ON THE COMPLETE INTEGRABILITY OF A NONLINEAR SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS IN TWO-DIMENSIONAL SPACE

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Complete solutions that depend on 2r arbitrary functions are obtained for a system of nonlinear partial differential equations of the form $\rho_{\alpha, z\bar{z}}=\sum_{\beta=1}^{r}k_{\alpha\beta}\exp\rho_{\beta}$, where k is the Cartan matrix of a semisimple algebra of rank r.

1. In the last century, Liouville [1] found the complete solution to the partial differential equation

$$\rho_{z\bar{z}}=e^{2\rho} \quad (z=x+t,\ \bar{z}=x-t \text{ or } z=x-it,\ \bar{z}=x+it), \tag{1}$$

this depending on two arbitrary functions:

$$e^{2\rho}=\Phi_z\bar{F}_{\bar{z}}/(1-\Phi\bar{F})^2, \tag{2}$$

where $\Phi\equiv\Phi(z)$ and $\bar{F}\equiv\bar{F}(\bar{z})$ are arbitrary functions of their arguments provided no reality restrictions are imposed on the solution of (1).

The present paper is devoted to the solution of the analogous problem for a nonlinear system of differential equations of the form

$$\rho_{\alpha,z\bar{z}}=\sum_{\beta=1}^{r}k_{\alpha\beta}\exp\rho_{\beta} \quad (1\leqslant\alpha\leqslant r), \tag{3}$$

where k is the Cartan matrix of an arbitrary semisimple algebra. We shall find explicitly solutions to (3) that depend on 2r arbitrary functions. Note that (1) is a special case of (3) for the algebra A_1, whose Cartan matrix consists of the unique element $k_{11}=2$.

The system of equations (3) with, in general, arbitrary matrix k is encountered in different fields of physics – in the theory of electrolytes, plasma theory, aerodynamics, and elsewhere [2]. A system with Cartan matrix was obtained in [3] in connection with solutions of the cylindrically symmetric Yang–Mills duality equations in the framework of a minimal embedding of SU(2) in a gauge group. In [4], Savel'ev and the author found complete centrally symmetric solutions $\rho_\alpha\equiv\rho_\alpha(z\bar{z})$ to (3) which depend on 2r arbitrary constants and identified solutions that lead to finite values of the topological charge. It follows from the results of the present paper that the system of duality equations is completely integrable under the assumption of cylindrical symmetry.

2. Making the change of variables $x=k^{-1}\rho$, we rewrite (3) in the form

$$(x_\alpha)_{z\bar{z}}=\exp(kx)_\alpha. \tag{4}$$

We begin our solution of the system (4) with the case of the series A_n; substituting in (4) the explicit form of the Cartan matrix (see, for example, [5]), we obtain

$$\begin{aligned}
(x_1)_{z\bar{z}}&=\exp(2x_1-x_2),\\
(x_2)_{z\bar{z}}&=\exp(-x_1+2x_2-x_3).\\
&\cdots\cdots\cdots\\
(x_\alpha)_{z\bar{z}}&=\exp(-x_{\alpha-1}+2x_\alpha-x_{\alpha+1}),\\
&\cdots\cdots\cdots\\
(x_n)_{z\bar{z}}&=\exp(-x_{n-1}+2x_n).
\end{aligned} \tag{5}$$

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We introduce the notation $\exp(-x_1) = X$. The first equation of the system (5) enables us to find $\exp(-x_2) = X_z X_{\bar z} - X X_{z\bar z}$; from the second equation of (5) we find

$$\exp(-x_3) = -\mathrm{Det}\begin{pmatrix} X & X_z & X_{zz} \\ X_{\bar z} & X_{z\bar z} & X_{zz\bar z} \\ X_{\bar z\bar z} & X_{\bar z\bar z z} & X_{\bar z\bar z zz} \end{pmatrix} = -\mathrm{Det}_3(X).$$

Continuing the reduction process, we obtain

$$\exp(-x_\alpha) = (-1)^{\alpha(\alpha-1)/2}\,\mathrm{Det}_\alpha(X) \quad (1 \leqslant \alpha \leqslant n), \tag{6a}$$

$$\exp(-x_{n+1}) = (-1)^{n(n+1)/2}\,\mathrm{Det}_{n+1}(X) \equiv 1. \tag{6b}$$

Equation (6b) is a direct consequence of the last equation of the system (5) – it is a nonlinear equation of order 2n for one unknown function X and is equivalent to the system (5). Any solution of (6b), when substituted in (6a) gives a solution of the system (5). We shall seek X in the form

$$X = \sum_{\alpha=1}^{n+1} \Phi^\alpha(z)\bar F^\alpha(\bar z). \tag{7}$$

The proposed form for X (7) is suggested on the one hand by the known complete cylindrically symmetric solutions of (5) (see [4]) and on the other by the known complete solution of the single-component Liouville equation (2).

Substitution of (7) in (6b) leads to a product of two determinants of order n + 1 of the matrices $\Phi_\beta{}^\alpha = \Phi^\alpha_{\underbrace{zz\ldots z}_\beta}$, $\bar F_\beta{}^\alpha = \bar F^\alpha_{\underbrace{\bar z\bar z\ldots\bar z}_\beta}$:

$$\mathrm{Det}\,\Phi_\beta{}^\alpha \times \mathrm{Det}\,\bar F_\beta{}^\alpha = (-1)^{n(n+1)/2}. \tag{8}$$

Thus, the system of partial differential equations (3) can be replaced by two ordinary differential equations of order n for the determination of Φ^{n+1}, $\bar F^{n+1}$ from known Φ^α, $\bar F^\alpha$, $1 \leqslant \alpha \leqslant n$.

For n = 1,

$$\Phi^1\Phi_z{}^2 - \Phi_z{}^1\Phi^2 = -1, \quad \bar F_{\bar z}{}^1\bar F^2 - \bar F^1\bar F_{\bar z}{}^2 = 1. \tag{8}$$

A particular solution of (9) is $\Phi^1 = z$, $\Phi^2 = 1$; $\bar F^1 = -\bar z$, $\bar F^2 = 1$. It follows from the conformal invariance of the system (3) that the functions

$$\Phi^1 = \Phi(\Phi_z)^{-1/2}, \quad \Phi^2 = (\Phi_z)^{-1/2}; \quad \bar F^1 = -\bar F(\bar F_{\bar z})^{-1/2}, \quad \bar F^2 = (\bar F_{\bar z})^{-1/2} \tag{10}$$

satisfy (9) (where $\Phi(z)$ and $\bar F(\bar z)$ are arbitrary functions of their arguments), which can be readily seen by direct substitution of (10) in (9). Thus, the general solution of the Liouville equation has the form $\exp(-x) = (\Phi\bar F - 1)/(\Phi_z\bar F_{\bar z})^{1/2}$, which agrees with (2).

We assume inductively that the functions Φ^α_{n-1} and $\bar F^\alpha_{n-1}$ $(1 \leqslant \alpha \leqslant n-1)$ satisfy (8) in the case of the algebra A_{n-1}. Then there is one obvious particular solution of (8) for the algebra A_n:

$$\Phi_n{}^\alpha = \int^z dv\,\Phi^\alpha_{n-1}(v), \quad \bar F_n{}^\alpha = \int^{\bar z} du\,\bar F^\alpha_{n-1}(u), \quad \Phi_n{}^n = -\bar F_n{}^n = 1 \quad (1 \leqslant \alpha \leqslant n-1).$$

The functions Φ^α_{n-1} depend functionally (through quadrature) on n – 1 arbitrary functions (the same is true of $\bar F^\alpha_{n-1}$). Using the conformal invariance of (5), we find that the functions

$$\Phi_n{}^\alpha = (\Phi_z)^{-n/2}\int^{\Phi(z)} du\,\Phi^\alpha_{n-1}, \quad \Phi^n{}_n = (\Phi_z)^{-n/2}, \quad \bar F_n{}^\alpha = (\bar F_{\bar z})^{-n/2}\int^{\bar F(\bar z)} dv\,\bar F^\alpha_{n-1}, \quad \bar F_n{}^n = (\bar F_{\bar z})^{-n/2}, \tag{11}$$

satisfy (8) and depend on n arbitrary functions, leading, thus, to the complete solution of the system (4) in the case of the algebra A_n. It is easily shown by induction that the explicit expressions for Φ^α_n and $\bar F^\alpha_n$ have the form

$$\Phi_n{}^\alpha = \varphi_0\int^z \varphi_1\,dz_2\int^z \varphi_2\,dz_3\ldots\int^{z_\alpha} \varphi_\alpha\,dz_{\alpha+1}; \quad (\varphi_0)^{-1} = \prod_1^n (\varphi_\alpha)^{\alpha/(n+1)},$$
$$\bar F_n{}^\alpha = (-1)^\alpha\bar\varphi_0\int^z \bar\varphi_1\,d\bar z_2\int^{\bar z_2} \bar\varphi_2\,d\bar z_3\ldots\int^{\bar z_\alpha} \varphi_\alpha\,dz_{\alpha+1}; \quad (\bar\varphi_0)^{-1} = \prod_1^n (\bar\varphi_\alpha)^{\alpha/(n+1)}. \tag{12}$$

In (12) $(0\leqslant\alpha\leqslant n)$ $\int^{z}\varphi_1\ldots\int^{z_\alpha}\varphi_0=1$; $(-1)^\alpha$ in the expression for F_n^α is needed to satisfy (8). Thus, (6a) and (12) solve the problem of complete integration of the system (4) in the case of the series A_n.

Note that any permutation of the function Φ_n^α with any simultaneous permutation of the function F_n^α such that the sign of the product of the determinants does not change also leads to a complete solution of (5) and reflects the invariance of the algebra A_n under transformations of the Weyl group (or rather, the Weyl group of the algebra GL(n, C) if we drop the requirement that the solutions of the system (5) be real).

To obtain complete solutions in the case of the series B_n and C_n, we can use a recursive procedure analogous to (6), the last equation for x_n being the equation for determining X:

$$(-1)^n\,\mathrm{Det}_{n+1}(X)=2\,\mathrm{Det}_n(X) \text{ in the case of } B_n,\quad \mathrm{Det}_{n+1}(X)=-\mathrm{Det}_{n-1}(X) \text{ in the case of } C_n.$$

It is however simpler to use the symmetry of the system (5) under the substitution $x_\alpha\rightleftharpoons x_{n+1-\alpha}$ and find conditions on the functions φ_α and $\overline{\varphi}_\alpha$ that lead to solutions with $x_\alpha=x_{n+1-\alpha}$; in the case of even n = 2k, the system (5) (after the substitution $x_k\to 2x_k$) goes over into the system of equations (2) with Cartan matrix of the series B_k, while in the case of odd n = 2k + 1 we obtain the system of equations (2) with Cartan matrix of the series C_k.

It follows from (6) and (7) that $(-1)^\alpha\exp(-x_\alpha)$ is equal to the sum of the mutual products of the minors of order α constructed from the first α rows of the matrices $\Phi^\alpha_{\underbrace{zz\ldots z}_{\beta}}$, $\overline{F}^\alpha_{\underbrace{\bar z\bar z\ldots\bar z}_{\beta}}$. For the minors of n-th order $\tilde{\Phi}_n{}^\alpha$, $\tilde{\overline{F}}_n{}^\alpha$ we find by induction

$$\tilde{\Phi}_n{}^\alpha=\tilde{\varphi}_0\int^{z}\varphi_n\,dz_2\int^{z_2}\varphi_{n-1}\,dz_3\ldots\int^{z_\alpha}\varphi_\alpha\,dz_{\alpha+1},\qquad(\tilde{\varphi}_0)^{-1}=\prod_1^n(\varphi_\alpha)^{(n+1-\alpha)/(n+1)}\tag{13}$$

For $\tilde{\overline{F}}_n{}^\alpha$, we obtain similar expressions with the obvious substitution $\varphi_\alpha\to\overline{\varphi}_\alpha$, etc.

Comparing (12) and (13), we find the condition $(\Phi_n{}^\alpha=\tilde{\Phi}_n{}^\alpha)$, which leads to solutions of the system (5) for which $x_\alpha=x_{n+1-\alpha}$,

$$\varphi_\alpha=\varphi_{n+1-\alpha},\quad \overline{\varphi}_\alpha=\overline{\varphi}_{n+1-\alpha}.\tag{14}$$

In the case of even n = 2k, (14) imposes precisely k additional conditions on φ_α and $\overline{\varphi}_\alpha$; in this case, (12) and (14) determine the solution of the system (4) for the case of the series B_k $(O(2k+1))$. In the case of odd n = 2k + 1, the number of additional conditions (14) is again equal to k (for α = k, (14) in this case is satisfied identically), and (12)-(14) lead to the solution of the system (4) for the series C_n.

The obtained solutions can be expressed in a unified manner in terms of the root spaces of the corresponding series A_n, B_n, C_n. We call the functions φ_α and $\overline{\varphi}_\alpha$ introduced in (14) simple roots of the corresponding series. In the case of the series A_n, α ranges over all values from 1 to n; for even n = 2k, we have by virtue of (14) $1\leq\alpha\leq k$ $(O(2k+1))$ and for odd $1\leq\alpha\leq k+1$ we have $Sp\,2(k+1)$.

Then (12) determines $\varphi_0{}^{-1}\Phi_n{}^\alpha$ as a multiple integral of the product of simple roots taken in an appropriate order and corresponding to all multiple roots in which the given simple root π_1 occurs, it being necessary in all multiple roots to add a zeroth root corresponding to α = 0, for which $\varphi_0{}^{-1}\Phi_n{}^0=1$.

Thus, to construct the functions Φ^α and $\overline{F}^\alpha$ it is necessary to find all the multiple roots of the algebra containing a simple root encountered a minimal number of times in them. The index α of the functions Φ^α and $\overline{F}^\alpha$ in (12) must be associated with one of these roots, after which Φ^α is determined as a multiple integral of the product of the simple roots from which the given multiple root is composed.

We consider as an example the algebra C_3. The system of multiple roots containing π_1 is π_1, $\pi_1+\pi_2$, $\pi_1+\pi_2+\pi_3$, $\pi_1+2\pi_2+\pi_3$, $2\pi_1+2\pi_2+\pi_3$. For brevity, we omit the sign of the multiple integral, but we retain the order in which the factors in the integrand are arranged:

$$\Phi_{\pi_1}\Rightarrow(\varphi_1),\quad \Phi_{\pi_1+\pi_2}\Rightarrow(\varphi_1,\varphi_2),\quad \Phi_{\pi_1+\pi_2+\pi_3}\Rightarrow(\varphi_1,\varphi_2,\varphi_3),\quad \Phi_{\pi_1+2\pi_2+\pi_3}\Rightarrow(\varphi_1,\varphi_2,\varphi_3,\varphi_2),\quad \Phi_{2\pi_1+2\pi_2+\pi_3}\Rightarrow(\varphi_1,\varphi_2,\varphi_3,\varphi_2,\varphi_1),$$

where we denote by φ_α the functions of a simple root introduced above in (14).

In the case of the series D_n (orthogonal group of even order $O(2n)$) the roots of the algebra determining X are*

* The last expression $2\pi_1+\ldots+2\pi_{n-2}+\pi_{n-1}+\pi_n$ is not a root of the algebra D_n; however, the corresponding function Φ^α must be added to obtain the correct expressions for x.

$$\pi_1, \pi_1+\pi_2, \ldots, \pi_1+\pi_2+\ldots+\pi_{n-2}, \pi_1+\pi_2+\ldots+\pi_{n-2}+\pi_{n-1}, \quad \pi_1+\pi_2+\ldots+\pi_{n-2}+\pi_n, \pi_1+\pi_2+\ldots+\pi_{n-1}+\pi_n,$$
$$\pi_1+\pi_2+\ldots+2\pi_{n-2}+\pi_{n-1}+\pi_n, \quad \pi_1+2\pi_2+\ldots+2\pi_{n-2}+\pi_{n-1}+\pi_n, 2\pi_1+2\pi_2+\ldots+2\pi_{n-2}+\pi_{n-1}+\pi_n. \tag{15}$$

The associating of the functions Φ^α and F^α in accordance with (15) undergoes some changes as compared with the series A_n, B_n, and C_n considered above. Up to the root $\pi_1+\pi_2+\ldots+\pi_{n-2}+\pi_{n-1}+\pi_n$ the construction law is the same as in the preceding cases:

$$\Phi_{\pi_1+\ldots+\pi_n} \Rightarrow (\varphi_1, \varphi_2, \ldots, \varphi_{n-1}, \varphi_n) + (\varphi_1, \varphi_2, \ldots, \varphi_n, \varphi_{n-1}).$$

For the following functions Φ_n^α, there is a symmetrization of the multiple integrals with respect to the roots π_n and π_{n-1}, i.e.,

$$\Phi_{\pi_1+\ldots+2\pi_{n-2}+\pi_{n-1}+\pi_n} \Rightarrow (\varphi_1, \varphi_2, \ldots, \varphi_{n-2}, \varphi_{n-1}, \varphi_n, \varphi_{n-2}) + (\varphi_1, \varphi_2, \ldots, \varphi_{n-2}, \varphi_n, \varphi_{n-1}, \varphi_{n-2}),$$

where the multiple integrals are taken in the order indicated in the brackets.

For the series D_n, we find

$$\exp(-x_\alpha) = (-1)^{\alpha(\alpha-1)/2}\,\mathrm{Det}_\alpha(X) \quad (1 \leqslant \alpha \leqslant n-2), \quad \exp(-x_{n-1}-x_n) = (-1)^{(n-1)(n-2)/2}\,\mathrm{Det}_{n-1}(X);$$
$$\exp(-2x_{n-1}) + \exp(-2x_n) = (-1)^{n(n-1)/2}\,\mathrm{Det}_n(X).$$

The two last relations determine $\exp(-x_{n-1})$, $\exp(-x_n)$, for which one can write down independent expressions analogous to the expression for X in terms of a chain of multiple roots containing π_{n-1} and π_n, respectively.

In the case of the algebra E_6, the augmented system corresponding to the first simple root has the form π_1, $\pi_1+\pi_3$, $\pi_1+\pi_3+\pi_4$, $\pi_1+\pi_3+\pi_4+\pi_5$, $\pi_1+\pi_2+\pi_3+\pi_4$, $\pi_1+\pi_3+\pi_4+\pi_5+\pi_6$, $\pi_1+\pi_2+\pi_3+\pi_4+\pi_5$, $\pi_1+\pi_2+\pi_3+\pi_4+\pi_5+\pi_6$, $\pi_1+\pi_2+\pi_3+2\pi_4+\pi_5$, $\pi_1+\pi_2+\pi_3+2\pi_4+\pi_5+\pi_6$, $\pi_1+\pi_2+2\pi_3+2\pi_4+\pi_5$, $\pi_1+\pi_2+\pi_3+2\pi_4+2\pi_5+\pi_6$, $\pi_1+\pi_2+2\pi_3+2\pi_4+\pi_5+\pi_6$, $2\pi_1+2\pi_2+2\pi_3+2\pi_4+\pi_5+\pi_6$, $2\pi_1+\pi_2+2\pi_3+2\pi_4+2\pi_5+\pi_6$, $2\pi_1+\pi_2+2\pi_3+3\pi_4+2\pi_5+\pi_6$, $2\pi_1+\pi_2+3\pi_3+3\pi_4+2\pi_5+\pi_6$, $2\pi_1+2\pi_2+2\pi_3+3\pi_4+2\pi_5+\pi_6$, $2\pi_1+2\pi_2+3\pi_3+3\pi_4+2\pi_5+\pi_6$, $2\pi_1+2\pi_2+3\pi_3+4\pi_4+2\pi_5+\pi_6$, $2\pi_1+2\pi_2+3\pi_3+4\pi_4+3\pi_5+\pi_6$, $2\pi_1+2\pi_2+3\pi_3+4\pi_4+3\pi_5+2\pi_6$. Then the expression for X is constructed in accordance with the standard rules:

$$\exp(-x_2) = -\frac{\mathrm{Det}_4(X)}{\mathrm{Det}_2(Y)}, \quad \exp(-x_3) = -\frac{\mathrm{Det}_4(Y)}{\mathrm{Det}_4(X)} \cdot \mathrm{Det}_2(Y) = -\mathrm{Det}_2(X),$$
$$\exp(-x_4) = -\mathrm{Det}_3(X) = -\mathrm{Det}_3(Y), \quad \exp(-x_5) = -\mathrm{Det}_2(Y), \quad Y = \exp(-x_6).$$

To find $\exp(-x_5)$ and $\exp(-x_6)$, it is convenient to use the symmetry of the system (4) (for E_6) under the transformations $x_6 \rightleftarrows x_1$, $x_5 \rightleftarrows x_3$. The solutions for the algebra F_4 are obtained from the case E_6 by setting $x_6 = x_1$, $x_3 = x_5$, which follows from comparison of the Cartan matrices for these algebras. The functions $\exp(-x_1)$ and $\exp(-x_2)$ for the algebra G_2 can be constructed by equating the functions x_3 and x_1 for the case of the algebra $B_3(O(7))$ (explicit expressions for the solutions in the algebra G_2 are given in [4]). We shall not write out the cumbersome systems of roots of the algebras E_7 and E_8. To solve the systems (4) in these cases, the results above are sufficient.

3. The complete integrability of the system (3) poses numerous interesting questions of both mathematical nature and relating to the physical applications of the obtained results; these call for further investigation. In the first place, we must consider to what extent the connection between complete integrability of the system (3) and the properties of semisimple algebras is or is not fortuitous; do there exist other nontrivial matrices k (which are not Cartan matrices) leading to complete integrability of (3). The method of integrating the system (3) developed in the present paper has a semi-invariant nature, since we succeeded in finding a solution in the case of the algebra A_n and the results for the other algebras (B, C, D, E, F, G) were actually obtained by the simple restriction of A_n to them. It would be interesting to rewrite the solutions of (3) (or rederive them) using only the invariant root technique, which would clarify the situation considerably.

The system (3) evidently possesses a Bäcklund transformation [6] relating the solutions of (3) to r free Laplace equations ($x_{z\bar{z}}{}^\alpha = 0$) by analogy with the case of the algebra A_1 [7]. It would be interesting to relate the integration of (3) to the inverse scattering problem [8]; it is well known that this can be done in the case of the Liouville equation (1). We note finally that the Liouville equation is intimately related to the sine–Gordon equation; they both describe two-dimensional spaces of constant curvature in different coordinate systems. Does there exist a connection between the system (3) and the many-component sine–Gordon equation? If this question has an answer in the affirmative, one could use the many-component sine–Gordon equations to construct exactly solvable two-dimensional models of quantum field theory with an exact S matrix [9]. The quantum numbers of the soliton states of such models would form multidimensional manifolds.

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VACUUM EXPECTATION VALUES OF THE ENERGY–MOMENTUM TENSOR OF QUANTIZED FIELDS IN A HOMOGENEOUS ISOTROPIC SPACE-TIME

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A unified method is used to calculate regularized vacuum expectation values of the energy–momentum tensors for scalar and spinor fields in a homogeneous isotropic metric with hyperbolic, spherical, and flat three-space. Allowance is made for the contribution of produced particles and also vacuum polarization by the gravitational field. For massless fields, the obtained expectation values have a nonvanishing trace (conformal anomalies).

1. Introduction

In recent years, much interest has been shown in effects that arise from the interaction of quantized fields with an external gravitational field. These effects include the production of particle–antiparticle pairs, vacuum polarization, and also spontaneous symmetry breaking.

The most important quantities for describing these effects are the expectation values of the operator of the energy–momentum tensor T_{ik} of the quantized field. In order to give expressions of the type $\langle\Psi|T_{ik}|\Psi\rangle$ a meaning, it is necessary to solve two related problems.

First, it is necessary to construct the Fock space of the states $|\Psi\rangle$ of the field and define particle creation and annihilation operators. In curved space-time, there is no Poincaré invariance, which provides the basis of the standard construction for a free field in Minkowski space [1], and the corpuscular interpretation of a field becomes ambiguous.

Second, for any choice of the Fock space the expectation values $\langle\Psi|T_{ik}|\Psi\rangle$ diverge. Such divergences also arise in Minkowski space, where they are eliminated by normal ordering of the energy–momentum tensor operator. In curved space-time, normal ordering requires definition of the creation and annihilation operators and the vacuum, which brings us back to the first problem.

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