expressions. For concrete calculations, this renormalization must be restored.

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KUBO–MARTIN–SCHWINGER STATES OF CLASSICAL DYNAMICAL SYSTEMS WITH INFINITE PHASE SPACE

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An example of a classical dynamical system with infinite phase space that satisfies the analog of the Kubo–Martin–Schwinger conditions for classical dynamics is constructed explicitly. Attention is drawn to the connection between the constructed system and the representation of dynamics in a Fock space.

1. The discovery of the role of Kubo–Martin–Schwinger (KMS) conditions as a stability criterion for dynamical systems [1-2] prompted the study of classical dynamical systems that satisfy these conditions [3-4]. In particular, in [5] an example was given (gas of noninteracting particles) of a classical dynamical system whose phase space is $\mathbf{R}^\infty$ and which satisfies the classical analog of the KMS conditions. The present paper is devoted to the construction of a somewhat different example of a dynamical system with infinite phase space that satisfies the classical analog of the KMS conditions.

2. We give the main definitions. Let $\mathfrak{A}$ be an algebra of complex-valued functions and $\{\cdot,\cdot\}$ be the bilinear operation (Poisson brackets) that defines the structure of a Lie algebra on $\mathfrak{A}$, with

$$\{AB, C\}=A\{B, C\}+B\{A, C\}. \tag{1}$$

Let α^t be the group of automorphisms of the algebra $\mathfrak{A}$, i.e., the action of an additive group $\mathbf{R}^1$ on $\mathfrak{A}$ such that

$$\alpha^{t_1+t_2}(A)=\alpha^{t_1}(\alpha^{t_2}(A)),\ \alpha^t(AB)=\alpha^t(A)\cdot\alpha^t(B),\ \alpha^t(aA+bB)=a\alpha^t(A)+b\alpha^t(B),\ \alpha^t(\{A, B\})=\{\alpha^t(A),\ \alpha^t(B)\}. \tag{2}$$

Let $\omega\colon\mathfrak{A}\to\mathbf{C}^1$ be a state on $\mathfrak{A}$, i.e., a linear non-negative normalized functional on $\mathfrak{A}$. We shall not assume that the algebra $\mathfrak{A}$ is equipped with any topology, and accordingly we shall not assume that the state ω is continuous.

<u>DEFINITION.</u> The state ω satisfies the KMS conditions with respect to the group of automorphisms α^t if

$$\forall A, B\in\mathfrak{A}\colon\ \omega(\alpha^t(A))=\omega(A),\quad \omega(\{\alpha^t(A),B\})=\beta\frac{d}{dt}\omega(\alpha^t(A)\cdot B);\quad \beta\in\mathbf{R}^1. \tag{3}$$

The aim of the present paper is to construct an example of an algebra $\mathfrak{A}$, automorphism α^t, and state ω for which the conditions (3) are satisfied (of course, only the case when $\alpha^t(A)\neq A$, $\{A, B\}\neq 0$ is of interest).

3. Let H be a real Hilbert space, $(\cdot,\cdot)$ be the scalar product in H, $H^{(2)}=H\oplus H$, and θ be a bilinear

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skew-symmetric form on $H^{(2)}: \theta\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix}\right) = (a, b') - (b, a')$. Let B be a self-adjoint operator in H; we shall assume that the set

$$D_0 = \bigcap_{\gamma=2,1,-1/2,-1} D(B^\gamma)$$

is dense in H (D(B) is the domain of definition of B). Let L be an operator in $H^{(2)}$: $L = \begin{pmatrix} 0, & E \\ -B^2, & 0 \end{pmatrix}$, $[\cdot, \cdot]$ be a bilinear form on $H^{(2)}$:

$$\left[\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix}\right] = \theta\left(L\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix}\right) = (Ba', Ba) + (b', b),$$

and let H_B be the Hilbert space that is the completion of the domain of definition of the form $[\cdot, \cdot]$ with respect to the metric induced by it. We assume that, as a set, $H_B \subset H^{(2)}$. In H_B, we consider the operator

$$U_t^0 : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix}, \quad \begin{matrix} x' = (\cos tB)x + B^{-1}(\sin tB)y, \\ y' = -B(\sin tB)x + (\cos tB)y. \end{matrix}$$

By direct calculation we verify

LEMMA 1. The operator U_t^0 is unitary in H_B: $[U_t^0 z, U_t^0 z'] = [z, z']$, $z, z' \in H_B$. Let $C_T(H_B)$ be the topological vector space of functions $R^1 \to H_B$ that are continuous in the weak topology of H_B; by definition, the base of neighborhoods of the origin in $C_T(H_B)$ consists of sets of the form $N(\varepsilon, \{f_i\}, T) = \{u; \sup_{t \in T} |\langle f_i, u\rangle| < \varepsilon\}$, where $\{f_i\}$ is a finite subset in H_B and T is a compactum in R^1. In $C_T^*(H_B)$, we consider a subset of the form

$$\tilde{D} = \left\{f; f = \sum_{i=1}^{n(f)} \alpha_i(\tau)\varphi_i\right\},$$

where $\alpha_i(\tau)$ are arbitrary continuous functions with compact support and $\varphi_i \in D(L) \subset H_B$. Let $\tilde{\tilde{D}}$ be the closure of the set $\tilde{D}$ in the metric

$$\rho(f, g) = \left\| \int_{-\infty}^{\infty} U_{-t}^0 (f-g)\,dt \right\|_{H_B} + \left\| L \int_{-\infty}^{\infty} U_{-t}^0 (f-g)\,dt \right\|_{H_B} + \int_{-\infty}^{\infty} \|f-g\|_{H_B} dt.$$

We set

$$D = \tilde{\tilde{D}} \cap C^*(H_B), \quad A_f(u) = \exp\left(i\int_{-\infty}^{\infty} [f, u]\,dt\right), \quad f \in D, \quad u \in C_T(H_B).$$

Let $\mathfrak{A}$ be the algebra of functions on $C_T(H_B)$ of the form

$$A(u) = \sum_{i=1}^{n(A)} \alpha_i A_{f_i}(u) \quad \alpha_i \in \mathbf{C}^1.$$

We set

$$\{A_f, A_g\}(u) = -\theta\left(L\int_{-\infty}^{\infty} U_{-t}^0 f(t)\,dt,\ L\int_{-\infty}^{\infty} U_{-t}^0 g(t)\,dt\right) A_{f+g}(u) = -\left[\int_{-\infty}^{\infty} U_{-t}^0 f(t)\,dt,\ L\int_{-\infty}^{\infty} U_{-t}^0 g(t)\,dt\right] A_{f+g}(u) \tag{4}$$

and extend the definite of the brackets $\{\cdot, \cdot\}$ to all elements of the algebra $\mathfrak{A}$ by linearity.

LEMMA 2. The algebra $\mathfrak{A}$ with bilinear operation $\{\cdot, \cdot\}$ is a Lie algebra.

Proof. It is sufficient to verify the Jacobi identity on the generators. We set

$$c(f, g) = -\theta\left(L\int_{-\infty}^{\infty} U_{-t}^0 f(t)\,dt,\ L\int_{-\infty}^{\infty} U_{-t}^0 g(t)\,dt\right).$$

We note that $c(f, g) = -c(g, f)$. From the definition (4), we obtain

$$\{\{A_f, A_g\}, A_h\} + \{\{A_g, A_h\}, A_f\} + \{\{A_h, A_f\}, A_g\} = (c(f, g)c(f, h) + c(f, g)c(g, h) + c(g, h)c(g, h) + c(g, h)c(h, f) +$$
$$c(h, f)c(h, g) + c(h, f)c(f, g))A_{f+g+h} = 0.$$

LEMMA 3. The elements of the algebra $\mathfrak{A}$ satisfy

$$\{AB, C\} = A\{B, C\} + B\{A, C\}. \tag{5}$$

Proof. It is sufficient to prove Eq. (5) on the generators:

$$\{A_f\cdot A_g, A_h\}=\{A_{f+g}, A_h\}=(c(f,h)+c(g,h))A_{f+g+h}=A_g\{A_f, A_h\}+A_f\{A_g, A_h\}.$$

On the generators A_f, we define the automorphism

$$\alpha^t: A_{f(\tau)}\to A_{f(\tau-t)} \tag{6}$$

and extend this definition to all elements of $\mathfrak{A}$ by linearity.

LEMMA 4. For the automorphism α^t Eqs. (2) are satisfied.

Proof. It follows from the definitions that

$$\{\alpha^t(A_f), \alpha^t(A_g)\}=\{A_{f(\tau-t)}, A_{g(\tau-t)}\}=-\theta\left(L\int_{-\infty}^{\infty}U_{-\tau}^0 f(\tau-t)\,d\tau,\ L\int_{-\infty}^{\infty}U_{-\tau}^0 g(\tau-t)\,d\tau\right)A_{f(\tau-t)+g(\tau-t)}=$$

$$-\theta\left(LU_{-t}^0\int_{-\infty}^{\infty}U_{-\tau}^0 f(\tau)\,d\tau,\ LU_{-t}^0\int_{-\infty}^{\infty}U_{-\tau}^0 g(\tau)\,d\tau\right)A_{f(\tau-t)+g(\tau-t)}=$$

$$-\theta\left(L\int_{-\infty}^{\infty}U_{-\tau}^0 f(\tau)\,d\tau,\ L\int_{-\infty}^{\infty}U_{-\tau}^0 g(\tau)\,d\tau\right)A_{f(\tau-t)+g(\tau-t)}=\alpha^t(\{A_f, A_g\}).$$

We have used the unitarity of the operator U_t^0. The verification of the remaining conditions is trivial.

We define the state ω on the generators by

$$\omega(A_f)=\exp\left(-\frac{1}{2\beta}\left[\int_{-\infty}^{\infty}U_{-\tau}^0 f(\tau)\,d\tau,\ \int_{-\infty}^{\infty}U_{-\tau}^0 f(\tau)\,d\tau\right]\right) \tag{7}$$

and extend this definition to all elements of the algebra $\mathfrak{A}$ by linearity. Equation (7) really does define a state, since the function $f\to\omega(A_f)$ is positive definite:

$$\sum \alpha_i\alpha_j^*\omega(A_{f_i-f_j})\geqslant 0.$$

LEMMA 5. The state ω satisfies the KMS conditions.

Proof. The invariance of the state is a consequence of the unitarity of the operator U_t^0. We check the fulfillment of the second condition. We have

$$\omega(\{\alpha^t(A_f), A_g\})=-\left[\int_{-\infty}^{\infty}U_{-\tau}^0 f(\tau-t)\,d\tau,\ L\int_{-\infty}^{\infty}U_{-\tau}^0 g(\tau)\,d\tau\right]\omega(A_{g+f(\tau-t)}),$$

$$\omega(\alpha^t(A_f)\cdot A_g)=\exp\left(-\frac{1}{2\beta}\left[\int_{-\infty}^{\infty}U_{-\tau}^0(f(\tau-t)+g(\tau))\,d\tau,\ \int_{-\infty}^{\infty}U_{-\tau}^0(f(\tau-t)+g(\tau))\,d\tau\right]\right)=\omega(A_{g+f(\tau-t)}),$$

$$\frac{d}{dt}\omega(\alpha^t(A_f)\cdot A_g)=\frac{1}{\beta}\left[L\int_{-\infty}^{\infty}U_{-\tau}^0 f(\tau-t)\,d\tau,\ \int_{-\infty}^{\infty}U_{-\tau}^0 g(\tau)\,d\tau\right]\omega(A_{g+f(\tau-t)})=\beta^{-1}\omega(\{\alpha^t(A_f), A_g\}).$$

From this our assertion follows. Thus, we have proved

THEOREM. The algebra $\mathfrak{A}$, automorphism α^t and state ω constructed above satisfy the KMS conditions.

4. Let us elucidate the origin of our system. Let $\tilde{\mathfrak{A}}$ be an algebra of functions of the form

$$\tilde{A}(u)=\sum_{|k|\leq m,j}\gamma_j^{(k)}D^k_{\alpha_1,\alpha_2,\ldots,\alpha_s}A_{\alpha_1 f_1+\alpha_2 f_2+\ldots+\alpha_s f_s+f_j}(u)\,|_{\alpha_i=0}.$$

Clearly $\mathfrak{A}\subset\tilde{\mathfrak{A}}$. We extend the definition of the Poisson brackets $\{\cdot,\cdot\}$, the automorphism α^t, and the state ω to the algebra $\tilde{\mathfrak{A}}$ by linearity. It is easily seen that

$$\tfrac{1}{2}\omega\left(A_g\cdot\alpha^t\left(\left\{A_f,\ \left(\int_{-\infty}^{\infty}[\varphi(\tau),u]\,d\tau\right)^2\right\}\right)\right)=\frac{1}{\beta}\left[\int_{-\infty}^{\infty}U_{-\tau}^0 f(\tau-t)\,d\tau,\ L\int_{-\infty}^{\infty}U_{-\tau}^0\varphi(\tau-t)\,d\tau\right]\times$$

$$\left[\int_{-\infty}^{\infty} U_{-\tau}^0 (f(\tau-t)+g(\tau))\,d\tau,\ \int_{-\infty}^{\infty} U_{-\tau}^0 \varphi(\tau-t)\,d\tau\right]\omega(A_{g(\tau)+f(\tau-t)}). \tag{8}$$

Let $\{\psi_j\}$ be a complete orthonormalized system in H_B. Setting $\varphi(\tau)=\psi_j\delta(\tau)$ in (8) and summing over j, we obtain formally

$$\sum_{j=1}^{\infty}\frac{1}{2}\omega(A_g\cdot\alpha^t(\{A_f,[\psi_j,u]^2\}))=\frac{1}{2}\omega(A_g\cdot\alpha^t(\{A_f,[u,u]\}))=$$

$$\frac{1}{\beta}\left[\int_{-\infty}^{\infty} U_{-\tau}^0 f(\tau-t)\,dt,\ L\int_{-\infty}^{\infty} U_{-\tau}^0 g(\tau)\,d\tau\right]\omega(A_{g(\tau)+f(\tau-t)})=(d/dt)\,\omega(A_g\cdot\alpha^t(A_f))$$

(only the first of these equations has not been proved). Thus, for all elements $A_f, A_g\in\mathfrak{A}$

$$(d/dt)\,\omega(A_g\cdot\alpha^t(A_f))=\omega(A_g\cdot\alpha^t(\{A_f,{}^1/_2[u,u]\})),$$

i.e., the state ω is concentrated on orbits of the group of automorphisms α^t that satisfies the equation

$$(d/dt)\,\alpha^t(A_f)=\alpha^t(\{A_f,\mathscr{H}(u)\}),\ \mathscr{H}(u)={}^1/_2[u,u]. \tag{9}$$

This equation describes a Hamiltonian system [6] with Hamiltonian $\mathscr{H}(u)$ and is equivalent to the wave equation $\partial_t\varphi=\pi,\ \partial_t\pi=-B^2\varphi$.

5. We note an interesting connection between our problem and quantum field theory. First, we recall some well-known facts [7, 8].

We transform the set $D_0\oplus D_0\subset H^{(2)}$ into a linear space over $\mathbf{C}^1$, setting by definition $i(x\oplus y)=(-B^{-1}y)\oplus(Bx)$ and on $D_0\oplus D_0$ we define a form that is complex-linear with respect to the second argument: $\langle z,z'\rangle=(B^{1/2}x, B^{1/2}x')+(B^{-1/2}y, B^{-1/2}y')+i\theta(z,z')$, $z=x\oplus y$. Let $\hat{H}$ be the Hilbert space obtained by completing the set $D_0\oplus D_0$ with respect to the scalar product $\langle\cdot,\cdot\rangle$. It is readily verified that the operator U_t^0 is unitary with respect to the form $\langle\cdot,\cdot\rangle$. Let $\mathscr{F}$ be a Hilbert space over $\mathbf{C}^1$ with scalar product $\langle\cdot,\cdot\rangle_{\mathscr{F}}$, $z\to W(z)$ be a mapping of the space $\hat{H}$ into the group of unitary operators on $\mathscr{F}$ that is continuous in the weak topology of $\hat{H}$ and the weak operator topology of $\mathscr{F}$, and

$$W(z)W(z')=\exp\left(\frac{i}{2}\operatorname{Im}\langle z,z'\rangle\right)W(z+z'). \tag{10}$$

It follows from the relation (10) that the mapping $t\to W(tz)$ for any $z\in\hat{H}$ is a group of unitary operators in $\mathscr{F}$. Let $\psi(z)$ be an infinitesimal operator of this group. For $x, y\in\hat{H}$, we set

$$Q(x)=\psi(x\oplus 0),\ P(y)=\psi(0\oplus y),\quad a^*(x)=\frac{1}{\sqrt{2}}(Q(x)+iP(x)),\quad a(y)=\frac{1}{\sqrt{2}}(Q(y)-iP(y)),$$

Then

$$Q(x)P(y)-P(y)Q(x)=i(x,y),\ a^*(x)a(y)-a(y)a^*(x)=(x,y).$$

Let Ω_0 be a vector in $\mathscr{F}$ such that Ω_0 is cyclic for $W(z)$, $\|\Omega_0\|=1$, $a(x)\Omega_0=0$. Then

$$\langle\Omega_0, W(z)\Omega_0\rangle_{\mathscr{F}}=\exp(-{}^1/_4\langle z,z\rangle). \tag{11}$$

Let $d\Gamma(L)$ be an operator which is self-adjoint in $\mathscr{F}$ and satisfies

$$\exp(itd\Gamma(L))\,W(z)\exp(-itd\Gamma(L))=W(U_t^0 z)$$

($d\Gamma(L)$ is the "second-quantized Hamiltonian"; if the operator B is the operator of multiplication by the function $\nu(k)$, then $d\Gamma(L)$ has the standard form $d\Gamma(L)=\int\nu(k)a^*(k)a(k)\,dk$.

We consider the algebra $\mathfrak{A}_0$ of functions on $\hat{H}\cap H_B$ whose generators are functions of the form $A_f^0(z)=\exp(i[f,z])$, $f, z\in\hat{H}\cap H_B$. We transform the algebra $\mathfrak{A}_0$ into a pre-Hilbert space, setting by definition $\langle\langle A_f^0(z), A_g^0(z)\rangle\rangle=\omega(A^*_{f\delta(\tau)}\cdot A_{g\delta(\tau)})$. Let $\hat{\mathfrak{A}}$ be the Hilbert space obtained by completing $\mathfrak{A}_0$ with respect to the scalar product $\langle\langle\cdot,\cdot\rangle\rangle$; we do not know if the state ω is faithful, and therefore the elements of $\hat{\mathfrak{A}}$ are in general equivalence classes with respect to the relation $A\sim B$: $\omega(|A-B|^2)=0$). We define the operator

$$\hat{U}_t:\ \langle\langle A_f^0,\hat{U}_t A_g^0\rangle\rangle=\omega(A^*_{f\delta(\tau)}\cdot A_{g\delta(\tau-t)}).$$

From the definition of the state ω we readily verify directly the equation $\langle\langle A_f, \hat{U}_t A_g\rangle\rangle = \langle\langle A_f, A_{(U^0_{-t}g)}\rangle\rangle$. Let $\hat{A}_g$ be the operator of multiplication by the function A_g acting on $\hat{\mathfrak{A}}$. Comparing (7) and (11), we obtain

$$\langle\langle 1, \hat{A}_g 1\rangle\rangle = \langle\Omega_0, W(K_g)\Omega_0\rangle_{\mathcal{F}}, \quad K = \sqrt{\frac{2}{\beta}}\begin{pmatrix} B^{1/2}, & 0 \\ 0, & B^{1/2} \end{pmatrix}. \tag{12}$$

On the left-hand side of (12) we have essentially the characteristic functional of the measure that is the projection onto the set of functions which depend only on the values of these functions at the point $\tau = 0$ of the measure concentrated on the trajectories of the system (9). On the right we have the characteristic functional of the Fock representation of commutation relations.

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UNIFIED FORMALISM FOR QUANTUM AND CLASSICAL SCATTERING THEORIES

Yu. M. Shirokov

Nonrelativistic scattering theory is formulated solely in terms of physical and mathematical concepts which have meaning in both classical and quantum mechanics. An integral equation is obtained whose iterations give the quantum corrections to classical scattering.

1. In the standard formulations of classical and quantum scattering theory, one uses completely different concepts, which do not "survive" the transition from one theory to the other. In classical theory, there is no analog of the S matrix or the scattering amplitude, whereas in the quantum theory there is no impact parameter. This hinders the development of systematic semiclassical methods. In [1-4], I developed a unified formalism for quantum and classical theories, called the combined quantum–classical algebra. In this formalism, one uses only physical and mathematical concepts which have meaning in both forms of mechanics. I show here that in this manner one can formulate naturally a theory that may be called Hamiltonian scattering theory. It too uses only concepts that have meaning in both classical and quantum mechanics. The theory is therefore suitable for both forms of mechanics.

For simplicity, we restrict ourselves to elastic potential scattering of nonrelativistic spinless particles. We consider Weyl quantization, which is distinguished by the fact that in it free (nonrelativistic) motion occurs in the same way in both the quantum and the classical cases.

Notation introduced without explanation is the same as in [2].

2. Following [2], we shall describe observables by c-number functions $A(\mathbf{p}, \mathbf{q})$ of the coordinates and the momenta and states by linear functionals $\rho(p, q)$. The corresponding topological vector spaces and their topologies are described in [2, 3]. Specific mathematical objects of the combined algebra are the quantum and classical operations of multiplication of observables. In the combined algebra, there are four operations of multiplication of observables: the ordinary classical multiplication π_0, the classical Poisson multiplication σ_0, and, respectively, the ordinary quantum multiplication $\Pi_\hbar$ and quantum Poisson multiplication $\sigma_\hbar$. These operations are defined in [2], in which it is shown that the operations of multiplication are the only objects that change on transitions from the one theory to the other. The Hamiltonian of scattering theory will

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