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SPECTRUM OF RENORMALIZATION GROUP DIFFERENTIAL

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The spectrum of the differential of Wilson's renormalization group at a non-Gaussian fixed point is described.

Introduction

In [1], Wilson's equations were solved for an effective scalar Hamiltonian with free part determined by the long-range potential $U(x)\sim\text{const}/|x|^a$, $x\to\infty$. A new non-Gaussian branch of fixed points of Wilson's renormalization group bifurcates from the Gaussian branch of fixed points at the point $a={}^3/_2 d$ and for d not a multiple of 4 describes the critical behavior of models with long-range interaction. The Hamiltonian is obtained by projecting the Hamiltonian of the φ^4_d theory onto the ball $\Omega=\{k\,|\,|k|<R\}$ by means of the operation of analytic renormalization: $H=\ln(\text{A.R.}:\exp(u(\varepsilon)\varphi^4):_{-\Delta(1-\chi)})$, where

$$\varphi^4=\int\sigma(\mathbf{k}_1)\ldots\sigma(\mathbf{k}_4)\delta(\mathbf{k}_1+\ldots+\mathbf{k}_4)d^{4d}\mathbf{k},$$

$:\ldots:_{-\Delta(1-\chi)}$ is the transition to Wick polynomials with respect to the Gaussian field with propagator $-\Delta(1-\chi)(\mathbf{k})=-|\mathbf{k}|^{d-a}(1-\chi_R(\mathbf{k}))$, $\chi_R(\mathbf{k})$ is the indicator of the ball Ω, $u(\varepsilon)=u_1\varepsilon+u_2\varepsilon^2+\ldots$ is a formal numerical series in ε, and A.R. denotes some variant of analytic renormalization. The Hamiltonian can be represented in the form of the power series $H=H_0+\varepsilon H_1+\varepsilon^2H_2+\ldots$, where $\varepsilon=a-{}^3/_2 d$, d is the dimension of space.

In [2], the leading eigenvalue of the renormalization group differential was calculated at a non-Gaussian point, and a corresponding eigenfunction was also constructed. This made it possible to find expressions for the critical exponents. Usually, the critical exponents are sought independently of the question of the existence of an effective Hamiltonian and the so-called Callan–Symanzik equations are used. In this case, the original Kadanoff–Wilson dynamical model of critical phenomena is neglected. The studies of [1-2] return to this original model and realize some of its basic propositions.

In this paper, we analyze the complete spectrum of the linearized renormalization group at a non-Gaussian point. We obtain a natural picture of the "bifurcation" of eigenspaces and the corresponding eigenvalues.

1. Renormalization Group Differential at a Gaussian Fixed Point

Let $H^{(0)}=\varepsilon H_1^{(0)}+\varepsilon^2H_2^{(0)}+\ldots$ be a formal smooth Hamiltonian. The action of the operator of the

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smoothed renormalization group on $H^{(0)}$ is represented in the form

$$\mathscr{R}^{a}_{\chi,\lambda}(H^{(0)}) = :\exp(\mathscr{R}_\lambda{}^a H^{(0)}):^{c}_{-\Delta(\chi_\lambda-\chi)}. \tag{1.1}$$

Here, $:\ldots:^{c}_{-\Delta(\chi_\lambda-\chi)}$ denotes the Wick operation with respect to the Gaussian measure with correlation function $\delta(\mathbf{k}_1+\mathbf{k}_2)(-\Delta(\chi_\lambda-\chi)(\mathbf{k}))$,

$$\Delta(\chi_\lambda-\chi)(\mathbf{k}) = |\mathbf{k}|^{d-a}(\chi(\mathbf{k}/\lambda)-\chi(\mathbf{k})), \tag{1.2}$$

where $\chi(\mathbf{k}) \in C_0^\infty(\mathbb{R}^d)$ is the smoothed indicator of the ball $\Omega=\{\mathbf{k}\,|\,|\mathbf{k}|<R\}$, $a={}^3/_2 d+\varepsilon$. The symbol c means that only connected diagrams are taken. The operator of such stretching of $\mathscr{R}_\lambda{}^a$ to an m-particle Hamiltonian

$$H = \int_{\Omega^m} h(\mathbf{k}_1,\ldots,\mathbf{k}_m)\,\delta(\mathbf{k}_1+\ldots+\mathbf{k}_m)\prod_{i=1}^{m}\sigma(\mathbf{k}_i)\,d\mathbf{k}_i$$

acts in accordance with the formula

$$\mathscr{R}_\lambda{}^a H = \lambda^{am/2-md+d}\int_{(\lambda\Omega)^m} h\left(\frac{\mathbf{k}_1}{\lambda},\ldots,\frac{\mathbf{k}_m}{\lambda}\right)\delta(\mathbf{k}_1+\ldots+\mathbf{k}_m)\,\sigma(\mathbf{k}_1)\ldots\sigma(\mathbf{k}_m)\,d\mathbf{k}_1\ldots d\mathbf{k}_m, \tag{1.3}$$

and can be extended by linearity to the complete space of formal Hamiltonians.

We denote by $D^{a}_{H^{(0)},\lambda}$ the differential of the nonlinear transformation $\mathscr{R}^{a}_{\chi,\lambda}$ of the renormalization group at the point $H^{(0)}$:

$$R^{a}_{H^{(0)},\gamma}H = \sum_{n=1}^{\infty}\frac{1}{(n-1)!}:(\mathscr{R}_\lambda{}^a H^{(0)})^n \mathscr{R}_\lambda{}^a H:^{c}_{-\Delta(\chi_\lambda-\chi)}. \tag{1.4}$$

The spectrum of the renormalization group differential at the Gaussian fixed point $H^{(0)}=0$ is constructed as follows. We are interested in only smooth and even (with respect to the spin variable) eigenforms.

PROPOSITION 1 (see [1-3]). Let $h_m(\mathbf{k}_1,\ldots,\mathbf{k}_m)$ be a homogeneous polynomial of the variables $\mathbf{k}_1,\ldots,\mathbf{k}_m$ of degree $\deg h_m=s$, $\mathbf{k}_i=(k_{i_1},\ldots,k_{i_d})$, $i=1,\ldots,m$. Then the Hamiltonian is an eigenform of the renormalization group differential:

$$D^{a}_{0,\lambda}:H_m:_{-\Delta(1-\chi)} = \lambda^{am/2-md+d-s}:H_m:_{-\Delta(1-\chi)}. \tag{1.5}$$

It can be seen from this that degeneracy of the spectrum is possible in the Gaussian case. We are particularly interested in the bifurcation value $a_0={}^3/_2 d$. Let h_m be a homogeneous polynomial of m variables of degree s. Then the exponent of the eigenvalue of the corresponding eigen-Hamiltonian is $\gamma(m,s) = ma_0/2 - md + d - s = d(1-m/4) - s$. The values of the exponents $\gamma(m,s)$ form a discrete series for $m = 2, 4, 6, \ldots$, $s = 0, 1, 2, \ldots$. We fix some value of γ in this series. We denote by $T(\gamma)$ the space of homogeneous even polynomials of m variables of degree s such that

$$d(1-m/4)-s=\gamma. \tag{1.6}$$

We recall that polynomials are regarded as equivalent if they are equal on the hyperplane $\mathbf{k}_1+\ldots+\mathbf{k}_m = 0$.

We denote by $:\mathscr{H}(\gamma):$ the eigenspace of the differential $D^{a_0}_{0,\lambda}$ consisting of forms of the type

$$:H_m:_{-\Delta(1-\chi)} = :\int h(\mathbf{k}_1,\ldots,\mathbf{k}_m)\,\delta(\mathbf{k}_1+\ldots+\mathbf{k}_m)\prod_{i=1}^{m}\sigma(\mathbf{k}_i)\,d\mathbf{k}_i:_{-\Delta(1-\chi)},$$

where $h(\mathbf{k}) \in T(\gamma)$. Clearly

$$:\mathscr{H}(\gamma): = \oplus\sum_{m,\,s:\,d(1-m/4)-s=\gamma}:\mathscr{H}(m,s):,$$

where $:\mathscr{H}(m,s):$ is the space of Hamiltonians of the form

$$:H:_{-\Delta(1-\chi)} = :\int h(\mathbf{k}_1,\ldots,\mathbf{k}_m)\,\delta(\mathbf{k}_1+\ldots+\mathbf{k}_m)\prod_{i=1}^{m}\sigma(\mathbf{k}_i)\,d\mathbf{k}_i:_{-\Delta(1-\chi)},$$

in which $h(\mathbf{k}_1,\ldots,\mathbf{k}_m)$ is a homogeneous polynomial of degree s.

Note that for $a={}^3/_2 d+\varepsilon$, $\varepsilon>0$ the degeneracy in the spectrum of the differential is partly lifted. The spaces of $\mathscr{H}(m, s)$ are eigenspaces of the renormalization group differential for all a.

We shall seek eigenforms of the renormalization group differential at the non-Gaussian fixed point $H^{(0)}=$A.R.:$\exp(u(\varepsilon)\varphi^4):^c_{-\Delta(1-\chi)}$ in the form

$$\text{A.R.}:H\exp(u(\varepsilon)\varphi^4):^c_{-\Delta(1-\chi)}, \tag{1.7}$$

where the coefficient function $h(k_1, \ldots, k_m)$ of the Hamiltonian H belongs to $T(\gamma)$.

A form of the type (1.7) bifurcates from the corresponding eigenforms at the Gaussian point.

2. Theorem on Analytic Renormalization

To determine how a Hamiltonian of the form (1.7) transforms under a renormalization transformation, we need to rewrite it in "counterterms." Let $:H_1:_{-\Delta(1-\chi)}, :H_2:_{-\Delta(1-\chi)}, \ldots, :H_{n(\gamma)}:_{-\Delta(1-\chi)}$ be some basis in the space $\mathscr{H}(\gamma)$, $n(\gamma)=\dim\mathscr{H}(\gamma)$. Without loss of generality, we can assume that the coefficient functions of the Hamiltonians H_i are chosen in the form of the monomials $h_i(\mathbf{k}_1, \ldots, \mathbf{k}_{m_i}) = \mathbf{k}_1^{\alpha_1^i} \ldots \mathbf{k}_{m_i}^{\alpha_{m_i}^i}$, $\mathbf{k}_j=(k_{j1}, \ldots, k_{jd})$, $j=1,\ldots,m_i$, α_j^i are multiple indices, $|\alpha_i|=\sum_{j=1}^{m_i}|\alpha_j^i|$, $d(1-m_i/4)-|\alpha_i|=\gamma$.

THEOREM 2.1.

$$\text{A.R.}:H_i\exp(u\varphi^4):^c_{-\Delta(1-\chi)} = \sum_{j=1}^{n(\gamma)} w_{ij}(u):H_j\exp(w(u)\varphi^4):_{-\Delta(1-\chi)}, \tag{2.1}$$

where $w(u)$ and $w_{ij}(u)$ are formal series in u:

$$w(u)=\sum_{n=1}^{\infty} C^{(n)}(\varepsilon)u^n, \quad w_{ij}(u)=\sum_{n=0}^{\infty} C_{ij}^{(n)}(\varepsilon)u^n,$$

$C^{(n)}(\varepsilon)$, $C_{ij}^{(n)}(\varepsilon)$ are polynomials in ε^{-1} with vanishing free terms, and $C_{ij}^{(0)}=0$, $i\neq j$, $C_{ii}^{(0)}=1$.

We note first that it is sufficient to prove this theorem for the case $\chi\equiv 0$, since A.R.:$H\exp(u\varphi^4)$:$^c_{-\Delta(1-\chi)}=$:A.R.:$H\exp(u\varphi^4)$:$^c_{-\Delta}$:$_{\Delta\chi}$. Let G be an arbitrary single-particle-irreducible graph and $\mathscr{F}_G$ be the corresponding amplitude. It follows from the additivity property that A.R.$\mathscr{F}_G=\sum_{A\in A(G)}\mathscr{F}_{G:A}$, where $A(G)=\{\{H_1,\ldots,H_k\}\,|\,H_i$ are pairwise different single-particle-irreducible subgraphs of $G\}$,

$$\mathscr{F}_{G:A}=(2\pi)^{-hd}\int d^d\mathbf{k}_1\ldots d^d\mathbf{k}_h \prod_{l\in L(G/A)}\Delta_l(\mathbf{q}_l)\prod_{H\in A}O(H).$$

Here, $A\in A(G)$, G/A is the graph G contracted with respect to A, L(H) is the set of internal lines of the graph H, $h=h(G/A)$ is the Betti number of the graph G/A, $\{\mathbf{k}_i\,|\,i=1,\ldots,h\}$ is a certain choice of cyclical variables, $\{\mathbf{q}_l\,|\,l\in L(G/A)\}$ ($\{\mathbf{p}_e\,|\,e\in E(G/A)\}$) are internal (respectively, external) momenta of the graph G/A, and O(H) is the vertex part corresponding to the subgraph H (in our case, a polynomial in **k** and **p** with coefficients that depend meromorphically on ε).

We denote by Λ the operator of the "last subtraction": $\Lambda\mathscr{F}_G=\mathscr{F}_{G:G}$. As in [1] (see also [4]), we use the counterterm formula: A.R.:$H\exp(u\varphi^4)$:^{c}=:Λ(:$H\exp(u\varphi^4)$:^{c}) $\exp\Lambda$(:$\exp(u\varphi^4)$:$^c-1$): (in what follows, we shall omit the propagator notation in the Wick operation).

For d = 1, 2, 3

$$\Lambda(:\exp u\varphi^4:^c-1)=w(u)\varphi^4,$$

where w(u) is a formal series in u.

Thus, Theorem 1.1 follows from the proposition

LEMMA 2.1.

$$\Lambda:H_i(\varphi^4)^n:^c=\sum_{j=1}^{n(\gamma)}C_{ij}(\varepsilon)H_j.$$

Schematic Proof of Lemma 2.1. Consider the expansion

$$:H_i(\varphi^4)^n:^c = \sum_G \int \mathcal{F}_G(\mathbf{p}_1, \ldots, \mathbf{p}_{|E(G)|})\,\delta\Big(\sum_{i=1}^{|E(G)|} \mathbf{p}_i\Big) \prod_{i=1}^{|E(G)|} \sigma(\mathbf{p}_i)\,d\mathbf{p}_i.$$

An arbitrary amplitude $\mathcal{F}_G$ can be represented in the form

$$\mathcal{F}_G(\mathbf{p}_1, \ldots, \mathbf{p}_{|E|}) = \prod_{i=1}^{|E|} \mathbf{p}_i^{\alpha_i} (2\pi)^{-\frac{h(G)d}{2}} \int \prod_{l=1}^{|L(G)|} \frac{Z_l(\mathbf{q}_l)}{|\mathbf{q}_l|^{d/2+\varepsilon}}\, d\mathbf{k}_1 \ldots d\mathbf{k}_{h(G)},$$

where $\mathbf{k}_1, \ldots, \mathbf{k}_{h(G)}$ is some set of cyclical variables, $Z_l(\mathbf{q}_l)$ is a monomial of degree r_l, and

$$\sum_{l \in L(G)} r_l + \sum_{e \in E(G)} |\alpha_e| = s_i. \tag{2.2}$$

We consider the generalized Feynman amplitude

$$\mathcal{F}_G' = \int \prod_{l \in L(G)} Z_l(\mathbf{q}_l)\,|\mathbf{q}_l|^{-(d/2+\varepsilon_l)}\, d\mathbf{k}_1 \ldots d\mathbf{k}_h,$$

and expand it with respect to distinguished s families: $\mathcal{F}_G' = \sum_{\mathscr{E}} \mathcal{F}_G'(\mathscr{E})$. We fix a particular s family and go over to scaling variables. We can show (see [5]) that $\mathcal{F}_G'(\mathscr{E})$ can be represented as a sum of terms of the form

$$\int_D \prod_{H \in \mathscr{E}} t_H^{\nu_H - 1}\, dt_H \prod_{l \in \sigma(\mathscr{E})} \beta_l^{d/4+\varepsilon_l/2-1}\, d\beta_l F(\beta, t, \mathbf{p}) \exp\Big(-\sum_{i,j=1}^{|E(G)|} A_{ij}(\beta, t)\,\mathbf{p}_i\mathbf{p}_j\Big). \tag{2.3}$$

We use here the α representation in the real form.

The integration is over the region D: $0 \le t_H \le \infty$ if H is a maximal element in $\mathscr{E}$; $0 \leqslant t_H \leqslant 1$, if $H \in \mathscr{E}$ is not maximal, $\sigma(\mathscr{E})$ is the set of distinguished lines, $0 \leqslant \beta_l \leqslant 1$, $F(\beta, t, \mathbf{p})$, $A_{ij}(\beta, t)$ are continuous functions in D,

$$\nu_H = \frac{1}{2} \sum_{l \in L(H)} \Big(\frac{d}{2} + \varepsilon_l\Big) - \frac{h(H)d}{2} - \frac{1}{2} \sum_{l \in L(H)} (r_l - a_l) \equiv \frac{1}{2} \sum_{l \in L(H)} \varepsilon_l - k_H, \tag{2.4}$$

$0 < a_l \leqslant r_l$, a_l is an integer. $F(\beta, t, \mathbf{p})$ is a polynomial in $\mathbf{p}$ of degree $\sum_{l \in L(G)} a_l$.

The vertex part corresponding to the graph G can be calculated in accordance with

$$O(G) = \sum_{H < G} \mathscr{P}(H) \mathcal{F}_G, \tag{2.5}$$

where the operator $\mathscr{C}(H) = \sum_{H' \subset H} (-1)^{|L(H) - L(H')|} V_{H'}$, $V_{H'}$ is the operator of the "analytic value" (see [5]).

The summation in (2.5) is over all single-particle-irreducible subgraphs H such that E(H) = E(G). We represent each factor $t_H^{\nu_H - 1}$ in the form

$$t_H^{\nu_H - 1} = \frac{\delta^{(k_H)}(t_H)}{k_H! \, \frac{1}{2} \sum_{l \in L(H)} \varepsilon_l} + r_H(t_H),$$

where the regularization $r_H(t_H)$ is analytic at 0 with respect to $\varepsilon_l = \varepsilon$, $l \in L(G)$. Expanding the brackets in (2.3), we obtain a sum of terms that depend meromorphically on ε_l, $l \in L(G)$. We can show that if $H \neq G$, then $\mathscr{P}(H)\mathcal{F}_G' = 0$. This follows from the fact that the operator $\mathscr{P}(H)$ carries the amplitude $\mathcal{F}_G'$ into the amplitude (with different coefficient functions) of the contracted graph G/H. If $H \neq G$, then in the graph G/H a loop is formed, and by virtue of the homogeneity of the propagator in our theory graphs with loops have zero amplitudes. Following [5], we can also show that the only nontrivial contribution to the vertex part is made by the term containing the product $\prod_{i=1}^{n} \delta^{(k_{H_i})}(t_{H_i})$, where $H_1, \ldots, H_n$ are the 2-connectedness components of the graph G. Using the fact that $A_{i,j}(\beta, t)|_{t_{H_i}=0,\ i=1,\ldots,n}=0$, we find that the corresponding term is a polynomial in $\mathbf{p}$ of degree

$$\sum_{l=1}^{|L(G)|} a_l + 2\sum_{i=1}^{n}\left(\sum_{l\in L(H_i)}\frac{1}{2}\left(\frac{d}{2}+r_l-a_l\right)-\frac{h(H_i)}{2}d\right)=d\left(\frac{|L(G)|}{2}-h(G)\right)+\sum_{l\in L(G)} r_l.$$

From this we find that $\Lambda\mathcal{F}_G(\mathbf{p}_1,\ldots,\mathbf{p}_{|E(G)|})$ is a homogeneous polynomial in $\mathbf{p}$ of degree

$$\deg\Lambda\mathcal{F}_G(\mathbf{p}_1,\ldots,\mathbf{p}_{|E(G)|})=d\left(\frac{|L(G)|}{2}-h(G)\right)+\sum_{l\in L(G)} r_l+\sum_{e\in E(G)}\alpha_e=d\left(\frac{|L(G)|}{2}-h(G)\right)+s_i.$$

Now, using the relations $h(G)=|L(G)|-n$, $4n+m=2|L(G)|+|E(G)|$, we find that $d(1-{}^1/_4|E(G)|)+\deg\Lambda\mathcal{F}_G=d(1-{}^1/_4m)-s_i=\gamma$, i.e., $\Lambda\mathcal{F}_G\in T(\gamma)$, which is what we had to prove.

The remaining propositions of Theorem 2.1 can be readily verified.

3. Spectrum of the Renormalization Group Differential

Suppose

$$H=\int h(\mathbf{k}_1,\ldots,\mathbf{k}_m)\,\delta(\mathbf{k}_1+\ldots+\mathbf{k}_m)\prod_{i=1}^{m}\sigma(\mathbf{k}_i)\,d\mathbf{k}_i, \tag{3.1}$$

where $h\in T(\gamma)$. We can directly verify

LEMMA 3.1.

$$D^{a}_{0,\lambda}{:}H(\varphi^4)^n{:}^c=\lambda^{\gamma+{}^1/_2 m\varepsilon}{:}H(\varphi^4)^n{:}^c. \tag{3.2}$$

Using this lemma, and also the relation

$$D^{a}_{H^0,\lambda}H=D^{a}_{0,\lambda}(H\exp H^0)\exp(-H^{(0)}), \tag{3.3}$$

we can calculate the action of the differential $D^{a}_{H^0,\lambda}$ on the Hamiltonian $\mathrm{A.\,R.}{:}H\exp(u\varphi^4){:}^c$.

Note that the differential $D_{H,\lambda}$ is a multiplicative group of linear operators with respect to λ. We make the substitution $\lambda=\exp(\tau/2)$. Then the operators $D^{a}_{H^0,\exp(\tau/2)}$ form an additive semigroup with respect to τ, and it is natural to calculate the spectrum of the generator of this group. We denote the generator of this group by $\underline{D}_{H^0}{}^{a}$.

LEMMA 3.2.

$$D^{a}_{H^0}\mathrm{A.R.}{:}H_i\exp(u\varphi^4){:}^c=\sum_{j=1}^{n(\gamma)}\frac{1}{2}w_{ij}{:}\left[\left(\gamma+m_j\frac{\varepsilon}{2}\right)+\varepsilon w\varphi^4\right]H_j\exp(w\varphi^4){:}^c.$$

This expression can be readily transformed to the form (3.4).

LEMMA 3.3.

$$D^{a}_{H^0}\mathrm{A.R.}{:}H_i\exp(u\varphi^4){:}^c=\rho(u)\frac{d}{du}\mathrm{A.R.}{:}H_i\exp(u\varphi^4){:}^c+\sum_{j=1}^{n(\gamma)}\alpha_{ij}{:}H_j\exp(w\varphi^4){:}^c, \tag{3.4}$$

where $\alpha_{ij}={}^1/_2w_{ij}(\gamma+{}^1/_2m_j\varepsilon)-\rho w_{ij}'$, $\rho(u)=\varepsilon w(u)/w'(u)$.

Inverting the relation 2.1, we can write

$$D^{a}_{H^0}\mathrm{A.R.}{:}H_i\exp(u\varphi^4){:}^c=\rho(u)\frac{d}{du}\mathrm{A.R.}{:}H_i\exp(u\varphi^4){:}^c+\sum_{j=1}^{n(\gamma)}\beta_{ij}(u)\,\mathrm{A.R.}{:}H_j\exp(u\varphi^4){:}^c, \tag{3.5}$$

where β_{ij} is the element of the matrix $B=AW^{-1}$, $A=(\alpha_{ij})_{i,j=1}^{n(\gamma)}$, $W=(w_{ij})_{i,j=1}^{n(\gamma)}$. Note that inversion of the matrix W is possible since $W=I+W_1$, where all the elements of the matrix W_1 are formal series in u with vanishing free terms.

The matrix A can be represented in the form $W\left(\frac{\gamma}{2}I+\varepsilon\Lambda_0\right)-\rho W_u'$, where Λ_0 is a diagonal matrix: $\Lambda_0=(\lambda_{ij}{}^0)_{i,j=1}^{n(\gamma)}$, $\lambda_{ii}{}^0={}^1/_4m_i$, $\lambda_{ij}^{(0)}=0$, $i\neq j$, and all the elements of the matrix $\rho W_u'$ are formal series in u with vanishing free terms. It follows that the matrix B can be represented in the form $B={}^1/_2\gamma I+\varepsilon\Lambda_0+B_1$.

THEOREM 3.1. All the elements of the matrix B_1 have the form

$$\beta_{ij}^1(u)=\sum_{n=1}^{\infty} b_{ij}^n u^n,$$

where the coefficients b_{ij}^n are constants that do not depend on ε.

The proof of this theorem follows from the linear independence of the Hamiltonians $:H_i:$, $i=1,\dots,n(\gamma)$ and can be verified by induction on n. We immediately obtain

COROLLARY. The spaces $\mathrm{A.R.}:\mathscr{H}(\gamma)\exp(u(\varepsilon)\varphi^4):^c$ that bifurcate from the spaces $:\mathscr{H}(\gamma):$ are invariant subspaces of the renormalization group differential at a non-Gaussian fixed point.

Here, $u(\varepsilon)$ is the effective coupling constant, $\rho(u(\varepsilon)) = 0$.

Thus, the spectral properties of the differential in the subspace $\mathrm{A.R.}:\mathscr{H}(\gamma)\exp(u(\varepsilon)\varphi^4):^c$ are determined by the matrix B. Let us consider it in more detail:

$$B=W\left(\frac{\gamma}{2}I+\varepsilon\Lambda_0\right)W^{-1}-\rho W_u'W^{-1}.$$

The expansion $W=I+u\Lambda_1+u^2\Lambda_2+\dots$ holds. Here, $\Lambda_1=\frac{1}{\varepsilon}\tilde{\Lambda}_1$, where the matrix elements of $\tilde{\Lambda}_1$ are constants. Substituting this expansion in (3.6), we obtain

$$B=\frac{\gamma}{2}I+\varepsilon\Lambda_0+\varepsilon u[\Lambda_1,\Lambda_0]-\varepsilon u\Lambda_1+u^2B_2+\dots\,. \tag{3.7}$$

It can be shown that $[\Lambda_1,\Lambda_0]=0$ in dimensions 1, 2, 3. Substituting the expansion $u(\varepsilon)=u_1\varepsilon+u_2\varepsilon^2+\dots$, we obtain

$$B=\frac{\gamma}{2}I+\varepsilon(\Lambda_0-u_1\tilde{\Lambda}_1)-\varepsilon^2T_2+\varepsilon^3T_3+\dots,$$

where T_2, T_3, ... are matrices that do not depend on ε. In accordance with the general theory of perturbations of finite-dimensional matrices, the eigenvalues of the matrix B can be expanded in powers of $\varepsilon^{1/p}$ $(p\geqslant 2)$ in Puiseux series (see [6]). But it is easy to see that if the spectrum of the matrix $\tilde{\Lambda}_1$ is nondegenerate, the eigenvalues can be expanded in powers of ε and the eigenvectors will also have an expansion in powers of ε. Expressions exist for the matrix $\tilde{\Lambda}_1$, but they are cumbersome and we omit them. We note that for values of γ and m that are not large the spectrum of the matrix $\tilde{\Lambda}_1$ is nondegenerate, but in the general case this question remains open.

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