ANALYTIC SOLUTION OF BOUNDARY-VALUE PROBLEMS FOR NONSTATIONARY MODEL KINETIC EQUATIONS

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A theory for constructing the solutions of boundary-value problems for nonstationary model kinetic equations is constructed. This theory was incorrectly presented in the recent well-known monographs of Cercignani [1,2]. After application of a Laplace transformation to the studied equation, separation of the variables is used, this leading to a characteristic equation. Eigenfunctions are found in the space of generalized functions, and the eigenvalue spectrum is investigated. An existence and uniqueness theorem for the expansion of the Laplace transform of the solution with respect to the eigenfunctions is proved. The proof is constructive and gives explicit expressions for the expansion coefficients. An application to the Rayleigh problem is obtained, and the corresponding result of Cercignani is corrected.

INTRODUCTION

In recent years, there has been considerable interest in nonstationary model kinetic equations. This interest is explained by the important applications of these equations in fields like the kinetic theory of gases and plasmas, theoretical astrophysics, solid-state theory (behavior of plasmas in metals), etc.

Besides the monographs [1,2], nonstationary kinetic equations were studied in [3,4] by the methods of the theory of semigroups and functional analysis. Note that in [1,2] the investigations were made by Case's method (see, for example, [5]) for the Laplace transform of the equation.

In this paper, we propose an analytic method for solving the half-space boundary-value problem for the model nonstationary kinetic equation

$$\left(\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial x}+1\right)Y(t,x,\mu)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(-\mu'^2)Y(t,x,\mu')d\mu'. \tag{0.1}$$

This equation arises, in particular, from linearization of the nonstationary Boltzmann equation with collision operator in the BGK (Bhatnagar, Gross, Krook) form (see, for example, [1,2]). The solution of the linearized Boltzmann equation can be sought in the form $f=f_0(1+C_yY)$, where f_0 is the local Maxwellian function, C_y is the projection of the molecular velocity onto the y axis, and the function Y satisfies Eq. (0.1).

As initial and boundary conditions, we take

$$Y(0,x,\mu)\equiv 0 \quad (x>0), \tag{0.2a}$$

$$Y(t,0,\mu)=Y_0(t,\mu) \quad (t>0,\ \mu>0), \tag{0.2b}$$

$$Y(t,\infty,\mu)\equiv 0 \quad (t>0). \tag{0.2c}$$

Here, $Y_0(t,\mu)$ is a given function that satisfies the conditions formulated below.

The conditions (0.2) mean that the initial-value—boundary-value problem posed for Eq. (0.1) is, in accordance with the widely adopted terminology (see, for example, [1,2,5]), a "half-space" problem in both the physical space, $x>0$, and in the velocity space: $\mu>0$.

Let

$$\hat{Y}(s,x,\mu)=\int_0^{\infty}e^{-st}Y(t,x,\mu)dt$$

be the Laplace transform of the function $Y(t,x,\mu)$.

Applying the Laplace transformation to Eq. (0.1) and Eqs. (0.2b) and (0.2c), we obtain in the space of Laplace transforms

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a boundary-value problem consisting of the solution of the equation

$$\left(s+1+\mu\frac{\partial}{\partial x}\right)\tilde{Y}(s, x, \mu)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(-\mu'^2)\tilde{Y}(s, x, \mu')d\mu' \tag{0.3}$$

($x>0$, $s=\sigma+i\omega\in\mathbb{C}$) with boundary conditions

$$\tilde{Y}(s, 0, \mu)=\tilde{Y}_0(s, \mu),\ \mu>0, \tag{0.4a}$$

$$\tilde{Y}(s, \infty, \mu)\equiv 0. \tag{0.4b}$$

1. EIGENFUNCTIONS OF CHARACTERISTIC EQUATION FOR THE CONTINUOUS SPECTRUM

For separation of the variables in Eq. (0.3), it is sufficient to use the following ansatz of Case:

$$\tilde{Y}_\eta(s, x, \mu)=\exp\left\{-\frac{s+1}{\eta}x\right\}\Phi(s, \eta, \mu),\quad s, \eta\in\mathbb{C}. \tag{1.1}$$

We then obtain the characteristic equation

$$(s+1)(\eta-\mu)\Phi(s, \eta, \mu)=\eta\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(-\mu'^2)\Phi(s, \eta, \mu')d\mu'. \tag{1.2}$$

We adopt the following "normalization" condition:

$$\int_{-\infty}^{\infty}\exp(-\mu^2)\Phi(s, \eta, \mu)d\mu=(s+1)n(s, \eta). \tag{1.3}$$

The characteristic equation then simplifies to

$$(\eta-\mu)\Phi(s, \eta, \mu)=\pi^{-1/2}\eta n(s, \eta). \tag{1.4}$$

Note that although the normalization (1.3) adopted here differs slightly from the normalization adopted in [1,2] and [6] it still leads to certain simplifications of the subsequent arguments.

The solution of Eq. (1.3) depends strongly on whether or not the parameter η belongs to the real axis $\mathbb{R}$.

Suppose first $\eta\notin\mathbb{R}$. From Eq. (1.4) we find

$$\Phi(s, \eta, \mu)=\pi^{-1/2}\frac{\eta}{\eta-\mu}n(s, \eta). \tag{1.5}$$

We substitute (1.5) in (1.3). We obtain the equation

$$(s+\lambda(\eta))n(s, \eta)=0. \tag{1.6}$$

Here

$$\lambda(z)=1+\pi^{-1/2}zt(z),\quad t(z)=\int_{-\infty}^{\infty}\exp(-\mu^2)\frac{d\mu}{\mu-z},$$

is a piecewise analytic function in the complex plane with cut along the real axis $\mathbb{R}$ whose elements can be calculated in accordance with the formula

$$\lambda^{\pm}(z)=1-2ze^{-z^2}\int_0^z e^{u^2}du\pm\sqrt{\pi}ize^{-z^2},\quad \text{if}\quad \pm\,\mathrm{Im}\,z>0.$$

Because Eq. (0.1) is homogeneous, the function $n(s, \eta)$ can be assumed identically equal to unity: $n(s, \eta)\equiv 1$. Further, we denote

$$\lambda(s, z)=s+\lambda(z).$$

With allowance for the foregoing convention, Eq. (1.6) can be written as

$$\lambda^{\pm}(s, z)=s+\lambda^{\pm}(z)=0, \quad \text{if} \quad \pm\operatorname{Im} z>0. \tag{1.7}$$

Setting $s=s_1+is_2$, we write Eq. (1.7) as a pair of equations for the real and imaginary parts:

$$s_1+\operatorname{Re}\left(1-2ze^{-z^2}\int_0^z e^{u^2}du\pm\sqrt{\pi}ize^{-z^2}\right)=0, \quad s_2+\operatorname{Im}\left(-2ze^{-z^2}\int_0^z e^{u^2}du\pm\sqrt{\pi}ize^{-z^2}\right)=0,$$

if $\pm\operatorname{Im} z>0$.

Letting z in these formulas tend to the real axis, Cercignani "investigates" the discrete spectrum and "finds" the discrete eigensolutions corresponding to it. However, such a "search" is incorrect, since in the case $z=x\in\mathbb{R}$ "discrete solutions" are in fact contained among the eigenfunctions of the continuous spectrum. The presence of discrete solutions must be investigated in a quite different way. The mode distinguished by Cercignani in [1,2] and [6] also exists in the stationary case, where, naturally, it is not specially separated, since it belongs to the continuous spectrum.

Bearing in mind that $|\lambda(z)|\le 1$ for all $z\in\mathbb{C}$, we conclude that for values of parameters s in the exterior of the unit disk ($|s|>1$) the discrete spectrum is empty. For values of the parameter s in the unit disk ($|s|<1$), discrete solutions of the form (1.5) can be present.

Now suppose the parameter $\eta\in\mathbb{R}$. Then in the class of generalized functions Eqs. (1.3) and (1.4) have the solution

$$\Phi(s, \eta, \mu)=\frac{1}{\sqrt{\eta}}\eta P\frac{1}{\eta-\mu}+e^{\eta^2}\lambda(s, \eta)\delta(\eta-\mu), \tag{1.8}$$

where the distribution Px^{-1} is the principal value of the integral of x^{-1}, $\delta(x)$ is the delta function, $\lambda(s, \eta)=s+\lambda(\eta)$, and

$$\lambda(x)=1-2xe^{-x^2}\int_0^x e^{u^2}du. \tag{1.9}$$

2. DISCRETE SPECTRUM OF CHARACTERISTIC EQUATION. FACTORIZATION OF DISPERSION FUNCTION

We first investigate the presence of isolated zeros of the characteristic equation (1.4), the set of which constitutes its discrete spectrum. For this, we shall use the principle of the argument and the formalism of boundary-value problems in the theory of functions of a complex argument (see [7]).

We consider the homogeneous Riemann boundary-value problem

$$X^{+}(s, \mu)=G(s, \mu)X^{-}(s, \mu), \quad 0<\mu<\infty, \tag{2.1}$$

with coefficient

$$G(s, \mu)=\lambda^{+}(s, \mu)/\lambda^{-}(s, \mu),$$

where

$$\lambda^{\pm}(s, x)=\lim_{y\to 0}\lambda^{\pm}(s, z), \quad z=x+iy, \quad y\gtrless 0,$$

and the function $\lambda^{\pm}(s, z)$ is determined by Eq. (1.7).

Suppose $s=\sigma+i\omega$. Then this coefficient can be written as

$$G(s, \mu)=\frac{\sigma+\lambda(\mu)+i(\omega+\sqrt{\pi}\mu e^{-\mu^2})}{\sigma+\lambda(\mu)+i(\omega-\sqrt{\pi}\mu e^{-\mu^2})}, \tag{2.2}$$

where the function $\lambda(\mu)$ is determined by Eq. (1.9).

In the expression (2.2), we separate the real and imaginary parts. We have

$$G(s, \mu)=\frac{(\sigma+\lambda(\mu))^2+\omega^2-(\sqrt{\pi}\mu e^{-\mu^2})^2+2i\sqrt{\pi}\mu e^{-\mu^2}(\sigma+\lambda(\mu))}{(\sigma+\lambda(\mu))^2+(\omega-\sqrt{\pi}\mu e^{-\mu^2})^2} \tag{2.3}$$

whence

$$\operatorname{Re} G(s, \mu)=\frac{(\sigma+\lambda(\mu))^2+\omega^2-(\sqrt{\pi}\mu e^{-\mu^2})^2}{(\sigma+\lambda(\mu))^2+(\omega-\sqrt{\pi}\mu e^{-\mu^2})^2}, \tag{2.4}$$

$$\operatorname{Im} G(s, \mu)=\frac{2\sqrt{\pi}\omega\mu e^{-\mu^2}(\sigma+\lambda(\mu))}{(\sigma+\lambda(\mu))^2+(\omega-\sqrt{\pi}\mu e^{-\mu^2})^2}. \tag{2.5}$$

We denote by $\gamma(s)=\gamma(s, \mu)$ the family of curves described by the equation $z=G(s, \mu)$, $0\le\mu\le\infty$. We note some properties of the curves $\gamma(s)$.

1. The curves $\gamma(s)$ are closed (at least, in the extended complex plane — the Riemann sphere).

This fact follows from the double equality

$$G(s, 0)=G(s, \infty)=1.$$

2. The curves $\gamma(s)$ lie outside the open disk $|z|<1$ if and only if $\operatorname{Im} s=\omega\ge 0$.

To prove this fact, it is sufficient to solve the inequality $|G(s, \mu)|\ge 1$.

3. If the parameter $s=\sigma$ is real, then all curves $\gamma(s)$ lie on the unit circle $|z|=1$.

Indeed, it is readily noted that the equation $|G(s, \mu)|=1$ is equivalent to the equation $\omega=0$, and this proves property 3. In the case $s=\sigma$, we have $\lambda^-(s, \mu)=\overline{\lambda^+(s, \mu)}$, where the bar denotes the complex conjugate. Therefore

$$G(\sigma, \mu)=\exp[2i\theta(\sigma, \mu)],$$

where $\theta(s, \mu)=\arg \lambda^+(S, \mu)$ is the principal value of the argument.

We denote by

$$\varkappa(s)=\operatorname{ind} G(s, \mu)=\frac{1}{2\pi}[\arg G(s, \mu)]_{\mathbb{R}_+}$$

the index of the coefficient $G(s, \mu)$, $\mathbb{R}_+=(0, \infty)$.

We establish for which values of the parameter $s\in\mathbb{C}$ $\varkappa(s)=1$, i.e., the curves $\gamma(s)$ pass once around the origin. We note first that $\varkappa(s)=0$ if the real or the imaginary part of the coefficient is either positive or negative for all $\mu>0$.

Suppose $\operatorname{Im} G(s, \mu)\ge 0$ for all $\mu\ge 0$. Then in accordance with (2.5) $\sigma+\lambda(\mu)\ge 0$ for all $\mu\ge 0$. We denote $\lambda(\mu_{\min})=\lambda_{\min}$. It is clear that $\varkappa(s)=0$ for all $\sigma\ge-\lambda_{\min}$. If $\operatorname{Im} G(s, \mu)\le 0$ for all $\mu\ge 0$, then $\sigma\le -1$. Thus, if $\varkappa(s)=0$, then $\sigma\in(-\infty, -1]\cup[-\lambda_{\min}, +\infty)$.

Let μ_σ and $\tilde{\mu}_\sigma$ $(\mu_\sigma\le\tilde{\mu}_\sigma)$ be zeros of the function $\sigma+\lambda(\mu)$ with $\mu_0=0.924141\ldots$ a zero of the function $\lambda(\mu)$. It is clear that $\mu_{-1}=0$ with $0=\mu_{-1}\le\mu_\sigma\le\mu_0$. As σ varies from -1 to 0, the zero of μ_σ varies from 0 to μ_0. If σ varies from 0 to $-\lambda_{\min}$, then in addition to μ_σ the function $\sigma+\lambda(\mu)$ has a second zero $\tilde{\mu}_\sigma$. Note that when $\sigma=-\lambda_{\min}$ these two zeros merge into one: $\mu_\sigma=\tilde{\mu}_\sigma=\mu_{\min}$.

If $s=\sigma$ $(\omega=0)$, then it is easy to see that $\varkappa(s)=1$ if $\sigma\in(-1, 0]$.

Suppose $s=\sigma+i\omega$, $\omega\ne 0$, $\sigma\in(-1, 0]$. In this case, the imaginary part of the coefficient changes sign once at the point μ_σ, being positive for $\mu\in[0, \mu_\sigma)$ and negative for $\mu\in(\mu_\sigma, +\infty)$. At the point μ_σ, the value of the coefficient is

$$G(s, \mu_\sigma)=\frac{\omega+\omega_\sigma}{\omega-\omega_\sigma}, \tag{2.6}$$

where

$$\omega_\sigma=\omega(\mu_\sigma), \quad \omega(\mu)=\sqrt{\pi}\mu\exp(-\mu^2).$$

We consider the real part of the coefficient. In accordance with (2.4), the equation $\operatorname{Re} G(s, \mu)=0$ is equivalent to the equation

$$g(s, \mu)=(\sigma+\lambda(\mu))^2+\omega^2-\omega^2(\mu)=0. \tag{2.7}$$

We rewrite Eq. (2.7) in the form

$$g(s, \mu)=(\sigma+\lambda(\mu)+\omega(\mu))(\sigma+\lambda(\mu)-\omega(\mu))+\omega^2=0.$$

It can be seen from this that if ω satisfies the condition $|\omega|<\omega_\sigma$ then in the intervals $(0, \mu_\sigma)$ and $(\mu_\sigma, +\infty)$ the real part of the coefficient vanishes at the points μ_1 and μ_2 $(0<\mu_1<\mu_\sigma<\mu_2<\infty)$, with $g(s, \mu)>0$ for $\mu\in[0, \mu_1)\cup(\mu_2, \infty]$ and $g(s, \mu)<0$ for $\mu\in(\mu_1, \mu_2)$. In addition, under the condition $|\omega|<\omega_\sigma$ we have $G(s, \mu_\sigma)<0$ in accordance with (2.6). Thus, if $s\in S_1$, then $\varkappa(s)=1$, where

$$S_1=\{s=\sigma+i\omega\colon \sigma\in(-1, 0], \ |\omega|<\omega_\sigma\}. \tag{2.8}$$

Now suppose $\sigma\in(0, -\lambda_{\min})$. In this case, as we have noted, the imaginary part of the coefficient has two zeros μ_σ and

$\tilde{\mu}_\sigma$, with Im $G(s, \mu)>0$ when $\mu\in[0, \mu_\sigma)\cup(\tilde{\mu}_\sigma, +\infty]$ and Im $G(s, \mu)<0$ when $\mu\in(\mu_\sigma, \tilde{\mu}_\sigma)$. We require $|\omega|<\omega_\sigma$. Then $G(s, \mu_\sigma)<0$. Therefore, the increment of the argument of the coefficient on the interval $[0, \mu_\sigma]$ is π. We require $|\omega|>\tilde{\omega}_\sigma$. This means that in the interval $(\mu_\sigma, \tilde{\mu}_\sigma)$ the real part of the coefficient vanishes at the one point μ_2 and $G(s, \tilde{\mu}_\sigma)>0$. Therefore, the increment of the argument of the coefficient on the interval $[0, \tilde{\mu}_\sigma]$ is 2π. For variation of μ from $\tilde{\mu}_\sigma$ to $+\infty$, Im $G(s, \mu)>0$, and the increment of the argument of the coefficient on this interval is zero. Note that for all $\sigma\in(0, -\lambda_{\min})$ we have $\mu_\sigma\in(\mu_0, \mu_{\min})$, and $\tilde{\omega}_\sigma<\omega_\sigma$, since μ_σ and $\tilde{\mu}_\sigma$ lie in the region of decrease of the function $\omega(\mu)$; moreover, as $\sigma\to+0$ $\tilde{\mu}_\sigma\to+\infty$ and $\tilde{\omega}_\sigma\to+0$. As $\sigma\to-\lambda_{\min}$, $\omega_\sigma\to\omega(\mu_{\min})$ and $\tilde{\omega}_\sigma\to\omega(\mu_{\min})$, since $\mu_\sigma\to\mu_{\min}$ and $\tilde{\mu}_\sigma\to\mu_{\min}$.

Thus, if $s\in S_2$, then $\varkappa(s)=1$, where

$$S_2=\{s=\sigma+i\omega:\ 0<\sigma<-\lambda_{\min},\ \tilde{\omega}_\sigma<|\omega|<\omega_\sigma\}. \tag{2.9}$$

Obviously, all the previous arguments can be inverted, i.e., if $\varkappa(s)=1$, then $s\in S_1\cup S_2$.

We summarize this analysis as a theorem.

THEOREM 2.1. *A necessary and sufficient condition for $\varkappa(s)=1$ is that $s\in S=S_1\cup S_2$, where S_1 and S_2 are introduced by Eqs. (2.8) and (2.9).*

Note that the boundary ∂S in the region S is specified parametrically:

$$\partial S=\{s=\sigma+i\omega:\ \sigma=-\lambda(\mu),\ |\omega|=\sqrt{\pi}\mu e^{-\mu^2},\ 0\leqslant\mu\leqslant\infty\},$$

and that its part ∂S^+ lying in the upper half-plane is described by the equations

$$\sigma=-\lambda(\mu),\ \omega=\sqrt{\pi}\mu e^{-\mu^2},\ 0\leqslant\mu\leqslant\infty,$$

and its part ∂S^- lying in the lower half-plane by the equations

$$\sigma=-\lambda(\mu),\quad \omega=-\sqrt{\pi}\mu e^{-\mu^2},\quad 0\leqslant\mu\leqslant\infty.$$

Corollary 2.1. *It follows from the theorem that $\varkappa(s)=0$ if $s\in\mathbb{C}\setminus S$.*

Corollary 2.2. *If the parameter $s\in S$, then by virtue of the parity the dispersion function $\lambda(s, z)$ has two zeros $\pm\eta_0$, which differ in their signs. In the case $s=0$, the dispersion function $\lambda(0, z)$ has two zeros that coincide at infinity: $\pm\eta_0=\infty$ (the case $s=0$ is the case of a stationary equation, which is not considered here because it is treated in* [1,2] *and in* [8]).

Thus, if $s\in S\setminus\{0\}$, then in accordance with (1.1) eigensolutions corresponding to the discrete spectrum are the functions

$$\Upsilon_{\pm\eta_0}(s, x, \mu)=\exp\left\{-\frac{s+1}{\pm\eta_0}x\right\}\frac{1}{\sqrt{\pi}}\frac{\eta_0}{\eta_0-\mu}. \tag{2.10}$$

We turn to the solution of the boundary-value problem (2.1). As solution that does not vanish at the origin, we take

$$X(s, z)=z^{-\varkappa(s)}\exp Y(s, z). \tag{2.11}$$

Here

$$Y(s, z)=\frac{1}{2\pi i}\int\limits_0^\infty[\ln G(s, \mu)-2\pi i\varkappa(s)]\frac{d\mu}{\mu-z}, \tag{2.12}$$

where the principal value of the logarithm is understood.

We how how the zero η_0 of the dispersion function can be found explicitly. We consider the function $\psi(s, z)=\lambda(s, z)/X(s, -z)$, which is continuous on the passage through the negative part of the real axis and satisfies the condition (2.1) on the positive part of that axis. The function $\psi(s, z)$ is analytic in z everywhere in $\mathbb{C}$, and as solution to the problem (2.1) it is equal to

$$\psi(s, z)=X(s, z)P(s, z), \tag{2.13}$$

where $P(s, z)$ is a polynomial in the variable z.

Since $X(s, z)\neq0$ in the plane $\mathbb{C}$, and $\lambda(s, \infty)=s$, $\lambda(s, \pm\eta_0)=0$ for $\forall s\in\mathbb{C}$, from the representation (2.13) we have

$$\lambda(s, z)=s(\eta_0^2-z^2)X(s, z)X(s, -z). \tag{2.14}$$

Note that if $s=0$ then it is possible to factorize only the boundary values of the dispersion function above and below on the real axis:

$$\lambda^\pm(0, \mu)=\tfrac{1}{2}X^\pm(0, \mu)X(0, -\mu).$$

Setting $z=0$ in (2.14), we obtain

$$\eta_0=\left(1+\frac{1}{s}\right)^{1/2}X^{-1}(s,0). \tag{2.15}$$

It is very inconvenient to calculate $X(s, 0)$ in accordance with (2.11) and (2.12). Therefore, we give without derivation an integral representation for the function $X(s, z)$:

$$X(s,z)=\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-\mu^2}\rho(s,\mu)\frac{d\mu}{\mu-z};$$

if $s\in S$, and

$$X(s,z)=X(s,\infty)+\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-\mu^2}\rho(s,\mu)\frac{d\mu}{\mu-z},$$

if $s\notin S$. Here

$$\rho(s,\mu)=\mu X^+(s,\mu)/\lambda^+(s,\mu).$$

To conclude this section, we note that in the case $s\notin S$ the dispersion function is factorized as follows:

$$\lambda(s,z)=sX(s,z)X(s,-z).$$

3. EXPANSION OF SOLUTION OF THE BOUNDARY-VALUE PROBLEM WITH RESPECT TO EIGENFUNCTIONS

Let $H(E)$ be a class of functions defined on the set E and satisfying on the closed set $\bar{E}$ a Hölder condition. From the set S we remove the interval $-1<\sigma<0$ and denote

$$S_0=S\setminus\{s=\sigma+i\omega:\ -1<\sigma<0,\ \omega=0\}.$$

THEOREM 3.1. *If the function* $\tilde{Y}_0(s,\mu)$ *satisfies the condition*

$$\mu\exp(-\mu^2)\tilde{Y}_0(s,\mu)\in H(\mathbb{R}_+), \tag{3.1}$$

then there exists a unique expansion of the solution $\tilde{Y}(s,x,\mu)$ *of the boundary-value problem* (0.3)—(0.4) *with respect to the eigenfunctions of the characteristic equation (1.4)*:

$$\tilde{Y}(s,x,\mu)=a_0(s)\frac{1}{\sqrt{\pi}}\frac{\eta_0}{\eta_0-\mu}\exp\left\{-\frac{s+1}{\eta_0}x\right\}+\int_0^\infty\exp\left\{-\frac{s+1}{\eta}x\right\}\Phi(s,\eta,\mu)a(s,\eta)\,d\eta, \tag{3.2}$$

where

$$\mathrm{Re}\,\frac{s+1}{\eta_0}>0.$$

Here, $a_0(s)$ is the expansion coefficient corresponding to the discrete spectrum, with $a_0(s)=0$ if $s\notin S_0$; $a(s,\eta)$ is the coefficient corresponding to the discrete spectrum. The proof given below provides explicit expressions for the expansion coefficients.

Proof. We substitute the eigenfunctions (1.8) in the expansion (3.2) for $x=0$. We obtain a characteristic singular integral equation with Cauchy kernel:

$$\tilde{Y}_1(s,\mu)=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{\eta a(s,\eta)}{\eta-\mu}d\eta+e^{\mu^2}\lambda(s,\mu)a(s,\mu),\quad \mu>0, \tag{3.3}$$

where

$$\tilde{Y}_1(s,\mu)=\tilde{Y}_0(s,\mu)-a_0(s)\frac{1}{\sqrt{\pi}}\frac{\eta_0}{\eta_0-\mu}. \tag{3.4}$$

If we introduce the auxiliary function

$$A(s,z)=\frac{1}{2\pi i}\int_0^\infty \frac{\eta a(s,\eta)}{\eta-z}d\eta, \tag{3.5}$$

then Eq. (3.3) can be reduced to the Riemann boundary-value problem

$$\lambda^+(s,\ \mu)A^+(s,\ \mu)-\lambda^-(s,\ \mu)A^-(s,\ \mu)=\mu e^{-\mu^2}\tilde{Y}_1(s,\ \mu),\ \ \mu>0. \tag{3.6}$$

The problem of factorizing the coefficient of this boundary condition is the problem

$$\frac{X^+(s,\mu)}{X^-(s,\mu)}=\frac{\lambda^+(s,\mu)}{\lambda^-(s,\mu)},\quad \mu>0, \tag{3.7}$$

considered in Sec. 2.

The index of the boundary-value problem (3.6) is

$$\varkappa(s)=\operatorname{ind}\frac{\lambda^-(s,\mu)}{\lambda^+(s,\mu)}$$

It differs in sign from the index introduced in Sec. 2. Using the solution of the problem (3.7) given in Sec. 2 [see formulas (2.11) and (2.12)], we turn to solution of the inhomogeneous problem (3.6), which we transform to a discontinuity problem. We shall have

$$X^+(s,\ \mu)A^+(s,\ \mu)-X^-(s,\ \mu)A^-(s,\ \mu)=e^{-\mu^2}\rho(s,\ \mu)\tilde{Y}_1(s,\ \mu),\quad \mu>0. \tag{3.8}$$

The solution to this problem is expressed by a Cauchy-type integral

$$A(s,\ z)=\Phi(s,\ z)/X(s,\ z), \tag{3.9}$$

where

$$\Phi(s,z)=\frac{1}{2\pi i}\int_0^\infty e^{-\mu^2}\rho(s,\mu)\tilde{Y}_1(s,\mu)\frac{d\mu}{\mu-z}. \tag{3.10}$$

If $s\in S_0$, then the problem (3.6) has index $\varkappa(s)=-1$. Therefore, on the solution (3.9) we must impose the solvability condition $\Phi(s,\ \infty)=0$, or

$$\int_0^\infty e^{-\mu^2}\rho(s,\mu)\tilde{Y}_1(s,\mu)d\mu=0. \tag{3.11}$$

Replacing $\tilde{Y}_1(s,\ \mu)$ in this equation by the expression (3.4), we find the coefficient $a_0(s)$ of the discrete spectrum:

$$a_0(s)=-\frac{1}{\eta_0 X(\eta_0)}\int_0^\infty e^{-\mu^2}\rho(s,\mu)\tilde{Y}_0(s,\mu)d\mu. \tag{3.12}$$

In the case $s\in S_0$, the solution (3.9) with the solvability condition (3.12) can now be taken as the function $A(s,\ z)$ introduced earlier by Eq. (3.5).

If $s\notin S_0$, then the solution (3.9) behaves at infinity as $O(1/z)$, and it can be immediately taken as the function $A(s,\ z)$ introduced by Eq. (3.5).

The coefficients $a(s,\ \mu)$ can be found in accordance with the Sochocki formula

$$\mu a(s,\ \mu)=A^+(s,\ \mu)-A^-(s,\ \mu),\ \mu>0.$$

Substituting in this equation the solution (3.9), we obtain

$$a(s,\mu)=\frac{\exp(-\mu^2)}{\sqrt{\lambda^+(s,\mu)\lambda^-(s,\mu)}}\left[-2\sqrt{\pi}\,i\frac{\Phi(s,\mu)}{X(s,\mu)}+\tilde{Y}_1(s,\mu)\right]. \tag{3.13}$$

Thus, the coefficients of the expansion (3.2) corresponding to the discrete and continuous spectra have been found explicitly and are determined by Eqs. (3.12) and (3.13), respectively.

The uniqueness of the expansion (3.2) follows from the orthogonality of the eigenfunctions, which is not proved here. In addition, uniqueness can also be proved by indirect proof as the impossibility of nontrivial expansion of zero with respect to the eigenfunctions of the characteristic equation. The theorem is proved.

Remark 3.1. The transition from the set S to the set S_0 is brought about solely by the boundary condition (0.4b). Indeed, if $s=\sigma$, $-1<\sigma<0$, then, as is readily seen from Eqs. (1.7), the discrete spectrum consists of two purely imaginary zeros.

Therefore, the contribution of the discrete spectrum to the expansion (3.2) oscillates with respect to x, and the boundary condition (0.4b) is not satisfied.

Remark 3.2. We note that an incorrect result is given by Cercignani in [1,2] (see Eq. (6.16) in Chap. 6 of [1]). The discrete mode is in fact absent from the expansion (6.16), as in the stationary case. Formulas (6.17) and (6.18) of the same chapter in [1] are also incorrect.

4. ANALYTIC SOLUTION OF SPECIFIC PROBLEMS

As illustration of the theory, we consider the following model problem. We consider a half-space $x>0$ filled with a monatomic gas and bounded by a wall in the plane $x=0$. We assume that the wall moves in its own plane with an exponentially damped velocity $U(t)=U_0e^{-s_0t}$, $s_0>0$. Then the function $Y(t, x, \mu)$ will satisfy the boundary conditions

$$Y(t, 0, \mu)=2U_0e^{-s_0t} \quad (\mu>0), \tag{4.1}$$

$$Y(t, \infty, \mu)=0, \tag{4.2}$$

where U_0 is the dimensionless velocity of the wall at the time $t>0$.

We are interested in the distribution function of the molecules of the gas. In this case, the solution of Eq. (0.1) can be sought in the form

$$Y(t, x, \mu)=e^{-s_0t}\Psi(s_0, x, \mu).$$

It is readily noted that $\Psi(s_0, x, \mu)=\tilde{Y}(-s_0, x, \mu)$. Therefore, in accordance with Theorem 1, the solution of the problem (0.1), (4.1), and (4.2) is given by the following "wave expansion":

$$Y(t,x,\mu)=\frac{1-s_0}{s_0}\int_0^\infty \exp\left\{-\frac{s_0}{v}(x+vt)\right\}\Phi\left(-s_0, \frac{1-s_0}{s_0}v, \mu\right)a\left(s_0, \frac{1-s_0}{s_0}v\right)dv.$$

Here

$$\eta=\frac{1-s_0}{s_0}v.$$

As an example of a half-space problem, we now consider the more general problem of the propagation of Rayleigh waves in a half-space. Under the conditions of the previous problem, we shall assume that the wall executes in its plane exponentially damped harmonic oscillations with frequency ω, i.e.,

$$U(t)=U_0\exp(-\sigma+i\omega)t, \ \sigma>0, \ t>0.$$

If we seek a solution of Eq. (0.1) in the form

$$Y(t, x, \mu)=\exp(-\sigma+i\omega)t\cdot\Psi(s, x, \mu) \quad (s=-\sigma+i\omega),$$

then the function $\Psi(s, x, \mu)$ satisfies Eq. (0.3). Therefore, in accordance with Theorem 3.1 we have the expansion

$$Y(t,x,\mu)=a_0(s)\frac{1}{\sqrt{\pi}}\frac{\eta_0}{\eta_0-\mu}\exp\left\{-\frac{s}{v_0}(x-v_0t)\right\}$$

$$+\frac{1+s}{s}\int_0^\infty \exp\left\{-\frac{s}{v}(x-vt)\right\}\Phi\left(s, \frac{1+s}{s}v, \mu\right)a\left(s, \frac{1+s}{s}v, \mu\right)dv$$

Here

$$a_0(s)=0, \quad \text{if} \quad s\notin S_0, \quad v=\frac{s}{1+s}\eta, \quad v_0=\frac{s}{1+s}\eta_0, \quad \mathrm{Re}\frac{1+s}{\eta_0}>0.$$

In contrast to the corresponding result of Cercignani (see [1], Chap. 6, §7), for $s\notin S_0$ there is no discrete mode [since $a_0(s)=0$], and therefore the general nature of the solution is also changed. The reason for this is that the index of the problem changes discontinuously when the point leaves the region S. At the same time, all the gas-dynamic processes are rearranged (also discontinuously). As we have already noted, an exception is the interval of values of the parameter s satisfying $-1<\sigma<0$, $\omega=0$, for which the index of the problem (3.6) is -1 but a discrete mode is absent by virtue of the boundary condition (4.2).

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