of interacting Yang–Mills fields and massless fermions.

LITERATURE CITED

1. L. S. Adler. Phys. Rev., **177**, 2426 (1969).
2. W. A. Bardeen, Phys. Rev. **184**, 1848 (1969).
3. D. J. Gross and J. Jackiw, Phys. Rev. D, **6**, 477 (1972).
4. R. Stora, "Algebraic structure and topological origin of anomalies," Preprint LAAP-TH-94 (1983). (Seminar given at the Cargese Institute of Theoretical Physics, 1983).
5. A. Andrianov, L. Bonora, and S. Pasti, "Anomalies, cohomology and finite mode regularization in higher dimensions," Preprint DFPD 30/83 (1983).
6. J. Wess and B. Zumino, Phys. Lett. B, **37**, 95 (1971).
7. S. P. Novikov, Dokl. Akad. Nauk SSSR, **260**, 31 (1981).
8. S. P. Novikov, Usp. Mat. Nauk, **37**, 3 (1982).
9. A. A. Slavnov and L. D. Faddeev, Introduction to the Quantum Theory of Gauge Fields [in Russian], Nauka, Moscow (1978).
10. S. S. Chern and J. Simons, Proc. Natl. Acad. Sci. USA, **68**, 791 (1971).
11. R. Jackiw, "Topological investigations of quantized gauge theories," Preprint CTP-1108 (1983).
12. S. Deser, R. Jackiw, and S. Templton, Ann. Phys. (N.Y.), **140**, 372 (1982).
13. E. Witten, Nucl. Phys. B, **223**, 422 (1983).
14. K.-C. Chou, H.-Y. Guo, K. Wu, and X.-C. Song, Phys. Lett. B, **134**, 67 (1984).
15. L. Alvarez-Gaumé and E. Witten, "Gravitational anomalies," Preprint HUTP-83/A039 (1983).
16. H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press (1956).
17. D. K. Faddeev, Dokl. Akad. Nauk SSSR, 8, 361 (1947).

RENORMALIZATION GROUP AND FUNCTIONAL SELF-SIMILARITY IN DIFFERENT BRANCHES OF PHYSICS

D. V. Shirkov

A general formulation of "different" renormalization groups is given for different branches of physics – quantum field theory, the theory of kinetic phenomena, turbulence theory, polymer physics, the theory of radiative transfer. The unified formulation uses the language of group transformations and functional equations. The transformations and equations are based on a simple property, functional self-similarity, which is a generalization of ordinary self-similarity. The difference between the physical basis of the renormalization-group transformations in systems with infinitely large number of degrees of freedom and the functional self-similarity of simple physical systems is discussed.

1. Introduction

During the last rather more than ten years there has been a remarkable extension of the ideas and methods of the renormalization group for different branches of physics. The renormalization group arose originally about 30 years ago in quantum field theory. The existence of a special group of continuous transformations associated with a finite arbitrariness that arises as a result of the quantum-field procedure of elimination of ultraviolet divergences was established in 1953 by Stueckelberg and Petermann [1], who introduced the term "la groupe de normalisation." Subsequently, in studies during 1954-1955 [2, 3], renormalization-group transformations were realized as specific transformations (of the type of Dyson transformations) of vertex functions and propagators with a simultaneous stretching transformation of the energy and momentum scales. It was found that these transformations could be expressed in the form of functional equations for the Green's functions and new specific quantities – the effective coupling constants. Very helpful from the practical point of view are the differential group equations corresponding to infinitesimally small group

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transformations; these were obtained for the first time in [3] (and for the multiply charged case in [4]). On the basis of these equations, Bogolyubov and Shirkov [3] proposed a regular procedure for improving the results of perturbation theory in the ultraviolet and infrared regions, i.e., in regions in which the solutions of the equations of motion have singular behavior. This procedure, which was first successfully applied [5] to the asymptotic behavior of the Green's functions of quantum electrodynamics, is now known as the renormalization-group method.

At the beginning of the seventies, the renormalization-group method was successfully used [6] in the theory of critical phenomena in statistical physics. The renormalization group here obtained a simpler physical formulation on the basis of the so-called Kadanoff mechanism [7]. The mechanism corresponds to the possibility of an equivalent description of the properties of a macroscopic sample in the neighborhood of a phase transition point by means of a sequence of different microscopic models related by a transformation that changes the magnitude of the "elementary" microscopic scale (for example, the lattice constant of a crystal) with a simultaneous appropriate change in the coupling constants.

This physical basis of the renormalization group, which is clearer than in quantum field theory, facilitated its further extension into other branches of physics. For example, the renormalization-group approach to turbulence theory [8, 9] is based on invariance of the macroscopic characteristics with respect to a change in the lower (in frequency) limit of the high-frequency fluctuations, which are not described by equations but are taken into account phenomenologically by certain suitable parameters (for example, an effective Reynolds number).

It is clear that a procedure for cutting off high frequencies is physically close to restrictions on minimal sizes. Similar considerations, which exploit the freedom in the choice of the "minimal block" in a long molecule, form the basis of the use of the renormalization-group method in the theory of polymers [10].

An original use of the idea of changing the "frequency limit" was recently made in [11] in the theory of radiative transfer in opaque media with a strong frequency dependence of the photon mean free path.

It is curious that at almost the same time renormalization-group associations arose in two-dimensional transport theory in an entirely different context. Using Ambartsumyan's invariance principle [12], it proved possible to obtain [13] a functional formulation of the corresponding symmetry properties in the real configuration space in a form identical to the renormalization-group functional equations. The transformation corresponding to the renormalization-group transformation changing the momentum or frequency scale is in the present case a shift with respect to the spatial coordinate.

The remarkable generality of the mathematical formalism based on functional group equations was found [14] to have a rather simple basis, which can be explicitly formulated for examples of classical mechanical systems. This basis, which we have called functional self-similarity (or automodelity) [14, 15], corresponds to the property of transitivity of the characteristics of a physical system with respect to the method of specifying their initial or boundary conditions. The renormalization-group transformations of quantum field theory and the theory of critical phenomena are special cases of functional self-similarity transformations.

In this paper, we begin by introducing functional self-similarity transformations in general form and establish the connection between functional self-similarity and ordinary self-similarity; we then discuss the correspondence between the complex construction of Kadanoff type that is the basis of the renormalization group in the theory of critical phenomena and the simpler symmetry property that leads to functional self-similarity of simple dynamical systems.

2. Functional Self-Similarity Transformations

We consider a set of one-parameter transformations of two quantities x and g characterized by a continuous positive parameter t,

$$x \to x' = x/t, \quad g \to g' = \bar{g}(t, g), \tag{1}$$

where $\bar{g}(t, g)$ is a single-valued function of its arguments that satisfies the condition

$$\bar{g}(1, g) = g. \tag{2}$$

It can be seen that with respect to x these transformations are scaling transformations with a group nature. The functional transformations of g form a group if the transformation function $\bar{g}$ satisfies the

composition condition

$$\bar{g}(t_1t_2, g)=\bar{g}(t_2, \bar{g}(t_1, g)), \tag{3}$$

which is a functional equation for $\bar{g}$. We shall call such transformations functional self-similarity transformations.

A function f of the arguments x and g forms a representation of the functional self-similarity group if it transforms under the transformations (1) in accordance with

$$f(x, g)\to f(x', g')=zf(x, g), \tag{4}$$

where z is a constant that depends on the transformation parameter. The special case z = 1 corresponds to an f that is an invariant of the transformation.

Setting $t_1t_2 = x$ and $t_1 = t$ in (3), we write it in the form

$$\bar{g}(x, g)=\bar{g}(x/t, \bar{g}(t, g)), \tag{5}$$

from which it can be seen that the function $\bar{g}(x, g)$ is an invariant of the transformation.

In quantum field theory, the transformations (1) and the relations (4) and (5) correspond to the simplest, so-called single-charge renormalization group in the massless case. The variable x is the square of the 4-momentum, and g is the coupling constant of the fields. The function $\bar{g}$ is called the effective, or invariant, coupling constant. Generalizations of the considered transformation correspond to more complicated quantum-field renormalization groups. We shall consider some of them.

Suppose that in addition to the argument x, the main physical variable of the problem, there is a parameter X having the same dimensions; in the quantum-field case, this is the square of the particle mass. Instead of (1), we write

$$x\to x'=x/t, \quad X\to X'=X/t, \quad g\to\bar{g}(t, X, g). \tag{6}$$

Thus, the transformation changes the scales of x and X in the same way. In addition, the parameter X occurs in the transformation function $\bar{g}$, the functional equation for which takes the form

$$\bar{g}(x, X, g)=\bar{g}(x/t, X/t, \bar{g}(t, X, g)). \tag{7}$$

Equation (4) is modified similarly. In quantum field theory, these transformations correspond to the renormalization group for a model of interaction with one coupling constant and one mass. The parameter X can be "multiplied." The case with several parameters X corresponds to quantum-field models with several different masses, for example, quantum chromodynamics.

One can also "multiply" the argument x, which does not occur in the transformation function $\bar{g}$. However, the nature of the group is not changed by this. Of greater consequence is "multiplication" of the argument g. Suppose that instead of one g there are two: g and h. We write the main transformation in the form

$$x\to x/t, \quad g\to\bar{g}(t; g, h), \quad h\to\bar{h}(t; g, h). \tag{8}$$

The conditions of group consistency of the transformations of the arguments g and h now have the form of the system of functional equations

$$\begin{cases}\bar{g}(x; g, h)=\bar{g}(x/t; g_t, h_t); & g_t=\bar{g}(t, g, h),\\ \bar{h}(x; g, h)=\bar{h}(x/t; g_t, h_t); & h_t=\bar{h}(t, g, h).\end{cases} \tag{9}$$

In quantum field theory, such transformations correspond to a so-called two-charge renormalization group, i.e., the group for a model of interacting fields with two coupling constants. It is not difficult to write down transformations with several arguments of the type X and several arguments of the type g simultaneously and so forth.

Leaving quantum field theory, we note that the transformations of the so-called Wilson renormalization group in the theory of critical phenomena can readily be given the form (1). It must only be borne in mind that the original formulation was based on the well-known Kadanoff considerations [7], which contain discrete transformations of the macroscopic scales. To this there correspond discrete (integral) values of the transformation parameter t in (1). Such transformations do not always have inverses and form a semigroup. The continuous renormalization-group transformations in the theory of turbulence and the other branches of physics noted above correspond exactly to the case (1) or to one of the given generalizations.

We now establish the connection with ordinary self-similarity. For this, we must consider solutions of some of the functional equations. We note in this connection that the general solutions of these equations found [16] in the middle of the fifties contain arbitrary functions with number of arguments one fewer than in the solution determined by them. For our purposes, it is sufficient to restrict ourselves to solutions linear in the second argument of the form $\bar{g}(x, g)=gf(x)$. Substituting such an expression in (5), we readily find $\bar{g}(x, g)=gx^{\nu}$. Using this expression in the main transformation (1), we see that it has taken on the form of a power-law similarity transformation, the basis of the self-similarity used in problems of hydrodynamics and gas dynamics. Thus, the transformations (1) are a functional generalization of ordinary self-similarity, and this justifies their name.

3. Physical Discussion

It follows from the above brief review of the development of the renormalization-group method in physics that, however paradoxical it may appear, the historical development has proceeded "from the more complicated to the simpler." Indeed, the basis of the renormalization group in quantum field theory is the least clear. In the pioneering paper [1], the renormalization-group transformations were introduced as formal transformations in a space of finite parameters without any discussion of their physical meaning. The following paper [2], devoted to quantum electrodynamics, used a fairly complicated momentum cutoff procedure, the cutoff momenta playing simultaneously the part of normalization momenta, in particular charge normalization. A physically perspicuous image corresponds to an electron of finite size "smeared" over a small region of radius $R_{\Lambda}=\hbar/c\Lambda$, the relation $\ln(\Lambda/m_e)\gg 1$ holding. The charge $e(\Lambda)$ of this electron is assumed to depend on the cutoff momentum Λ in such a way that this dependence reflects the effects of vacuum polarization due to virtual e^+e^- pairs at distances less than R_{Λ} from the real point electron. When the radius is increased to the electron Compton wavelength, we arrive at the well-known experimental value of the charge: $e^2(m_e)=\hbar c/137$. In this language, the renormalization-group transformation is a transition from one smearing radius, i.e., one cutoff momentum, to another, $R_1 \to R_2$, with a simultaneous corresponding change in the charge: $e(\Lambda_1)\to e(\Lambda_2)$. One can say that this is a transition from one nonlocal physical model to another, each of them being equivalent to a local model at large distances $r\gtrsim\hbar/m_e c$.

The so-called Wilson renormalization group in the theory of critical phenomena considers a Kadanoff sequence of "block" lattices with increasing size a, $2a$, $3a$, ..., na, ..., of the block, the coupling constants of neighboring blocks forming a sequence $K_1, K_2, K_3, \ldots, K_n, \ldots$, whose general term can be represented in the form $K_n = \bar{K}(n, K)$. A renormalization-group transformation is a set of transformations changing the scale of the microscopic distances, $r \to nr$, and a simultaneous transformation $K \to \bar{K}(n, K)$ of the coupling constant. The explicit form of the dependence $\bar{K}$, like the function $e(\Lambda)$ in the previous example, can be determined from the equations of motion. Like the foregoing case, the transition from a system consisting of blocks of one size to a system composed of blocks of another size is a transition between different models, i.e., between approximations of the same physical system, each of which correctly reproduces the macroscopic characteristics in the neighborhood of the phase transition point.

The use of the renormalization-group approach in the theory of turbulence [8, 9], polymer physics [10], and the theory of incoherent radiative transfer [11] also uses a similar logic. In each case, one constructs (or understands) a sequence of related models $\{\mathcal{M}_n\}$, these differing from one another in the scale of some variable x (the cutoff limit of high-frequency fluctuations, the length of an "elementary" block in a long molecule, etc.), and also in the values of some parameter (or set of parameters) g, the analog of the coupling constants e and K, whose transformation law is determined by the properties of the "original" physical model. Thus, the renormalization group is a group of transformations within a set of models $\{\mathcal{M}_n\}$.

In contrast to this, the property of functional self-similarity of simple classical systems (flexible rod, two-dimensional problem of radiative transfer, ...) considered in [13, 14, 15] is a property of a symmetry of the (single!) model itself formulated in the language of its natural variables and simple physical characteristics determined by the equations of motion.

It is interesting to note that the formulation of the quantum-field renormalization group given by Bogolyubov and the present author [3] (see also Chap. 9 of our monograph [17]) is in a certain sense intermediate between these two extremes. It is based on concepts and functions (propagators and vertices) relating to one local renormalizable quantum-field model. At the same time, it uses auxiliary variables (normalization momenta) and functions (invariant or effective coupling constants) that do not occur in the equations of motion.

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LITERATURE CITED

1. E. Stueckelberg and A. Petermann, Helv. Phys. Acta, **26**, 499 (1953).
2. M. Gell-Mann and F. Low, Phys. Rev., **95**, 1300 (1954).
3. N. N. Bogolyubov and D. V. Shirkov, Dokl. Akad. Nauk SSSR, **103**, 203 (1955).
4. D. V. Shirkov, Dokl. Akad. Nauk SSSR, **105**, 972 (1955).
5. N. N. Bogolyubov and D. V. Shirkov, Dokl. Akad. Nauk SSSR, **103**, 391 (1955).
6. K. Wilson, Phys. Rev. B, **4**, 3174, 3184 (1971).
7. L. P. Kadanoff, Physics (Long Island City, N.Y.), **2**, 263 (1966).
8. C. De Dominicis and P. Martin, Phys. Rev. A, **19**, 419 (1979).
9. G. Pelletier, J. Plasma Phys., **24**, 421 (1980).
10. P. G. de Gennes, Scaling Concepts in Polymer Physics, Ithaca (1979).
11. T. Bell, U. Frisch, and H. Frisch, Phys. Rev. A, **17**, 1049 (1978).
12. M. A. Mnatsakanyan, Soobshch. Byurak. Obs., Akad. Nauk Arm. SSR, **50**, 59 (1978).
13. M. A. Mnatsakanyan, Dokl. Akad. Nauk SSSR, **262**, 856 (1982).
14. D. V. Shirkov, Dokl. Akad. Nauk SSSR, **263**, 63 (1982).
15. D. V. Shirkov, Preprint E2-83-790 [in English], JINR, Dubna (1983).
16. L. V. Ovsyannikov, Dokl. Akad. Nauk. SSSR, **109**, 1112 (1956).
17. N. N. Bogolyubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields [in Russian], Third Edition, Nauka, Moscow (1976), Fourth Edition, Nauka, Moscow (1984). (English translation published by Interscience (1959)).

DYNAMIC STOCHASTICITY AND QUANTIZATION

B. V. Medvedev

It is shown that after quantization of a classical dynamically stochastic system 1) the spectrum can be purely discrete, 2) stationary states correspond to simple closed classical trajectories, 3) stochastically entangled motions are "pushed" upward in energy to infinity.

1. Introduction

In recent years, much attention has been paid to the phenomenon of so-called dynamic stochasticity, which means that for a completely deterministic dynamical system, frequently with only very few degrees of freedom, the overwhelming majority of the motions are extremely irregular, chaotically entangled, and unpredictable, the source of these irregularities being in no way connected with external random perturbations but in the extremely strong (exponential) divergence of trajectories that begin at phase points with arbitrarily small separation. There is already a rich literature on the subject (see, for example, the reviews [1-4]), and numerous applications in physics, astrophysics, and celestial mechanics have been found. In particular, Matinyan et al. [5-7] (see also [8-10]) have found that dynamic stochasticity is exhibited by non-Abelian gauge fields in the long-wavelength approximation, when the dependence on the spatial coordinates can be ignored and it can be assumed that the fields and potentials are functions of only the time.

The phenomenon of dynamic stochasticity was discovered and investigated in the framework of classical mechanics. It is of extreme interest to consider how classical dynamic stochasticity affects the properties and behavior of the corresponding quantum system. Various comments have been made about this in the literature. It was stated long ago (see, for example, [11]) that in quantum dynamics stochasticity is not possible at all, since for a closed system confined in the phase space the energy spectrum is always discrete. In [4] there was an investigation of the specific manifestations of stochasticity that can be observed in a quantum system in the process of establishment of a stationary state. The features of the

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