as in the derivation of Eq. (2.9). Regarding $\mathbf{v}\in R^3$ as a random variable, we note further that the main tool for calculating the mean values of functions of such a variable is the probability measure $\omega(\mathbf{v};t)$ in R^3, this depending in the present case parametrically on the time $t\geq 0$. Then the mean value of the function $h(\mathbf{v})$ is determined as an integral with respect to the measure,

$$\langle h(\mathbf{v})\rangle=\int d\omega(\mathbf{v};t)h(\mathbf{v}), \tag{3.2}$$

and we define the mean value $\langle\!\langle H(\mathbf{v},\mathbf{w})\rangle\!\rangle$ on the right-hand side of (3.1) as an integral with respect to a product of measures,

$$\langle\langle H(\mathbf{v},\mathbf{w})\rangle\rangle=\int d\omega(\mathbf{v};t)\,d\omega(\mathbf{w};t)\,H(\mathbf{v},\mathbf{w}), \tag{3.3}$$

regarding **v** and **w** as independent random variables with the same distribution. Such a rule for calculating $\langle\langle H(\mathbf{v},\mathbf{w})\rangle\rangle$ in (3.1) is obviously equivalent to Boltzmann's main hypothesis that the colliding particles can be assumed to be independent.

Equation (3.1) in the notation (3.2)-(3.3) can be taken as the basis with the requirement that it be satisfied on any continuously differentiable function $h(\mathbf{v})$ of compact support; differentiability of $h(\mathbf{v})$ is needed only when one considers true Maxwellian molecules, for which $g(\mu)$ has a nonintegrable singularity at $\mu=1$. The dependence of the measure $\omega(\mathbf{v};t)$ on the parameter $t\geq 0$ must be such as to ensure continuous differentiability with respect to t of $\langle h(\mathbf{v})\rangle$. The initial condition can also be assumed to be given in the form of the measure $\omega(\mathbf{v};0)=\omega_0(\mathbf{v})$.

In such a formulation, the main object of the investigation is the probability measure $\omega(\mathbf{v};t)$, which describes the velocity distribution of a randomly chosen molecule of the gas. However, it is less convenient to work with the measure, which is a function of a set, than with an ordinary function of a point. This can be done in two ways. First, it can be assumed that the measure $\omega(\mathbf{v};t)$ has a density $f(\mathbf{v},t)$ – called in kinetic theory a distribution function – that is integrable with respect to **v** and differentiable with respect to t. We then naturally return to the original Boltzmann equation (2.9). Another – more general – method is to consider the characteristic function of the measure $\omega(\mathbf{v};t)$, i.e., its Fourier transform

$$\varphi(\mathbf{k},t)=\langle e^{-i\mathbf{k}\mathbf{v}}\rangle=\int d\omega(\mathbf{v};t)\,e^{-i\mathbf{k}\mathbf{v}}; \tag{3.4}$$

this way obviously leads to the simpler equation (2.14) and does not require any additional restrictions on the local properties of $\omega(\mathbf{v};t)$. In order not to have to give up the usual expression of the Boltzmann equation, we introduce in the necessary manner the concept of a generalized solution of it.

DEFINITION 1. Let $\omega(\mathbf{v})$ be a probability measure in R^3. We shall call the linear functional f defined on functions $h(\mathbf{v})$ that are integrable with respect to the measure $\omega(\mathbf{v})$ by the equation

$$(f,h)=\langle h(\mathbf{v})\rangle=\int d\omega(\mathbf{v})\,h(\mathbf{v}) \tag{3.5}$$

the distribution function or generalized density of the measure $\omega(\mathbf{v})$; we shall call the Fourier transform of the measure $\omega(\mathbf{v})$ the characteristic function $\varphi(\mathbf{k})$, i.e., $\varphi(\mathbf{k})=\langle\exp(-i\mathbf{k}\mathbf{v})\rangle$.

We shall use the notation $f(\mathbf{v})$ for the distribution function f and write (3.5) in the form

$$(f,h)=\langle h(\mathbf{v})\rangle=\int d\mathbf{v}\,f(\mathbf{v})\,h(\mathbf{v}),$$

irrespective of whether the generalized function f is regular, i.e., $f(\mathbf{v})\in L(R^3)$, or not.

We now consider the Cauchy problem (2.9)-(2.10), making the assumption that $f_0(\mathbf{v})$ is a distribution function.

DEFINITION 2. We shall say that the distribution function $f(\mathbf{v},t)$, which depends parametrically on $t\geq 0$, is a (generalized positive) solution of the Cauchy problem (2.9)-(2.10) if the corresponding characteristic function $\varphi(\mathbf{k},t)$ for all $\mathbf{k}\in R^3$, $t\geq 0$, satisfies Eq. (2.14) and for any $\mathbf{k}\in R^3$

$$\lim_{t\to+0}\varphi(\mathbf{k},t)=\varphi_0(\mathbf{k})=(f_0,e^{-i\mathbf{k}\mathbf{v}}).$$

It is obvious that in order to construct in this manner a definite solution $f(\mathbf{v},t)$ to the problem (2.9)-(2.10) it is sufficient to 1) construct a classical solution $\varphi(\mathbf{k},t)$ to the problem (2.14)-(2.15), 2) show that $\varphi(\mathbf{k},t)$ is a characteristic function for any $t>0$, and 3) use the well-known fact of the one-to-one correspondence between a probability measure (distribution function) and its characteristic function [29]. Such an approach makes it possible, exploiting the simplicity of Eq. (2.14) and the well-known properties of characteristic functions, to prove fairly easily an existence and uniqueness theorem for a generalized positive solution to the Cauchy problem (2.9)-(2.10) and a theorem of stabilization as $t\to\infty$ of this solution to the Maxwell distribution (2.12). We shall not do this here in the general case, since the aim of the paper is to describe more subtle properties of the solutions for at least a comparatively small class of distribution functions. Restricting ourselves to isotropic functions $f=f(|\mathbf{v}|)$, we introduce this class B_* as follows [15].

Rapidly Decreasing Distribution Functions. By B_* we denote the set of isotropic distribution functions $f(|\mathbf{v}|)$ for which for some $r > 0$ there is convergence of the integral

$$\Psi(r)=\int d\mathbf{v}\, f(|\mathbf{v}|)\exp(r v^2/2)<\infty$$

and the two following normalization conditions are satisfied:

$$\int d\mathbf{v}\, f(|\mathbf{v}|)=1, \quad \int d\mathbf{v} f(|\mathbf{v}|)v^2=3. \tag{3.6}$$

The function $f\in B_*$ has a natural characteristic asymptotic behavior as $|\mathbf{v}| \to \infty$, which we shall call the tail temperature τ and determine by the equation

$$\tau^{-1}=\sup \tau, \tag{3.7}$$

where the upper bound is taken over the $r > 0$ for which $\Psi(r) < \infty$.

In particular, the class B_* contains all non-negative functions of the Hilbert space L_2 with the norm (2.29) for which the conditions (3.6) are satisfied. Such functions correspond to $\tau \leq 2$.

We now consider the Cauchy problem for the Boltzmann equation (2.9) with the initial condition $f_0(|\mathbf{v}|)\in B_*$ (2.10), making the assumption that the inequality (2.8) is satisfied.

PROPOSITION 1. A solution $f(|\mathbf{v}|, t)$ to the problem (2.9)-(2.10) exists and $f(|\mathbf{v}|, t)\in B_*$ for all $t \geq 0$. In the limit $t \to \infty$, the solution $f(|\mathbf{v}|, t)$ converges weakly to the Maxwell distribution $f_M(|\mathbf{v}|)$ (2.12), i.e., $(f, h)\to(f_M, h)$ as $t \to \infty$ for any bounded continuous function $h(\mathbf{v})$.

The main stages of the proof will be briefly described below in the process of constructing a solution of the corresponding Cauchy problem in the Fourier representation. We note first some properties of characteristic functions corresponding to rapidly decreasing distribution functions.

We denote by A_* the set of functions $\varphi(x)$ defined for $x \geq 0$ by the equation

$$\varphi(x)=4\pi\int_0^\infty dv\, v^2 f(v)\frac{\sin v\sqrt{x}}{v\sqrt{x}}=\left[\int d\mathbf{v} f(|\mathbf{v}|)e^{-i\mathbf{k}\mathbf{v}}\right]_{x=k^2/2}, \quad f\in B_*. \tag{3.8}$$

An important property of the functions $\varphi(x) \in A_*$ is that $\varphi(x)$ can be analytically continued to the complete plane of the complex variable x and is an entire function of exponential type [29], the type σ of this function being equal to the tail temperature τ of the corresponding distribution function $f(|\mathbf{v}|)\in B_*$. If $\varphi\in A_*$ and $f\in B_*$ are related by the transformation (3.8), then

$$\varphi(x)=\sum_{n=0}^\infty(-1)^n z_n\frac{x^n}{n!}, \quad z_n=\frac{4\pi}{(2n+1)!!}\int_0^\infty dv f(v)v^{2(n+1)}, \quad n=0,1,\ldots,$$
$$\sigma=\limsup_{x\to\infty}\frac{\ln\varphi(-x)}{x}=\limsup_{n\to\infty}\sqrt[n]{z_n}=\tau<\infty. \tag{3.9}$$

We also define a natural extension of the class A_*, denoting by A the set of functions that can be represented in the form

$$\varphi(x)=\sum_{n=0}^\infty(-1)^n z_n\frac{x^n}{n!}, \quad z_0=z_1=1, \quad \sup\sqrt[n]{|z_n|}<\infty. \tag{3.10}$$

It is obvious that $A_* \subset A$ consists of functions of the class A that are characteristic (with allowance for the replacement of x by $k^2/2$).

To solve the Boltzmann equation (2.9) with the initial condition (2.10) $f_0\in B_*$, it is necessary to consider the following Cauchy problem for the function $\varphi(x, t)$:

$$\varphi_t=\int_0^1 ds\,\rho(s)\,\{\varphi(sx)\varphi[(1-s)x]-\varphi(0)\varphi(x)\}, \quad \varphi|_{t=0}=\varphi_0(x)\in A_*, \tag{3.11}$$

where $\varphi_0(x)$ and $f_0(|\mathbf{v}|)$ are related by the transformation (3.8). It is natural to seek the solution of (3.11) in the form of the series

$$\varphi(x,t)=\sum_{n=0}^{\infty}(-1)^n z_n(t)\frac{x^n}{n!};\quad z_n(0)=z_n^{(0)},\quad n=0,1,\ldots. \tag{3.12}$$

Substituting this series in (3.11), we obtain the system (2.26), which it is here convenient to rewrite in the notation (2.27) and (2.27a) in the form

$$\dot z_0=\dot z_1=0,\quad \dot z_n+\lambda_n z_0 z_n=\sum_{k=1}^{n-1}H_{k,n-k}z_k z_{n-k},\quad n=2,\ldots. \tag{3.13}$$

The solution of these equations corresponding to the initial conditions (3.12) can be written down immediately:

$$z_n(t)=z_n^{(0)}e^{-\lambda_n t}+\sum_{k=1}^{n-1}H_{k,n-k}\int_0^t d\tau\, e^{-\lambda_n(t-\tau)}z_k(\tau)z_{n-k}(\tau),\quad n=2,\ldots. \tag{3.14}$$

The series (3.12) with coefficients (3.14) obviously determines a formal solution $\varphi(x, t)$ of the Cauchy problem (3.11), and it remains for us to estimate the growth of $|z_n(t)|$ and $|\dot z_n(t)|$ as $n \to \infty$ (it is convenient not to make use yet of the fact that $z_n(t) > 0$ for $\varphi_0 \in A_*$ but consider the more general case $\varphi_0 \in A$ in (3.11)). Such an estimate leads to the inequalities

$$|z_n(t)|\leqslant a^n,\quad |\dot z(t)|\leqslant 2\delta n a^n;\quad a=\sup\sqrt[n]{|z_n^{(0)}|},\quad \delta=\int_0^1 ds\,\rho(s)s, \tag{3.15}$$

which hold for all $t \geq 0$, $n = 2, \ldots$. The proof of (3.15) is based on elementary application of induction to (3.14) using the connection (2.27a) between λ_n and $H_{k,l}$ and the simple estimate $\lambda_n\leqslant n\delta$ for $n = 2, \ldots$.

Thus, we have constructed the solution $\varphi(x, t)$ to the Cauchy problem (3.11) for $\varphi_0 \in A$ and showed that $\varphi(x, t) \in A$ for all $t > 0$. The constructed solution is obviously unique in the class of functions that can be represented by convergent power series. It remains to prove that $\varphi(x, t) \in A_*$ for all $t > 0$ provided $\varphi_0 \in A_*$. In other words, it is necessary to show that the constructed function $\varphi(x, t)$ will be the characteristic function (3.8) for all t if it was such at $t = 0$. If $\rho(s) \geq 0$ in (3.11) is a function integrable on $[0, 1]$, then for the proof one can use representation of $\varphi(x, t)$ in the form of the Wild's sum [30]

$$\varphi(x,t)=e^{-\rho_0 t}\sum_{n=0}^{\infty}(1-e^{-\rho_0 t})^n\varphi_n(x),\quad \rho_0=\int_0^1 ds\,\rho(s), \tag{3.16}$$

where $\varphi_0(x)$ is the initial condition from (3.11), and

$$\varphi_{n+1}(x)=\frac{1}{(n+1)\rho_0}\sum_{k=0}^{n}\int_0^1 ds\,\rho(s)\varphi_k(sx)\varphi_{n-k}[(1-s)x]. \tag{3.17}$$

The convergence of the series (3.16) for $0 \leq x < \infty$ and the identity of (3.16), (3.17) to the solution (3.12), (3.14) constructed above is readily verified. It follows from the well-known properties of characteristic functions [29] that for all $n = 0, 1, \ldots$ the functions $\varphi_n(x)$ are characteristic functions and, therefore, so is the sum of the series (3.16) for any $t \geq 0$; this is what we had to prove. To prove $\varphi(x, t) \in A_*$ in the case of a nonintegrable function $\rho(s)$ (see the condition (2.8)), it is sufficient to consider a sequence of solutions $\varphi(x, t; \varepsilon_n)$ of problems (3.11) in which the lower limit in the integral is replaced by $s=\varepsilon_n$, $0<\varepsilon_n<1$, $n=0, 1,\ldots$. Going then to the limit $\varepsilon_n \to 0$ and using the fact that the limit of a sequence of characteristic functions that is continuous at $x = 0$ is also a characteristic function, we can readily show that $\varphi(x, t)=\varphi(x, t; +0)\in A_*$.

To investigate the behavior of $\varphi(x, t)$ as $t \to \infty$, we can use the representation of this function in the form of the series (2.30) with coefficients $u_n(t)$, $n = 0, 1, \ldots$. From the recursion relations (2.32) we readily conclude that $u_n \to 0$ as $t \to \infty$ for any fixed $n = 2, \ldots$. On the other hand, an estimate analogous to (3.15) shows that $|u_n(t)|\leqslant(a+1)^n$, $n=0, 1,\ldots$. It follows that $\varphi(x, t) \to \exp(-x)$ as $t \to \infty$, the convergence being uniform on any finite interval $[0, X]$ (or in any disk $|x| < R$ if x is regarded as a complex variable).

To complete the proof of Proposition 1, it is now sufficient to take into account the well-known connection between the convergence of characteristic functions and the convergence of the corresponding probability measures [29].

Thus, we have established above unique solvability (in the large) of the Cauchy problem (2.9)-(2.10) on functions of the class B_* and weak convergence in the limit $t \to \infty$ of its solution $f(|\mathbf{v}|, t)$ to the Maxwell distribution (2.12). For the physics applications of the Boltzmann equation, the main interest attaches to the asymptotic properties of the distribution function $f(|\mathbf{v}|, t)$ as $|\mathbf{v}| \to \infty$ (the formation of Maxwellian tails) and as $t \to \infty$ (relaxation rate). We turn to the investigation of these properties.

Asymptotic Behavior as $|\mathbf{v}| \to \infty$. The asymptotic behavior as $|\mathbf{v}| \to \infty$ of the solution $f(|\mathbf{v}|, t) \in B_*$ to the problem (2.9)-(2.10) can be formulated as follows: What can be said about the tail temperature $\tau(t)$ determined by the equation

$$\tau^{-1} = \sup r: \quad \int d\mathbf{v} f(|\mathbf{v}|, t) \exp(r v^2/2) < \infty, \quad r>0, \tag{3.18}$$

for given initial condition $f_0(|\mathbf{v}|) \in B_*$? We describe here the simplest method of constructing upper and lower bounds of $\tau(t)$ [15].

The method is based on (3.9), which relates $\tau(t)$ to the normalized moments $z_n(t)$,

$$\tau(t) = \limsup_{n\to\infty} \sqrt[n]{z_n(t)}, \quad z_n(t) = \frac{4\pi}{(2n+1)!!} \int_0^\infty dv f(v,t) v^{2(n+1)}, \tag{3.19}$$

and on the monotonic dependence of the non-negative solution $z_n(t)$, $n = 0, 1, \ldots$, of the system (3.13) on the coefficients λ_n, $H_{k,l}$ and the initial conditions $z_n(0)$. Applying induction to the recursion relations (3.14), we can readily establish the following lemma.

LEMMA. Let $\tilde{z}_n(t)$ be a solution to the system of equations obtained from (3.12)-(3.13) by replacing λ_n, $H_{k,l}$, and $z_n^{(0)}$, respectively, by $\tilde{\lambda}_n$, $\tilde{H}_{k,l}$, and $\tilde{z}$ for all $n = 0, 1, \ldots$; $k, l = 1, 2, \ldots$. Then if $\tilde{\lambda}_n \geq \lambda_n$, $0 \leq \tilde{H}_{k,l} \leq H_{k,l}$, $0 \leq \tilde{z}_n \leq z_n$, then $\tilde{z}_n(t) \leq z_n(t)$ for all $t > 0$; conversely, if $\tilde{\lambda}_n \leq \lambda_n$, $\tilde{H}_{k,l} \geq H_{k,l}$, $\tilde{z}_n^{(0)} \geq z_n^{(0)}$, then $\tilde{z}_n(t) \geq z_n(t)$ for all $t > 0$.

The lemma makes it possible to obtain upper and lower bounds for the complete set of normalized moments $z_n(t)$, $n = 0, 1, \ldots$, but for brevity we shall restrict ourselves below to formulations of the results only for the tail temperature $\tau(t)$ (3.19). It is convenient to have in mind already some characteristic values of this quantity: 1) $\tau(0) = 0$ for initial conditions $f_0 \in B_*$ of compact support, i.e., in the case when $f_0(|\mathbf{v}|)=0$ for $|v| > v_0$, where v_0 is some limiting velocity; 2) $\tau = \tau_M = 1$ for the Maxwell distribution (2.12); 3) $\tau(t) \leq 2$ for non-negative functions $f(|\mathbf{v}|, t) \in L_2$ with the norm (2.29); 4) for the exact solution (2.41)

$$\tau(t) = 1 - \theta e^{-\lambda t}, \quad \lambda = \int_0^1 ds\, \rho(s) s(1-s). \tag{3.20}$$

The properties of the tail temperature that are common to all solutions $f(|\mathbf{v}|, t) \in B_*$ of the problem (2.9)-(2.10) are as follows.

PROPOSITION 2. The function $\tau(t)$ does not decrease with increasing t, and for all $t \geq 0$

$$1 - \exp(-\lambda t) \leq \tau(t) \leq \sup_{n=0,1,\ldots} \sqrt[n]{z_n(0)}, \tag{3.21}$$

where λ is defined in (3.20).

Proof. The upper bound in (3.21) follows immediately from the analogous estimate in (3.15). The lower bound is obtained by comparing the solution of the system (3.13) for the initial conditions (3.12) with the exact solution (2.42) of this system, this corresponding for $\theta = 1$ in the notation of the lemma to initial conditions $\tilde{z}_0^{(0)} = \tilde{z}_1^{(0)} = 1$, $\tilde{z}_n^{(0)} = 0$ for $n = 2, \ldots$. Finally, to prove the monotonic dependence of $\tau(t)$ on the time we note that a trivial consequence of the lemma is the inequality $z_n(t) \geq z_n(0) \exp(-\lambda_n t)$, $n=2,\ldots$, which for $\tau(t)$ (3.19) leads to the estimate

$$\tau(t) \geq \limsup_{n\to\infty} [\sqrt[n]{z_n(0)} \exp(-\lambda_n t/n)]. \tag{3.22}$$

But $(\lambda_n/n) \to 0$ as $n \to \infty$, which can be readily deduced from the explicit expression (2.27a) for λ_n by using the upper bound (2.8) for $g(\mu)$. Note that for true Maxwell molecules $\lambda_n \sim n^{1/4}$ as $n \to \infty$ [3]. Thus, it follows from (3.22) that $\tau(t) \geq \tau(0)$ for any $t \geq 0$. This inequality is equivalent to the inequality $\tau(t_1) \geq \tau(t_2)$ for all $t_1 \geq t_2 \geq 0$, since the origin of time can be chosen arbitrarily. This proves Proposition 2.

For applications, an interesting question is that of the manner in which the Maxwell tail is formed in the process of relaxation of initial conditions of compact support when $\tau(0) = 0$. The lower bound in

(3.21) shows that $\tau(0) = 0$ is possible only for $t = 0$, while for $t > 0$ the distribution function $f(|\mathbf{v}|, t)$ decreases roughly speaking as $|\mathbf{v}| \to \infty$ not faster than $\exp[-v^2/2(1-e^{-\lambda t})]$. The fact that every non-negative continuous solution of the Boltzmann equation for $t > 0$ is bounded below by a function of the form $a\exp(-bv^{2+\varepsilon})$ for any $\varepsilon > 0$ was already found by Carleman [31], however for Maxwellian molecules we can obtain much more accurate estimates. Moreover, since in general we consider generalized solutions, it is not the distribution function $f(|\mathbf{v}|, t)$ that is estimated but rather an intergral characteristic of it – the tail temperature $\tau(t)$.

We give an example of more accurate estimates of $\tau(t)$ for one class of initial conditions of compact support, a typical representative of which is the monoenergetic distribution

$$f_0(|\mathbf{v}|)=\frac{1}{2\pi\sqrt{3}}\delta(v^2-3),$$

where the constants are chosen to make the conditions (3.6) satisfied. Assuming the possibility of "smearing" the δ function over a small finite interval, we obtain the class of initial data to which the following proposition applies.

PROPOSITION 3. Suppose in (2.10) $f_0(|\mathbf{v}|)=0$ for $|\mathbf{v}| > v_0$, where $3 \leq v_0^2 \leq 5$. Then for $t \geq 0$

$$1-e^{-\lambda t}\leqslant\tau(t)\leqslant1-\theta e^{-\lambda t},\quad \theta=(1-v_0^2/5)^{1/2},$$

where λ is the same as in Proposition 2.

For the proof, it is sufficient to estimate the normalized moments for $t = 0$ as follows:

$$z_0^{(0)}=z_1^{(0)}=1,\quad 0\leqslant z_n^{(0)}\leqslant\frac{3v_0^{2(n-1)}}{(2n+1)!!},\quad n=2,3,\ldots,$$

and then make an elementary but somewhat lengthy comparison with the exact solution (2.64) for $t = 0$ and $\theta=(1-v_0^2/5)^{1/2}$ and, finally, apply the lemma and the expression (3.19) for $\tau(t)$.

Thus, we have found a class of initial conditions for which the evolution of $\tau(t)$ takes place in accordance with a law close to (3.20) for the exact solution (2.63). At the first glance one might think that for other initial conditions of compact support satisfying the normalization (3.6) $\tau(t)$ will increase with increasing $t \geq 0$ monotonically (see Proposition 2) from the initial value $\tau(0) = 0$ to the value $\tau_M = 1$, which corresponds to the Maxwell distribution. However, it can be shown that there exist initial conditions of compact support for which $\tau(t)\to\tau_\infty>1$ as $t \to \infty$. This, of course, does not contradict the relaxation of the function $f(|\mathbf{v}|, t)$ itself to the equilibrium distribution (2.12). The extent and duration of the exceeding of the equilibrium value $\tau_M = 1$ in the process of relaxation of initial conditions of compact support can be estimated from the following proposition.

PROPOSITION 4. Suppose that in (2.9) $g(\mu) \geq \varepsilon$ for all $-1 \leq \mu \leq 1$ and some $\varepsilon > 0$. Then for all numbers $N > 0$ and $t_0 > 0$ one can find an initial condition $f_0(|\mathbf{v}|)$ (2.10) of compact support such that for all $t > t_0$ the inequality $\tau(t) > N$ holds.

Proof. Using the lemma, we find a lower bound of the solution to the Cauchy problem (3.12)-(3.13) as follows:

1) we replace the initial conditions (3.12) by $\tilde{z}_0^{(0)}=1$, $\tilde{z}_2^{(0)}=z_2^{(0)}$, $\tilde{z}_1^{(0)}=z_3^{(0)}=\ldots=\tilde{z}_n^{(0)}=\ldots=0$;

2) we replace the coefficients λ_n and $H_{k,l}$ of the system (3.13) with allowance for the condition in the proposition by the quantities $\tilde{\lambda}_n=n\delta\geqslant\lambda_n$, $\delta=\int\limits_0^1 ds\,\rho(s)s$, $n=2,\ldots$; $\tilde{H}_{k,l}=\varepsilon(k+l+1)^{-1}\leqslant H_{k,l}$, $k, l=1, 2, \ldots$.

The solution $\{\tilde{z}_n(t),\ n=0, 1,\ldots\}$ to the Cauchy problem "spoilt" in this manner can be expressed, as is readily seen, by

$$\tilde{z}_0=1,\ \tilde{z}_{2m+1}=0,\ \tilde{z}_{2m}=y_m[z_2^{(0)}]^m t^{m-1}e^{-2m\delta t},\quad m=1,\ldots, \tag{3.23}$$

where the numbers y_m are determined recursively: $y_1 = 1$,

$$y_m=\frac{\varepsilon}{(m-1)(2m+1)}\sum_{k=1}^{m-1}y_k y_{m-k},\quad m=2,\ldots.$$

Hence, after application of induction and simple calculations based on the identity $\sum\limits_{k=1}^{m-1}k(m-k)=$

$^1/_6 m(m^2-1)$, we obtain the lower bound $y_m \geqslant m(\varepsilon/12)^{m-1}$ for all m = 1, ... We now substitute this estimate in (3.23), note further that in accordance with the lemma $z_n \geq \tilde{z}_n$ (n = 0, 1, ...), and, finally, we use the representation (3.19) of the function $\tau(t)$. We obtain the inequality

$$\tau(t) \geqslant e^{-\delta t}\sqrt{t\varepsilon z_2(0)/12}, \tag{3.24}$$

which relates the lower bound of the tail temperature $\tau(t)$ to the moment of fourth order of the initial distribution function $f_0 \in B_*$, $z_2(0) = \frac{1}{15}\int d\mathbf{v}\, f_0(|\mathbf{v}|)v^4$.

But for fixed $z_0(0) = z_1(0) = 1$ the quantity $z_2(0)$ can attain arbitrarily large values even for functions of compact support. Indeed, one can specify a function $\tilde{f}_0(v)$ that decreases as $v \to \infty$ in proportion to v^{-6} and then "cut off" its tail at a sufficiently large distance $R \gg 1$ from the coordinate origin. Then the moments z_0 and z_1 will hardly depend on R, but the moment z_2 will increase unboundedly with increasing R. Choosing now an arbitrarily short time interval t_0, we can always make the cutoff radius R sufficiently large that the right-hand side of the inequality (3.24) for $t = t_0$ exceeds a given number $N \gg 1$. By Proposition 2, $\tau(t) \geq \tau(t_0)$ for $t \geq t_0$, and therefore Proposition 4 is proved.

Remark 1. The condition $g(\mu) > \varepsilon$ of Proposition 4 can be weakened, but we shall not do this, since it is very clear and satisfied for the models most frequently used: $g(\mu) \equiv (4\pi)^{-1}$ for isotropic scattering and $g(\mu) \geqslant g(-1)$ for true Maxwellian molecules.

Remark 2. Proposition 4 shows that the class B_* we have chosen is in a certain sense the minimal class of distribution functions containing all solutions of the Cauchy problem (2.9)-(2.10) for initial conditions of compact support. In other words, the restriction (3.5) on the asymptotic behavior as $|\mathbf{v}| \to \infty$ cannot be weakened.

Remark 3. The representation of the solution in the form of the series (2.28) for all $t \geq 0$ is meaningful only when the initial condition $f_0(|\mathbf{v}|)$ is in a comparatively small neighborhood of equilibrium, $\|f_0 - f_M\| < r_0$ (we shall not dwell here on the estimate of r_0). Otherwise, as Proposition 4 shows, the necessary condition $\tau(t) \leq 2$ of convergence of the integral (2.29) can be violated for all $t > t_0$, where $t_0 > 0$ can be chosen arbitrarily small. It is clear that the example of an initial distribution $f_0(|\mathbf{v}|)$ constructed in the Proof of Proposition 4 belongs to the space L_2.

With this, we conclude the study of the asymptotic behavior as $|\mathbf{v}| \to \infty$. Nonmonotonic (in time) behavior of the distribution function at large velocities was noted for the first time in numerical experiments [32] and has often been discussed in the literature at a physical level of rigor [13, 16]. The precise significance of the effect (Proposition 4) became clear only after the introduction in [15] of the concept of the tail temperature and the construction of an asymptotic theory based on this concept.

Asymptotic Behavior as $t \to \infty$. To study this question, it is convenient to transform Eq. (3.11), setting in it

$$\varphi(x, t) = e^{-x}[1 + u(x, t)]. \tag{3.25}$$

We then obtain

$$u_t + \int_0^1 ds\, \rho(s)\{u(x) - u(sx) - u[(1-s)x]\} = \int_0^1 ds\, \rho(s) u(sx) u[(1-s)x]. \tag{3.26}$$

By A_0 we shall denote the set of functions $u(x)$ representable in the form

$$u(x) = \sum_{n=2}^{\infty} u_n \frac{x^n}{n!}, \quad \sup \sqrt[n]{|u_n|} < \infty. \tag{3.27}$$

If $\varphi(x)$ and $u(x)$ are related by the transformation (3.25), then the inclusions $\varphi \in A$ (3.10) and $u \in A_0$ are obviously equivalent.

Since $u(x, t) \to 0$ as $t \to \infty$, it is to be expected that the right-hand side of (3.26) makes an unimportant contribution to the asymptotic behavior. Setting $u(x, t) \approx y(x, t)$ in (3.26) and retaining only the linear terms, we obtain the linearized equation

$$y_t + \int_0^1 ds\, \rho(s)\{y(x) - y(sx) - y[(1-s)x]\} = 0. \tag{3.28}$$

If $y(x, 0) \in A_0$, then the solution of the Cauchy problem for this equation can be written down directly:

$$y(x,t)=\sum_{n=2}^{\infty} y_n(0)e^{-\lambda_n t}\frac{x^n}{n!}, \quad \lambda_n=\int_0^1 ds\,\rho(s)\,[1-s^n-(1-s)^n], \tag{3.29}$$

from which it is clear that $y(x, t) \in A_0$ for all $t \geq 0$. In the following section, we construct an exact transformation relating solutions $u(x, t) \in A_0$ of the nonlinear equation to solutions $y(x, t) \in A_0$ of the linearized equation; here we restrict ourselves to $t \to \infty$ asymptotic estimates.

It is clear from (3.29) that $y(x, t) \sim \exp(-\lambda_2 t)$ as $t \to \infty$. A similar estimate can be expected for the solution $u(x, t) \in A_0$ of the nonlinear equation (3.26). To obtain such an estimate, it is sufficient to represent $u(x, t)$ in the form of the series (3.27), whose coefficients $u_n = u_n(t)$ are determined by (2.32). On the basis of these expressions, it is easy to show that for all $t \geq 0$

$$|u_n(t)|\leqslant\left(\frac{7}{3}\right)^{n/2-1} b^n e^{-\lambda_2 t}, \quad b=\sup_{n=2,\dots}\sqrt[n]{|u_n(0)|}, \quad n=2,\dots. \tag{3.30}$$

Hence, for the solution of the original problem (3.11) we have

PROPOSITION 5. Let $\varphi(x, t) \in A_*$ be the solution of the Cauchy problem (3.11). Then for any $X > 0$ for all $t \geq 0$

$$\|\varphi(x,t)-e^{-x}\|=\sup_{x\in[0,X]}|\varphi(x,t)-e^{-x}|\leqslant K(X)e^{-\lambda_2 t}, \tag{3.31}$$

where $K(X)$ is a positive number that depends on X and the initial condition $\varphi_0(x)$.

From the inequalities (3.30) there follows the validity of Proposition 5 for $K(X)=\exp[X(b\sqrt{7/3}-1)]$. The exponential growth of this quantity does not in general make it possible to obtain an estimate of the type (3.31) for distribution functions $f(|\mathbf{v}|, t)\in B_*$ by simple application of the inversion formula (2.16). However, for a very small class of functions, when $b^2 < 3/7$ in (3.30), this can be done and we obtain as a result the inequality

$$|f(|\mathbf{v}|,t)-(2\pi)^{-3/2}e^{-v^2/2}|<\text{const}\, e^{-\lambda_2 t}, \tag{3.32}$$

which can be regarded as an example. In the general case, the problem of estimating the proximity of distribution functions on the basis of an estimate of the proximity of the corresponding characteristic functions is rather difficult [29], and we shall not go into it, limiting ourselves to the inequality (3.31).

An interesting effect – appreciable slowing down of the relaxation rate already at the level of the characteristic function $\varphi(x, t)$ – occurs in the case of a strong extension of the class B_* of rapidly decreasing distribution functions. Indeed, the linearized equation (3.28) admits solutions of the form

$$y(x,t;p)=x^p e^{-\lambda(p)t}, \quad \lambda(p)=\int_0^1 ds\,\rho(s)\,\{1-s^p-(1-s)^p\}, \tag{3.33}$$

where $\lambda(p) \to +0$ as $p \to 1$, so that $\lambda(p) > 0$ can be made arbitrarily small. From this it is clear how one must choose the initial conditions for the nonlinear equation (3.26) if one is to expect analogous behavior of its solution $u(x, t)$ as $t \to \infty$. We set

$$u_0(x)=\theta x^p, \quad \theta>0, \quad 1<p<2, \tag{3.34}$$

and then in accordance with (3.25) and (3.8) the corresponding initial condition for the Boltzmann equation is

$$f_0(|\mathbf{v}|)=(2\pi)^{-3/2}e^{-v^2/2}\left\{1+\frac{2}{\sqrt{\pi}}\theta\,{}_1F_1\left(-p,\frac{3}{2},\frac{v^2}{2}\right)\right\}, \tag{3.35}$$

where ${}_1F_1(\dots)$ is the confluent hypergeometric function. It can be concluded from the known properties of this function that: 1) for sufficiently small $\theta > 0$ the quantity $f_0(|\mathbf{v}|)$ is non-negative and 2) $f_0(|\mathbf{v}|)\sim|\mathbf{v}|^{-(3+2p)}$ as $|\mathbf{v}| \to \infty$. Thus, $f_0(|\mathbf{v}|)$ is an isotropic distribution function satisfying the normalization conditions (3.6) but, in contrast to functions of the class B_*, $f_0(|\mathbf{v}|)$ decreases as $|\mathbf{v}| \to \infty$ in accordance with a power law.

To construct a solution of the Boltzmann equation with the initial condition (3.35), it is necessary to consider Eq. (3.26) with the initial condition (3.34). It is natural to seek a solution in the form of the series

$$u(x,t)=\sum_{n=1}^{\infty} u_n(t)(\theta x^p)^n, \quad u_n(0)=\delta_{n,1}, \tag{3.36}$$

substitution of which in (3.26) gives us recursion relations of the type (2.32):

$$u_n(t)=\sum_{k=1}^{n-1} h[kp,(n-k)p]\int_0^t d\tau\, u_k(\tau)u_{n-k}(\tau)\exp[-\lambda(np)(t-\tau)], \quad n=2,\ldots,$$
$$u_1(t)=\exp[-\lambda(p)t], \quad h(\alpha,\beta)=\int_0^1 ds\,\rho(s)s^\alpha(1-s)^\beta. \tag{3.37}$$

Assuming for simplicity of the proof that the function $\rho(s)$ is bounded (pseudo-Maxwellian molecules: $0\leqslant\rho(s)\leqslant\rho_M$), we can readily obtain by induction from (3.37) the estimate

$$0\leqslant u_n(t)\leqslant\left[\frac{\rho_M}{\lambda(p)}\right]\frac{[\Gamma(p-1)]^n}{\Gamma(np+1)}e^{-\lambda(p)t}, \quad n=1,\ldots. \tag{3.38}$$

Hence, taking into account (3.36) and (3.25), we obtain for the characteristic function $\varphi(x,t)$ the inequality

$$\theta x^p e^{-x-\lambda(p)t}\leqslant\varphi(x,t)-e^{-x}\leqslant Axe^{(D-1)x-\lambda(p)t}, \quad A=\lambda(p)D/\rho_M, \quad D^p=\theta\rho_M\Gamma(p+1)/\lambda(p). \tag{3.39}$$

For sufficiently small $\theta>0$, the function $\varphi(x,t)$ decreases exponentially as $x\to\infty$ and the corresponding distribution function $f(|v|,t)$ can be obtained from the inversion formula

$$f(|v|,t)=(2\pi)^{-3}\int d\mathbf{k}\,e^{-i\mathbf{k}\mathbf{v}}\varphi\left(\frac{k^2}{2},t\right), \tag{3.40}$$

which completes the construction of the solution of the Boltzmann equation (2.9) with initial condition (3.35). The estimate (3.39) enables us to prove readily the following proposition.

PROPOSITION 6. Suppose that in the Boltzmann equation (2.9) $g(\mu)$ is a bounded function. Then for any $0\leq\delta<\lambda_2$ it is possible to find a non-negative solution $f(|v|,t)$ of this equation satisfying the normalization (3.6) and such that for all $t\geq 0$

$$\|M-M_M\|_L=\int d\mathbf{v}\,|f(|\mathbf{v}|,t)-(2\pi)^{-3/2}e^{-v^2/2}|\leqslant C_1e^{-\delta t}, \tag{3.41}$$

the corresponding characteristic function $\varphi(k^2/2,t)$ satisfying the inequalities

$$C_1e^{-\delta t}\leqslant\|\varphi-\varphi_M\|_C\leqslant C_2e^{-\delta t}, \quad \|\varphi-\varphi_M\|_C=\sup_{0\leqslant x<\infty}|\varphi(x,t)-e^{-x}|, \tag{3.42}$$

where C_1 and C_2 are positive constants.

The proof has actually already been given above by the explicit construction of such a solution. Indeed, it is simply necessary to choose in (3.34)-(3.35) the quantity $p_0>1$ such that $\lambda(p_0)=\delta$. Then (3.42) follows from (3.39), and (3.41) from the well-known relation $\|\varphi-\varphi_M\|_C\leqslant\|f-f_M\|_L$ between an integrable function and its Fourier transform.

Remark. Proposition 5 also remains true in the general case (2.8), but the proof of an estimate of the type (3.38) in this case becomes rather lengthy.

Thus, initial conditions with power-law tails can lead to equilibrium much more slowly than distribution functions that decrease rapidly as $|\mathbf{v}|\to\infty$. The existence of such solutions was suggested by the linear theory [4, 8]. A general method of solving the nonlinear Boltzmann equation in the class of slowly decreasing distribution functions based on formal "reduction" of this equation to a system of ordinary differential equations of the type (2.31) is described in [33]. For a different approach to this problem, see [34].

4. Equivalence Theorem and Consequences

We now establish an exact correspondence between the nonlinear equation (3.26) and the linearized equation (3.28), these being considered on functions of the class A_0, i.e., on functions $u(x,t)$ and $y(x,t)$ that for all $t\geq 0$ can be represented by power series:

$$u(x,t)=\sum_{n=2}^{\infty} u_n(t)\frac{x^n}{n!},\quad y(x,t)=\sum_{n=2}^{\infty} y_n(t)\frac{x^n}{n!},\tag{4.1}$$

$$\sup \sqrt[n]{|u_n(t)|}<\infty,\quad \sup \sqrt[n]{|y_n(t)|}<\infty.\tag{4.2}$$

For such functions we shall write $u(x, t) \in A_0$ and $y(x, t) \in A_0$ and omit the variable t when it is not important. Restricting ourselves to the case of pseudo-Maxwellian molecules, we formulate the main result of this section in the form of a theorem.

THEOREM. Suppose $\rho(s) \geq 0$ and that $\rho(s)$ is integrable on $[0, 1]$. Then there exist nonlinear operators (transformations) $\hat{R}$ and $\hat{S}$ acting on the variable x from A_0 to A_0 and such that:

1) if $y(x, t) \in A_0$ is a solution of (3.28), then $\hat{R}y = u(x, t) \in A_0$ is a solution of (3.26);

2) if $u(x, t) \in A_0$ is a solution of (3.26), then $\hat{S}u = y(x, t) \in A_0$ is a solution of (3.28);

3) $\hat{R}\hat{S} = \hat{S}\hat{R} = \hat{I}$, where $\hat{I}$ is the identity operator;

4) the corresponding transformations of the coefficients of the series (4.1) are polynomial:

$$u=\hat{R}y \Leftrightarrow u_n=y_n+P_n(y_2,\dots,y_{n-2}),\ n=2,\dots,\qquad y=\hat{S}u \Leftrightarrow y_n=u_n+Q_n(u_2,\dots,u_{n-2}),\ n=2,\dots,$$

where $P_n(\dots)$ and $Q_n(\dots)$ are polynomials of degree $[n/2]$ not containing terms of zeroth and first powers.

The proof can be done in two stages – first the construction in explicit form of the transformations $\hat{R}$ and $\hat{S}$ and then the verification that all the propositions of the theorem do indeed hold for these transformations.

Considered on functions of the class A_0, Eqs. (3.26) and (3.28) reduce to corresponding systems of equations for the coefficients of the series (4.1):

$$\dot{u}_n+\lambda(n)u_n=\sum_{\substack{k_1,k_2\geq 2\\ k_1+k_2=n}} H(k_1,k_2)u_{k_1}u_{k_2},\quad n=2,3,\dots,\tag{4.4}$$

$$\dot{y}_n+\lambda(n)y_n=0,\quad n=2,3,\dots,\tag{4.5}$$

where we have used the notation (2.27), (2.27a). For n = 2, 3 the summation condition $k_1 + k_2 = n$ on the right-hand side of (4.4) cannot be satisfied, and we assume in such cases by definition that the right-hand side of (4.4) is zero. We consider the reduction of the system (4.4) to normal form. Applying the usual method [35], we can readily conclude that for reduction of this system to its linear part the condition

$$\Delta_m(k_1,\dots,k_m)=\lambda\left(\sum_{j=1}^{m} k_j\right)-\sum_{j=1}^{m}\lambda(k_j)\neq 0\tag{4.6}$$

for all natural $m\geq 2$, $1\leq j\leq m$, $k_j\geq 2$ is sufficient. In other words, because of the special form of the right-hand side of (4.4) only resonances of a certain type that violate the condition (4.6) are important. From the expression (2.27a) for the eigenvalues there follow the inequalities

$$\lambda(n)+\lambda(m)-\lambda(n+m)\geq 2\lambda(2)-\lambda(4)=2\int_0^1 ds\rho(s)s^2(1-s)^2\tag{4.7}$$

for n, m = 2, 3, ... Therefore, the condition (4.6) is indeed satisfied. In addition, it can readily be verified that the system (4.4) can be reduced to the normal form (4.5) by polynomial transformations, i.e., the Poincaré series terminate. If we were talking about finite systems (4.4), (4.5) for n = 2, ..., N, our arguments would be sufficient to conclude that these systems are equivalent. However, since we are interested in the equivalence of Eqs. (3.26) and (3.28) themselves, it is necessary to establish that the reduction to normal form does not take us outside the class A_0, i.e., that it preserves inequalities of the type (4.2). This requires a more detailed knowledge of the normalizing transformation.

We seek such a transformation – a change of variables in (4.4) – in the form

$$u_n=\sum_{m=1}^{[n/2]}\sum_{\substack{k_1,\dots,k_m\geq 2\\ k_1+\dots+k_m=n}} r_m(k_1,\dots,k_m)\prod_{j=1}^{m} y_{k_j},\quad r_1(k_1)\equiv 1,\quad n=2\dots$$

Here, the coefficient $r_1(k_1)$ of the linear term is taken equal to unity, as always in the method of normal forms [35], and the structure of the nonlinear terms is suggested by the form of the right-hand side of (4.4). Substituting (4.8) in (4.4) and requiring that the transformation (4.8) carry an arbitrary solution of the system (4.5) into a solution of the system (4.4), we obtain after simple but lengthy calculations the following expressions for the recursive calculation of $r_m(\ldots)$:

$$r_1(k_1)=1,\quad r_m(k_1,\ldots,k_m)\Delta_m(k_1,\ldots,k_m)=$$

$$\sum_{j=1}^{m-1} H(k_1+\ldots+k_j,k_{j+1}+\ldots+k_m)r_j(k_1,\ldots,k_j)r_{m-j}(k_{j+1},\ldots,k_m),\quad m=2,\ldots, \tag{4.9}$$

where we have used the notation (4.6).

It is clear that $r_m(k_1,\ldots,k_m)$ in (4.8) are determined only up to any permutation of the arguments $k_1,\ldots,k_m$. The lack of uniqueness can be eliminated by symmetrization, but we do not do this but rather choose $r_m(k_1,\ldots,k_m)$ in such a way that they are determined by the simplest expressions (4.9).

We construct similarly a transformation that carries a solution of the system (4.4) into a solution of the system (4.5). The result has the form

$$y_n=\sum_{m=1}^{[n/2]}\sum_{\substack{k_1,\ldots,k_m\geq 2\\ k_1,\ldots,k_m=n}} s_m(k_1,\ldots,k_m)\prod_{j=1}^{m}u_{k_j},\quad n=2,\ldots; \tag{4.10}$$

$$s_1(k_1)=1,\quad s_m(k_1,\ldots,k_m)\Delta_m(k_1,\ldots,k_m)=-\sum_{j=1}^{m-1}H(k_j,k_{j+1})s_{m-1}(k_1,\ldots,k_{j-1},k_j+k_{j+1}+k_{j+2},\ldots,k_m),\ m=2,\ldots. \tag{4.11}$$

The fact that the transformations (4.8)-(4.9) and (4.10)-(4.11) are mutually invertible follows from the uniqueness and one-to-one invertibility of the normalizing transformation. The expressions (4.8)-(4.11) determine the explicit form of the polynomials P_n and Q_n in (4.3), and the corresponding transformations $\hat{R}$ and $\hat{S}$ of the functions $y(x,t)\in A_0$ and $u(x,t)\in A_0$ can obviously be determined on the basis of representation of these functions in the form of the series (4.1).

To complete the proof of the theorem, it is sufficient to establish that the transformations $\hat{R}$ and $\hat{S}$ map from A_0 to A_0. This can be done as follows. First, from the recursion relations (4.9) and (4.11), using the condition of the theorem and the inequality (4.7), we obtain by induction the estimate

$$|r_m(k_1,\ldots,k_m)|\leqslant A^{m-1},\quad |s_m(k_1,\ldots,k_m)|\leqslant A^{m-1},\quad m=1,\ldots, \tag{4.12}$$

where

$$A=\int_0^1 ds\rho(s)/2\int_0^1 ds\rho(s)s^2(1-s)^2. \tag{4.13}$$

Now suppose $y(x)\in A_0$; then in (4.1) $|y_n|\leqslant a^n$ for some $a>0$ and all $n=2,\ldots$. We consider a function $u=\hat{R}y$ such that the coefficients u_n and y_n of the series (4.1) are related by Eqs. (4.8). The inclusion $u(x)\in A_0$ is equivalent to the generating function

$$F(z)=\sum_{n=2}^{\infty}u_nz^n \tag{4.14}$$

being analytic in some neighborhood of the point $z=0$. Substituting (4.8) in (4.14) and estimating $|F(z)|$ with allowance for (4.12), we obtain the simple inequality

$$|F(z)|\leqslant a^2|z|^2[1-a|z|-Aa^2|z|^2]^{-1},$$

which guarantees convergence of the series (4.14) in some disk $|z|<r$. Therefore, $u(x)\in A_0$, i.e., the transformation $\hat{R}$ maps from A_0 to A_0. The proof of the analogous fact for the inverse transformation $\hat{S}$ reduces to the argument given here by a simple change of the notation. The theorem is proved.

Remark. The local – in a sufficiently small neighborhood of the equilibrium solution – equivalence of the nonlinear and linearized Kac equations [24] on functions of the Hilbert space L_2 was established in [36], and then in [37] an analogous proposition for the Boltzmann equation (2.9) was formulated without proof. The theorem proved here differs from the results of [36, 37] above all in the global nature of the equivalence, and also by the choice of a class of functions different from L_2 (in L_2 only local equivalence is possible) and by the explicit representation of the normalizing transformation.

It follows from this theorem that Eq. (3.26) admits extensive families of particular solutions that generalize the class of self-similar solutions described in Sec. 2.

COROLLARY 1. Equation (3.26) for any N = 1, ... and any set of natural numbers $n_1 \geqslant 2, \ldots, n_N \geqslant 2$ has solutions of the form

$$u_N(x, t) = U_N[\gamma_1 x^{n_1} e^{-\lambda_{n_1} t}, \ldots, \gamma_N x^{n_N} e^{-\lambda_{n_N} t}], \quad u_n \in A_0, \tag{4.15}$$

where $U_N(z_1, \ldots, z_N)$ is an analytic function of N variables, and $\gamma_1, \ldots, \gamma_N$ are arbitrary constants. These constants can be chosen sufficiently small in absolute magnitude for substitution of (4.15) in (3.25) and (3.40) to determine the solution $f_N(|\mathbf{v}|, t)$ of the Boltzmann equation (2.9).

The solution $u_N(x, t)$ can be constructed as follows. In (4.8), we set

$$y_{n_1} = \gamma_1 \exp(-\lambda_{n_1} t), \ldots, y_{n_N} = \gamma_N \exp(-\lambda_{n_N} t), \quad y_n = 0, \quad n \neq n_j, \tag{4.15a}$$

for all j = 1, ..., N. Substituting the result in the series (4.1) for u(x, t), we obtain

$$u_N(x, t) = \sum_{m=1}^{\infty} \sum_{i_1=1}^{N} \frac{r_m(n_{j_1}, \ldots, n_{j_m})}{(n_{j_1} + \ldots + n_{j_m})!} \prod_{k=1}^{m} \gamma_{j_k} x^{n_{j_k}} \exp(-\lambda_{n_{j_k}} t), \tag{4.16}$$

which determines the right-hand side of (4.15). The growth of $|u_n(x, t)|$ as $|x| \to \infty$ can be estimated in the same way as in the proof of the theorem. It is clear that for N = 1 the result reduces to the self-similar solutions constructed in Sec. 2. The expression (4.16) determines a solution of Eq. (3.26) for nonintegral $n_1 \geqslant 2, \ldots, n_N \geqslant 2$ as well if the substitution $z! = \Gamma(z + 1)$ is made for nonintegral z [14].

Thus, the direct transformation $u = \hat{R}y$ makes it possible to distinguish special classes of particular solutions. We now give an example of the use of the inverse transformation $y = \hat{R}u$.

COROLLARY 2. Let $f(|\mathbf{v}|, t) \in B_*$ be a solution of the Boltzmann equation (2.9). There exists a countable set of functionals $\Gamma_n[f]$, $n = 2, 3, \ldots$, which are conserved in time.

To prove this, we note that with every solution $f(|\mathbf{v}|, t) \in B_*$ of Eq. (2.9) it is possible to associate a formal (in general, divergent) series of the form (2.28), where

$$u_n(t) = \frac{(2n)!!}{(2n+1)!!} \int d\mathbf{v} f(|\mathbf{v}|, t) \mathscr{L}_n^{1/2}\left(\frac{v^2}{2}\right), \quad n = 2, \ldots. \tag{4.17}$$

Irrespective of the convergence with respect to the metric L_2 of the series (2.28), its coefficients (4.17) uniquely determine in accordance with (2.30) a characteristic function $\varphi(x, t) \in A_*$, and, therefore, a distribution function $f(|\mathbf{v}|, t) \in B_*$. The time evolution of quantities (4.17) can be described by the system (4.4), and in accordance with the theorem there exists a set of polynomials (4.10) of the form

$$y_n(t) = u_n(t) + Q_n[u_2(t), \ldots, u_{n-2}(t)], \quad n = 2, 3, \ldots, \tag{4.18}$$

these varying in time purely exponentially,

$$y_n(t) = y_n(0) e^{-\lambda_n t}, \quad \lambda_n = \int_0^1 ds \rho(s) [1 - s^n - (1-s)^n], \quad n = 2, 3, \ldots. \tag{4.19}$$

If $y_{n_0}^{(0)} \neq 0$ for some n_0, then obviously

$$\Gamma_n = |y_n(t)|^{\lambda_{n_0}} / |y_{n_0}(t)|^{\lambda_n} = \text{const}, \quad n = 2, 3, \ldots, \tag{4.20}$$

and Eqs. (4.17)-(4.18) show that Γ_n is indeed a functional defined on functions of the class B_*. The case $y_n \equiv 0$ for all n = 2, 3, ... is of no interest, since it corresponds to the trivial solution (2.12).

It is not the invariants (4.20) that are of practical interest but the functions $y_n(t)$ (4.18), which may be called normal coordinates of the solution $f(|\mathbf{v}|, t)$ of the Boltzmann equation. The transformation that is the inverse of (4.18) is also polynomial (4.8) and can be expressed in the form

$$u_n(t) = y_n(t) + P_n[y_2(t), \ldots, y_{n-2}(t)], \quad n = 2, 3, \ldots. \tag{4.21}$$

We describe the scheme of solution of the Cauchy problem (2.9)-(2.10) for $f_0 \in B_*$ by the method of transformation to normal coordinates. In the first step, we calculate in accordance with the expressions

(4.17) the sequence $\{u_n(0),\ n = 2, \ldots\}$, and then in accordance with the expressions (4.18) the sequence $\{y_n(0),\ n = 2, \ldots\}$ of initial values of the normal coordinates. In the second (trivial) stage, the expressions (4.19) are used to calculate the sequence $\{y_n(t),\ n = 2, \ldots\}$ for all $t > 0$. In the third stage, the expressions (4.21) are used to calculate the sequence $\{u_n(t),\ n = 2, \ldots\}$ for all $t > 0$, and then, in accordance with (2.30), the characteristic function $\varphi(x, t) \in A_*$ is constructed. In this paper we nowhere consider the problem of recovering the distribution function $f(|\mathbf{v}|, t) \in B_*$ from its characteristic function $\varphi(x) \in A_*$, restricting ourselves to the remark that there is a one-to-one correspondence between these functions. Therefore, the third stage can be regarded as the final one.

From the practical point of view, such a scheme is unnecessarily complicated, since it is possible to calculate $\{u_n(t),\ n = 2, \ldots\}$ using the recursion relations (2.32) without recourse to normal coordinates. However, this scheme is helpful not only for understanding the structure of the general solution of the nonlinear Boltzmann equation but also for comparison of its properties with the properties of other equations, in particular equations of Korteweg–de Vries type. It is easy to see an analogy between the scheme we have described and the classical scheme for integrating the KdV equation by the inverse scattering method. The normal coordinates, whose evolution has the form (4.19), correspond here to the scattering data, and the transformations (4.18) and (4.21) correspond, respectively, to the solution of the direct and inverse scattering problems. It is well known that N-soliton solutions of the KdV equation correspond to the case when the set of scattering data reduces to a finite set of numbers (reflectionless potentials). In the considered scheme, this case corresponds to a finite set of nonzero normal coordinates (4.15a), and the analog of N-soliton solutions are for the Boltzman equation the solutions $f_N(|\mathbf{v}|, t)$ described in the formulation of Corollary 1. Other aspects of the formal analogy between the equations of KdV and Boltzmann type are pointed out in [14, 38].

5. Conclusions

We mention some generalizations and applications of the methods described here. The simplest generalizations of Eq. (2.9) are the system of equations for a mixture of Maxwellian gases [33, 39] and the analog of this equation in Euclidean space of arbitrary dimension [10]. Fourier transformation leads here to the same simplifications, and the theory of relaxation can be constructed in exactly the same way. However, for a system, as for one equation in the general – anisotropic – case [25], resonances may arise, and, therefore, the nonlinear and linearized equations are not in general equivalent. Less trivial is the generalization to the case of a velocity-dependent collision frequency. The expression (2.5) indicates the simplifications that arise in the case when $g(u, \cos\theta)$ is a polynomial in u^2. Indeed, the two-dimensional Boltzmann equation for $g(u, \cos\theta) = u^2|\sin\theta|$ is exactly solvable for isotropic distribution functions [40, 41]. Generalizations to equations associated with the theory of polymers are discussed in [16, 17].

A natural application of the exact solutions is to the analysis of approximate methods. In connection with Grad's method, we mention the examples constructed in Sec. 3 of convergence and divergence of the series (2.28) in the relaxation problem. Examples of convergent and divergent Hilbert–Chapman–Enskog series encountered in model nonlinear problems are described in [39, 42]. However, the main question for the Chapman–Enskog method is not the problem of the convergence of the series but the problem of making the Navier–Stokes hydrodynamics more accurate. Can one assume, ignoring boundary-value problems and making a restriction to Cauchy problems, that the Burnett equations are more accurate forms of the Navier–Stokes equations at sufficiently small Knudsen numbers? The answer is evidently in the negative. The equilibrium solutions of the Burnett equations are unstable with respect to small periodic perturbations with wavelength less than or of the order of the mean free path [43]. This unphysical instability has the consequence that the behavior of the solutions of the Burnett equations differs qualitatively from the behavior of the solutions of the Navier–Stokes and Boltzmann equations. Thus, despite its well-known logical elegance, the Chapmann–Enskog method requires certain modifications.

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