radius of this core is R. The height of the potential for R = 0.3 F is $V_0 = 2m^2R \approx 270$ MeV. This is an order of magnitude greater than the characteristic depth of the potential well that describes the attraction between nucleons at distances of order 2.4 F [10]; at the same time, for the radius of the nucleon core estimates $R_C = 0.4$ F are given, and the core is assumed to be absolutely hard. In our case, the core is not absolutely hard but its radius is somewhat greater. It can be estimated as $2R \simeq 0.6$ F. In any case, repulsion is needed to prevent collapse of nuclei.

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INTEGRABLE INITIAL–BOUNDARY-VALUE PROBLEMS

I. T. Khabibullin

Initial–boundary-value problems on a half-axis for integrable equations are considered. Solutions of an initial–boundary-value problem for the nonlinear Schrödinger equation and the sine–Gordon equation for definite types of boundary conditions are described in terms of scattering data.

The inverse scattering method is an effective method for investigating nonlinear equations of mathematical physics [1]. It enables one to construct particular solutions of an equation and also to analyze its general solution, for example, the Cauchy problem, in a given class of initial data. However, the investigation of mixed problems on an interval and on a half-axis in the framework of the inverse problem has until recently been restricted to the case of periodic boundary conditions and variants of them (see [2–4]). Interest in mixed problems was sharply increased by [5], in which a class of nontrivial boundary conditions compatible with integrability of the equations was found. The analytic aspect of the "impure" generalization of the inverse scattering method proposed by Sklyanin and its application to problems of mixed type were considered in [6,7].

Alternative approaches to boundary-value problems for integrable equations can be found in [8,9].

In [10,11], a connection was found between boundary conditions compatible with integrability of an equation and its Bäcklund transformations. An initial–boundary-value problem on a half-axis with boundary conditions compatible with complete integrability of the equation can be reduced to a Cauchy problem on the complete axis by continuation of the initial condition by virtue of a Bäcklund transformation. On the other hand, in the "general solution" of the equation one can identify a subclass of solutions that certainly satisfy the boundary condition. In this paper, we obtain an explicit description of such subclasses in terms of scattering data for the nonlinear Schrödinger equation with linear inhomogeneous boundary condition and two different types of mixed problem for the

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sine-Gordon (SG) equation. One of these three problems can be reduced by a Bäcklund transformation to the problem of evolution of initial data of "step" type (see Sec. 3 below), and the other two can be reduced to a Cauchy problem in the class of rapidly decreasing functions (Secs. 2 and 4).

In contrast to [6,7], we use the ordinary version of the inverse scattering method to investigate initial-boundary-value problems in this paper.

1. Bäcklund Transformation and Boundary Condition

We assume that the partial differential equation

$$u_t = f(u, u_1, \dots, u_k) \tag{1.1}$$

admits a representation in the form of a zero-curvature condition:

$$U_t = V_x + [V, U], \tag{1.2}$$

where $u = u(x, t)$, $u_l = \partial^l u / \partial x^l$. The matrix-valued functions $U = U(u\ \lambda)$, $V = V(u, u_1, \dots, u_m, \lambda)$ depend rationally on the parameter $\lambda \in C$. Equation (1.2) is a condition for the existence of a simultaneous solution of the following system of linear differential equations:

$$\Psi_x = U\Psi, \qquad \Psi_t = V\Psi, \tag{1.3}$$

where $\Psi = \Psi(x, t, \lambda)$.

Two solutions $u(x, t)$ and $\bar{u}(x, t)$ of Eq. (1.1) are related by a Bäcklund transformation if the eigenfunctions Ψ and $\bar{\Psi}$ of the linear system (1.3) and the system $\bar{\Psi}_x = \bar{U}\bar{\Psi}$, $\bar{\Psi}_t = \bar{V}\bar{\Psi}$, where $\bar{U} = U(\bar{u}, \lambda)$, $\bar{V} = V(\bar{u}, \bar{u}_1, \dots, \bar{u}_m, \lambda)$, satisfy the condition [12]

$$\bar{\Psi}(x, \lambda) = F(\lambda)\Psi(x, \lambda), \tag{1.4}$$

where $F(\lambda)$ is a matrix polynomial of the parameter λ with coefficients that depend on u, $\bar{u}$, and their derivatives.

In terms of the solutions u and $\bar{u}$, the Bäcklund transformation takes the form of a system of differential constraints:

$$\bar{u}_x = p(\bar{u}, u, u_1, \dots, u_n), \qquad \bar{u}_t = q(\bar{u}, u, u_1, \dots, u_l). \tag{1.5}$$

Suppose the system of constraints (1.5) withstands the reduction $\bar{u}(x) = h(u(-x))$; it is then obvious that $n = 1$. Imposing this reduction on the relations (1.5) and then setting $x = 0$, we obtain a boundary condition of the form (see [10])

$$u_x = -\frac{1}{h_u(u)} p(h(u), u, u_x)\big|_{x=0},$$

which is compatible with integrability of Eq. (1.1).

2. Nonlinear Schrödinger Equation

It is known [1] that the following system of nonlinear partial differential equations, which generalizes the nonlinear Schrödinger equation:

$$-v_t + v_{xx} = 2v^2 w, \qquad w_t + w_{xx} = 2vw^2, \tag{2.1}$$

admits representation in the form of the zero-curvature condition (1.2) with potentials

$$U = \begin{bmatrix} \lambda & w \\ v & -\lambda \end{bmatrix}, \qquad V = \begin{bmatrix} vw - 2\lambda^2 & -w_x - 2\lambda w \\ v_x - 2\lambda v & -vw + 2\lambda^2 \end{bmatrix} \tag{2.2}$$

The system of equations (2.1) withstands an even reduction of the form $u(x) = u(-x)$, where $u = (v, w)$. A Bäcklund transformation that admits the reduction $\bar{u}(x) = u(-x)$ is determined by a polynomial of first degree in λ (see (1.4)):

$$F(\lambda) = \begin{bmatrix} 2\lambda + k(c, u, \bar{u}) & w - \bar{w} \\ \bar{v} - v & 2\lambda - k(c, u, \bar{u}) \end{bmatrix} \tag{2.3}$$

where $k(c, u, \bar{u}) = (c^2 + (\bar{w} - w)(\bar{v} - v))^{1/2}$, and it has the form of the differential constraint

$$\bar{u}_x = u_x + (\bar{u} + u)k(c, u, \bar{u}). \tag{2.4}$$

The even reduction of type $\bar{u}(x) = u(-x)$ in (2.4) leads to the linear boundary condition (see also [5])

$$u_x(0,t)+cu(0,t)=0. \tag{2.5}$$

We shall describe this reduction in terms of scattering data of the system of linear equations $\Psi_x = U\Psi$. We shall assume that the solution u(x, t) decreases sufficiently rapidly as $|x| \to \infty$. (It will be shown below that for a self-conjugate reduction $w^* = -v$, where the asterisk denotes the complex conjugate, decrease at $\pm\infty$ is ensured by decrease at one of the infinite limits and by the reduction condition $\bar{u}(x) = u(-x)$ (see Proposition 1).) By virtue of the reduction condition, the function $k(c, u, \bar{u})$ has the same limit as $x \to \pm\infty$, and therefore the limit

$$F_0(x,\lambda)=\lim_{|x|\to\infty} F(x,t,\lambda)=2\begin{bmatrix}\lambda+\lambda_0 & 0\\ 0 & \lambda-\lambda_0\end{bmatrix},\quad \lambda_0^2=\frac{c^2}{4},$$

exists.

We recall necessary results from general scattering theory (see [1]). For all λ, Re $\lambda = 0$, we determine matrix solutions of the system of linear differential equations

$$y_x=Uy \tag{2.6}$$

that satisfy the asymptotic conditions

$$y_\pm(x,\lambda)\to\exp(\sigma_3 x\lambda),\quad x\to\pm\infty, \tag{2.7}$$

where the potential U has the form (2.2). Here and below, we denote by σ_i the Pauli matrices. It is well known that the first column of the matrix $y_-(x, \lambda)\exp(-\sigma_3 x\lambda)$ and the second column of the matrix $y_+(x, \lambda(\exp(-\sigma_3 x\lambda)$ can be continued analytically with respect to λ into the half-plane Re $\lambda > 0$. The remaining columns of these matrices admit analytic continuation into the left half-plane Re $\lambda < 0$ of the complex plane. The diagonal elements $s_{11}(\lambda)$ and $s_{22}(\lambda)$ of the S matrix $s(\lambda) = y_+^{-1}(x, \lambda)y_-(x, \lambda)$ can also be analytically continued into the half-planes Re $\lambda > 0$ and Re $\lambda < 0$, respectively. At the zeros η_{1k} and η_{2k} of the functions $s_{11}(\lambda)$ and $s_{22}(\lambda)$, repsectively, which are situated in the indicated regions of analyticity, the columns of the solutions $y_\pm(x, \lambda)$ satisfy the conditions of proportionality $y_-^1(x,\eta_{1k})=\gamma_{1k}y_+^2(x,\eta_{1k})$, $y_+^1(x,\eta_{2k})=\gamma_{2k}y_-^2(x,\eta_{2k})$, where $\gamma_{1k},\gamma_{2k}\neq 0$.

Suppose that in the analyticity regions Re $\lambda > 0$ and Re $\lambda < 0$ the functions $s_{11}(\lambda)$ and $s_{22}(\lambda)$ have only simple zeros and do not have zeros on the line Re $\lambda = 0$. Then the S matrix $s(\lambda)$ and the coefficients γ_{ij} corresponding to the zeros η_{ij} form a complete set of scattering data of the linear system (2.6) and uniquely determine its potential $U(x, \lambda)$.

Let u(x, t) and $\bar{u}(x, t)$ be two solutions of the system of equations (2.1) related by the Bäcklund transformations (2.4). Then the corresponding Jost solutions determined by asymptotic behaviors of the form (2.7) satisfy the conditions $\bar{y}_\pm(x,\lambda)=F(\lambda)y_\pm(x,\lambda)F_0^{-1}(\lambda)$. From this, it obvious follows that the scattering data of the two solutions u and $\bar{u}$ are related by

$$\bar{s}(\lambda)=F_0(\lambda)s(\lambda)F_0^{-1}(\lambda),\qquad \bar{\eta}_{ij}=\eta_{ij},\qquad \bar{\gamma}_{ij}=\frac{\eta_{ij}-\lambda_0}{\eta_{ij}+\lambda_0}\gamma_{ij}. \tag{2.8}$$

We now find how the scattering data change under the substitution $x \to -x$. All notation relating to the potential $\tilde{u}(x) = u(-x)$ will be identified by a tilde. It is obvious that $\tilde{U}\overset{\text{def}}{=}U(\tilde{u}(x),\lambda)=-\sigma_3U(u(-x),-\lambda)\sigma_3$, and therefore the Jost solutions of the two linear systems of equations are related by the conditions $\tilde{y}_\pm(x,\lambda)=\sigma_3y_\mp(-x,-\lambda)\sigma_3$. Therefore, we have

$$\tilde{s}(\lambda)=\sigma_3s^{-1}(-\lambda)\sigma_3,\quad \tilde{\eta}_{ij}=-\eta_{2j},\quad \tilde{\eta}_{2j}=-\eta_{1j},\quad \tilde{\gamma}_{1j}=-\gamma_{2j},\quad \tilde{\gamma}_{2j}=-\gamma_{1j}. \tag{2.9}$$

Returning to the original problem of describing the reduction $\bar{u}(x) = u(-x)$, we note that the condition of identity of the scattering data $\bar{s}(\lambda) = \tilde{s}(\lambda)$, $\bar{\gamma}_{ij} = \tilde{\gamma}_{ij}$ and formulas (2.8) and (2.9) immediately show that the scattering data of the solution $u(x, t) = \bar{u}(-x, t)$ satisfy the simple involution*

$$s(\lambda)=F_0^{-1}(\lambda)\sigma_3s^{-1}(-\lambda)\sigma_3F_0(\lambda),\qquad \gamma_{2j}=-\frac{\eta_{1j}-\lambda_0}{\eta_{1j}+\lambda_0}\gamma_{1j}, \tag{2.10}$$

*Equations (2.10) were obtained in collaboration with R. F. Bikbaev.

where the matrix $F_0(\lambda)=2\,\mathrm{diag}(\lambda+\lambda_0, \lambda-\lambda_0)$ is determined explicitly up to the choice of the root $2\lambda_0 = \sqrt{c^2}$.

As an example, we consider the soliton solution of the system (2.1) with the following scattering data (here $\lambda_0=c/2$): $s_{11}(\lambda)=s_{22}(-\lambda)=(\lambda-\eta)/(\lambda+\eta)$, $\mathrm{Re}\,\eta>0$, $s_{12}=s_{21}=0$, $\gamma_{21}=-\gamma_{11}(2\eta-c)/(2\eta+c)$,

$$w(x, t)=4\eta\exp(4i\eta^2 t)/r(x), \quad v(x, t)=\gamma_{11}\gamma_{21}w(x, -t), \quad r(x)=\gamma_{21}\exp(2\eta x)-\gamma_{11}\exp(-2\eta x). \tag{2.11}$$

By direct calculation we readily verify that this solution satisfies the boundary condition (2.5) for any $\eta \neq 0$.

The involution (2.10) is completely consistent with self-conjugate reductions of the system of equations (2.1) of the type $w=\varkappa v^*$, $\varkappa=\pm 1$, which lead to the nonlinear Schrödinger equation

$$iv_\tau=v_{xx}-2\varkappa|v|^2v, \tag{2.12}$$

where $\tau = it$. The scattering data corresponding to these reductions satisfy the conditions $s_{11}^*(-\lambda^*)=s_{22}(\lambda)$, $s_{12}^*(-\lambda^*)=\varkappa s_{21}(\lambda)$, $\eta_{2j}=-\eta_{1k}^*$, $\gamma_{2j}=\varkappa\gamma_{1k}^{-1*}$. The involution (2.10) for the self-conjugate case has the form

$$s_{21}(\lambda)=s_{21}(-\lambda)\frac{\lambda+\lambda_0}{\lambda-\lambda_0}, \quad \eta_{1j}=\eta_{1k}^*, \quad s_{11}(\lambda)=s_{11}^*(\lambda^*), \quad \gamma_{1k}^{-1*}=-\varkappa\frac{\eta_{1j}-\lambda_0}{\eta_{1j}+\lambda_0}\gamma_{1j}. \tag{2.13}$$

Thus, for any $\lambda_0 \in R$ the scattering data (2.13) correspond to a certain solution of the boundary-value problem with a condition of the form (2.5), where $c = 2\lambda_0$ or $c = -2\lambda_0$.

It should be noted that the involution (2.13) describes the "general" solution of Eq. (2.12) with the boundary condition

$$v_x(0, \tau)\pm cv(0, \tau)=0, \qquad c\in R. \tag{2.14}$$

More precisely, for Eq. (2.12) with $\varkappa=-1$ the following proposition is proved in the Appendix.

PROPOSITION 1. For any smooth rapidly decreasing initial condition

$$v(x, 0)=v_0(x), \tag{2.15}$$

defined for $x \geqq 0$ and consistent at the origin with the boundary condition (2.14), the solution of the half-axis problem (2.12), (2.14), (2.15), $\varkappa=-1$ is the restriction to the half-axis of the solution of the Cauchy problem for (2.12) with some smooth rapidly decreasing initial condition.

Thus, the considered initial-boundary-value problem reduces to the well-investigated Cauchy problem on the complete line. An alternative, possibly more laborious method of integrating the problem (2.12), (2.14) is developed in [6].

3. The Sine-Gordon Equation in Light Cone Coordinates

The classical SG equation $u_{xt} = \sin u$ admits a representation in the form of a zero-curvature condition (see [1]), where

$$U=\begin{bmatrix} \lambda & iu_x/4 \\ iu_x/4 & -\lambda \end{bmatrix}, \quad V=\frac{1}{4\lambda}\begin{bmatrix} -\cos u & i\sin u \\ -i\sin u & \cos u \end{bmatrix}.$$

A Bäcklund transformation for this equation is determined by the polynomial

$$F(\lambda)=-\frac{2}{a}\lambda\sigma_3+\sigma_0\cos\frac{\bar{u}-u}{2}+i\sigma_1\sin\frac{\bar{u}-u}{2},$$

where σ_0 is the unit matrix, and it has the form [13]

$$\bar{u}_x=2a\sin\frac{u-\bar{u}}{2}-u_x, \quad \bar{u}_t=u_t-2a^{-1}\sin\frac{\bar{u}+u}{2}. \tag{3.1}$$

Imposing on the solution $u(x, t)$ of the SG equation the reduction $\bar{u}(x, t) = \pi - u(-x, t)$, we arrive at a certain local condition along the characteristic $x = 0$ of this equation (see [10]):

$$u_x(0, t)+a\cos u(0, t)=0. \tag{3.2}$$

We describe in terms of scattering data the class of solutions of the SG equation that satisfy the condition (3.2). Suppose the initial function u(x, 0) is smooth everywhere on the half-axis $x \geqq 0$, decreases rapidly at infinity, and is consistent with (3.2). Solving the first of Eqs. (3.1), we construct a function $\bar{u}(x)$ such that $\bar{u}(0) = \pi - u(0, 0)$. We continue u(x, 0) to the entire axis by means of the equation $u(-x, 0) = \pi - \bar{u}(x)$. Arguments similar to those in the Appendix show that the obtained function will be smooth for all x, and $u(-\infty) = \pi$. Thus, the initial-boundary-value problem is reduced to the Cauchy problem of the evolution of an initial condition of "step" type:

$$u(x, 0) \to {}^1/_2\pi \mp {}^1/_2\pi, \quad x \to \pm\infty, \tag{3.3}$$

where the limiting values are attained by the function u(x, 0) quite rapidly. Many studies (see, for example, [14]) have been devoted to analysis of solutions of such type and also their physical interpretation.

Since the equation with respect to x for the Ψ function in the case of the SG equation has exactly the same form as (2.6), we can use the definition of the scattering data from the previous sections. The main difference between the "step" problem (3.3) and the rapidly decreasing case is in the unusual time dynamics of the scattering data:

$$s_{11}(\lambda, t) = s_{11}(\lambda, 0) \exp\left(-\frac{1}{2\lambda}\right), \quad s_{12}(\lambda, t) = s_{12}(\lambda, 0), \quad \gamma_{ij}(x, t) = \gamma_{ij}(0, 0), \quad \eta_{ij}(t) = \eta_{ij}(0). \tag{3.4}$$

Completely repeating the arguments of the previous section, we can show that the reduction $u(x, t) = \pi - \bar{u}(-x, t)$ corresponds to the following additional involution of the scattering data of the solution:

$$s_{12}(\lambda) = s_{12}(-\lambda)\frac{2\lambda + a}{-2\lambda + a}, \quad s_{11}(\lambda) = s_{22}(-\lambda), \quad \eta_{1j} = -\eta_{2j},$$

$$h_{ij} = \begin{bmatrix} -2\eta_{2j} + a & 0 \\ 0 & -2\eta_{2j} - a \end{bmatrix} h_{2j}, \quad h_{ij}^T = (1, -\gamma_{ij}).$$

4. The Sine–Gordon Equation in Laboratory Coordinates

When reduced to the form

$$u_{tt} - u_{xx} + \sin u = 0, \tag{4.1}$$

the sine–Gordon equation has various physical applications (see, for example, [13]). The linear system of equations associated with this equation has the following form in the framework of the inverse scattering method (see [5]):

$$y_x = \frac{1}{4i}\left(u_t\sigma_3 + k_0\sigma_1 \sin\frac{u}{2} + k_1\sigma_2 \cos\frac{u}{2}\right) y, \quad y_t = \frac{1}{4i}\left(u_x\sigma_3 + k_1\sigma_1 \sin\frac{u}{2} + k_0\sigma_2 \cos\frac{u}{2}\right) y, \tag{4.2}$$

where $k_0 = \lambda + \lambda^{-1}$, $k_1 = \lambda - \lambda^{-1}$, λ is a complex parameter. A Bäcklund transformation for Eq. (4.1) is specified by the differential relations

$$\bar{u}_x = u_t + c \sin\frac{\bar{u} + u}{2} + c^{-1} \sin\frac{\bar{u} - u}{2}, \quad \bar{u}_t = u_x + c \sin\frac{\bar{u} + u}{2} - c^{-1} \sin\frac{\bar{u} - u}{2}. \tag{4.3}$$

In a formula of the type (1.4), this transformation corresponds to a polynomial of the form

$$F(\lambda, c) = \exp(\sigma_3 u/4i) \begin{bmatrix} \lambda & c \\ -c & \lambda \end{bmatrix} \exp(-\sigma_3 \bar{u}/4i). \tag{4.4}$$

It is shown in [5] that the boundary condition

$$u_x(0, t) = p_0 \sin\frac{u(0, t)}{2}, \quad p_0 \in R, \tag{4.5}$$

is compatible with integrability of Eq. (4.1). This boundary condition has a remarkably simple connection to the Bäcklund transformation (4.3) (see [11]). Suppose the solution u(x, t) of Eq. (4.1) is odd in the sense that $u(x, t) = 2m\pi - u(-x, t)$. Then for x = 0 we have $u = \pi m$, $u_t = 0$, and therefore the new solution $\bar{u}(x, t)$ of this equation obtained by applying to u(x, t) the Bäcklund transformation (4.3) will satisfy either the boundary condition (4.5) or the boundary condition

$$u_x(0,t)=q_0\cos\frac{u(0,t)}{2}, \tag{4.5'}$$

the choice depending on the parity of the number m (the condition (4.5′) was also found independently by V. O. Tarasov). For example, setting $u \equiv 0$ in (4.3), we obtain the single-soliton solution of Eq. (4.1)

$$\bar{u}(x,t)=4\,\mathrm{arctg}\exp{}^1/_2(p_0x+q_0t+x_0),$$

$p_0=\pm(c+c^{-1})$, $q_0=\pm(c-c^{-1})$, which, as is readily verified, satisfies the boundary condition (4.5).

Thus, among all solutions of the SG equation there are subclasses of solutions that withstand the condition (4.5) or (4.5′). We describe these two subclasses in terms of the scattering data of the linear system (4.2). On the initial functions $u(x, 0)$ and $u_t(x, 0)$ we impose the usual requirements of smoothness and rapid tending to the limiting values at infinity, which are assumed to be multiples of 2π. A simple analysis of the system of differential equations (4.3) analogous to the one given in the Appendix shows that $\bar{u}(x, t) \to 2\pi m_\pm$ as $x \to \pm\infty$, where $m_\pm$ are certain integers. Therefore, the polynomial (4.4) will have the following limiting values as $x \to \pm\infty$:

$$\lim_{x\to+\infty}F(\lambda,c)=\exp\left(-\frac{i}{2}\pi\sigma_3m_1\right)F_0(\lambda,c)\exp\left(\frac{i}{2}\pi\sigma_3m_+\right),$$

$$\lim_{x\to-\infty}F(\lambda,c)=F_0(\lambda,c)\exp\left(\frac{i}{2}\pi\sigma_3m_-\right),\qquad F_0(\lambda,c)=\begin{bmatrix}\lambda & c\\ -c & \lambda\end{bmatrix}.$$

Following the monograph [15], we determine solutions of the first of Eqs. (4.2) for $\mathrm{Im}\,\lambda = 0$ by means of the asymptotic conditions

$$y_+(x,\lambda)\xrightarrow[x\to+\infty]{}\exp\left(\frac{i\pi}{2}\sigma_3m_1\right)E(x,\lambda),\qquad y_-(x,\lambda)\xrightarrow[x\to-\infty]{}E(x,\lambda),$$

$$E(x,\lambda)=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ i & 1\end{bmatrix}\exp\left(\frac{\lambda-\lambda^{-1}}{4i}x\sigma_3\right). \tag{4.6}$$

The ratio of these two solutions of the problem (4.2) determines its S matrix: $S(\lambda) = y_+^{-1}y_-$. The elements of the S matrix satisfy the involution conditions $s_{22}(\lambda)=s_{11}^*(\lambda)=s_{11}(-\lambda)$, $s_{12}(\lambda)=-s_{21}^*(\lambda)=-s_{21}(-\lambda)$. The zeros λ_j of the function $s_{11}(\lambda)$ in the upper half-plane form the discrete spectrum of the problem (4.2). At the points λ_j the following conditions hold:

$$y_-^1(x,\lambda_j)=\gamma_jy_+^2(x,\lambda_j),\ j=1,\ldots,N.$$

We assume that all the zeros λ_j are simple and that $s_{11}(\lambda) \neq 0$ for $\mathrm{Im}\,\lambda = 0$. Then the S matrix and the coefficients of proportionality γ_j form a complete set of scattering data.

We establish how the scattering data change under Bäcklund transformation. By direct calculation we verify the formula $F_0(\lambda, c)E(x, \lambda)=E(x, \lambda)F_d(\lambda, c)$, where $F_d(\lambda, c)=\mathrm{diag}(\lambda+ic, \lambda-ic)$ is a diagonal matrix. Solutions $\bar{y}_\pm(x, \lambda)$ of a linear problem of the form (4.2) corresponding to the function $\bar{u}(x, t)$ are related to $y_\pm$ by the conditions $\bar{y}_\pm(x,\lambda)=F(\lambda,c)y_\pm(x,\lambda)$ $F_d^{-1}(\lambda,\varkappa_\pm c)$, where $\varkappa_+=(-1)^{m_+-m}$, $\varkappa_-=(-1)^{m_-}$. It immediately follows from this that the scattering data transform as follows:

$$\bar{s}(\lambda)=F_d(\lambda,\varkappa_+c)s(\lambda)F_d^{-1}(\lambda,\varkappa_-c),\qquad \bar{\gamma}_j=\gamma_j(\lambda_j-i\varkappa_+c)/(\lambda_j+i\varkappa_-c). \tag{4.7}$$

In addition, a) an additional eigenvalue $\bar{\lambda}_{N+1}=-ic\varkappa_+$ appears for $c\varkappa_+<0$, $c\varkappa_->0$; b) the eigenvalue $\lambda_k=-ic\varkappa_-$ is removed for $c\varkappa_+>0$, $c\varkappa_-<0$; c) the discrete spectrum is not changed for $\varkappa_-=\varkappa_+$.

We reformulate the odd reduction of the form $u(x, t) = 2\pi m - n(-x, t)$ in the language of scattering data. We first find the connection of scattering data of the two potentials u and ũ such that $\tilde{u}(x, t) = 2\pi m - n(-x, t)$. All variables relating to the potential ũ will be identified by the tilde. It is easy to verify fulfillment of the condition $U(\tilde{u}(x),\lambda)=-U(u(-x),(-1)^m\lambda^{-1})$, where

$$U(u,\lambda)=\frac{1}{4i}\left(u_t\sigma_3+k_0\sigma_1\sin\frac{u}{2}\mp k_1\sigma_2\cos\frac{u}{2}\right).$$

Therefore, the matrix solutions of the two linear systems of equations $y_x = Uy$ and $\tilde{y}_x = \tilde{U}\tilde{y}$ determined by the asymptotic behaviors (4.6) satisfy the condition $\tilde{y}_\pm(x,\lambda)=y_\mp(-x,(-1)^m\lambda^{-1})\sigma_\mp$, where $\sigma_\pm = 1$ for even m and $\sigma_\pm = \pm i\sigma_2$ for odd m. From this we obtain the following connection between the two S matrices: $\tilde{s}(\lambda)=\sigma_-^{-1}(s(\lambda^{-1}(-1)^m))^{-1}\sigma_+$ and also a connection between the two sets of proportionality coefficients: $\tilde{\gamma}_k=\gamma_j^{-1*}$ for even m and $\tilde{\gamma}_k = \gamma_j^{-1}$ for odd m. Therefore, the scattering data of the potential $u(x, t) = 2\pi m - u(-x, t)$ satisfy for $\mathrm{Im}\,\lambda = 0$, $\lambda \neq 0$ the involution

$$s_{11}(\lambda)=s_{11}^*(\lambda^{-1}),\quad s_{21}(\lambda)=-s_{21}(\lambda^{-1}),\quad \forall j\quad \exists k\quad \lambda_j=\lambda_k^{-1*},\quad \gamma_j=\gamma_k^{-1*} \tag{4.8}$$

for even m and involutions of the form

$$s_{11}(\lambda)=-s_{11}(-\lambda^{-1}),\quad s_{21}(\lambda)=s_{21}^*(-\lambda^{-1}),\quad \forall_j\quad \exists k\quad \lambda_j=-\lambda_k^{-1},\quad \gamma_j=\gamma_k^{-1} \tag{4.8$'$}$$

for odd m.

Thus, suppose we are given an arbitrary set of scattering data that satisfy the involution (4.8) (or (4.8′)). Using the rule (4.7), we construct a new set of scattering data. The parameters $\varkappa_\pm$, whose possible values are 1 or −1, are chosen such that the function $\bar{s}_{11}(\lambda)=(\lambda+ic\varkappa_+)s_{11}(\lambda)/(\lambda+ic\varkappa_-)$ is bounded in the upper half-plane $\mathrm{Im}\,\lambda > 0$. Then the solution of Eq. (4.1) corresponding to this new set of scattering data will satisfy the boundary condition (4.5) (the condition (4.5′)). For example, the S matrix $s(\lambda) \equiv 1$ of the zeroth solution of Eq. (4.1) obviously satisfies the conditions (4.8). Setting $\varkappa_\pm=\pm\operatorname{sign} c$ and choosing an arbitrary real $\gamma \neq 0$, we construct in accordance with (4.7) a complete set of scattering data $s_{11}(\lambda)=(\lambda-i|c|)/(\lambda+i|c|)$, $\lambda_1=i|c|$, $\gamma_1=\gamma$, which determines a single-soliton solution of the SG equation satisfying the condition (4.5).

On the other hand, any solution of the boundary-value problem (4.1), (4.5) (or (4.5′)) is the restriction to the half-plane of some solution of the Cauchy problem with scattering data constructed in accordance with the above rule.

Appendix

The algorithm for reducing the initial-boundary-value problem (2.12), (2.14), (2.15) to a Cauchy problem on the complete axis is as follows. The initial condition $v_0(x)$, specified on the half-axis $x \geqq 0$, is extended to the entire axis by virtue of the condition $v_0(x) = \bar{v}(-x)$, where $\bar{v}$ is the solution of the equation

$$\bar{v}_x=v_{0x}+(\bar{v}+v_0)\sqrt{c^2+\varkappa|\bar{v}-v_0|^2},\quad \bar{v}(0)=v_0(0),\quad c\in R, \tag{A.1}$$

which realizes a Bäcklund transformation for Eq. (2.12).

LEMMA. Suppose $v_0(x)\in C^2(0,\infty)$; then for $\varkappa=-1$ there exists a solution $\bar{v}(x)$ of the problem (A.1) that also belongs to the class $C^2(0, \infty)$. If at the same time $v_0(x)$ decreases exponentially, $|v_0(x)| \leqq c \exp(-\alpha x)$, α, $x > 0$, then the function $\bar{v}(x)$ satisfies the same condition (possibly, with different α).

Proof. By the substitution $y(x) = \bar{v}(x) - v_0(x)$ we reduce the problem (A.1) to the more convenient form $y_x=(y+2v_0)\sqrt{c^2-|y(x)|^2}$, $y(0)=0$. It is clear that this problem satisfies all the conditions of the Cauchy theorem outside the surface $|y| = c$, where the condition of uniqueness of the solution breaks down. From the original equation, we readily deduce the relation

$$\frac{d}{dx}|y(x)|^2=2(|y(x)|+g(x))|y(x)|\sqrt{c^2-|y(x)|^2},\quad g(x)=2\,\mathrm{Re}\,\frac{y(x)v^*(x)}{|y(x)|}. \tag{A.2}$$

In the neighborhood of the point $x = x'$, where the equality $|y(x')| = c$ is attained, we go over by means of the substitution $|y(x)| = c \sin h$ from Eq. (A.2) to an equation of the form

$$h_x=c\sin h+g(x), \tag{A.3}$$

which satisfies the conditions of Cauchy's theorem on the complete plane. It is clear that for the x for which $c \sin h(x) > 0$ the solution $h(x)$ of this equation corresponds to a certain smooth solution of Eq. (A.2). Therefore, for such x the function $y(x)$ can be constructed as the solution of a linear equation, since $|y(x)|$ is already determined by virtue of Eq. (A.2). In the neighborhood of the line $y = 0$ the solution can be extended by virtue of the original equation to $y(x)$. Thus, the function $y(x)$ can be continued to the entire half-axis $x > 0$.

We now show that $v(x)$ decreases exponentially. We choose an arbitrary positive $\varepsilon < |c|$. We assume first that the inequality $|y(x)| \leqq \varepsilon$ holds for all sufficiently large x. Then in Eq. (A.2) the factor $k(x)=\sqrt{c^2-|y(x)|^2}$ is bounded, nonzero, and, therefore, has unchanged sign. If $k(x) > 0$, then, comparing (A.2) with the model equation

$$z'=2\sqrt{c^2-\varepsilon^2}(z-\varepsilon|g(x)|),$$

we can show that the inequality $|y(x)| \leqq \varepsilon$ is violated as $x \to \infty$. Therefore, the function $k(x)$ is negative, and it is obvious that $k(x)\leqslant-|\sqrt{c^2-\varepsilon^2}|$. In this case, using the majorizing equation

$$z'=-2|\sqrt{c^2-\varepsilon^2}|z+\exp(-\alpha x)\cdot\mathrm{const},$$

we can readily obtain the estimate required in the lemma.

It remains to show that there do not exist arbitrarily large x for which the inequality $|y(x)| > \varepsilon$ holds. Suppose otherwise. We choose $x_0 \gg 0$ on the basis of the condition $|y(x_0)| > \varepsilon$ and such that for all $x > x_0$ the inequality $|g(x_0)| < \varepsilon/4$ holds. Suppose for definiteness $c > 0$. If now $k(x_0) = c \cos h(x_0) > 0$ (since $\sin h > 0$, this is equivalent to $0 < h < \pi/2$), then the function $|y(x)|$ will increase until $k(x)$ changes sign. Indeed, in the interval $0 < h < 2\pi/3$ the estimate $\sin h > h/2$ holds. Therefore, comparing (A.3) with the equation $z' = z/2 - \varepsilon/2$, we can show that the function $h(x)$ increases and passes through the point $h = \pi/2$, i.e., the function $c \cos h(x) = k(x)$ changes sign to become negative. Therefore, without loss of generality, we can directly assume that $k(x_0) < 0$. But then the function $|y(x)|$ will decrease with nonzero rate until the inequality $|y(x)| > \varepsilon/2$ is violated. Suppose that the inequality holds for all $x\in(x_0, x_1)$, and $y(x_1) = \varepsilon/2$. We denote by X a neighborhood of the point x_1 such that for $\forall x\in X$ the estimate $|y(x)| < \varepsilon$ holds. Directly from the relation (A.2) we find

$$(|y(x)|^2)_x\leqslant-2|\sqrt{c^2-\varepsilon^2}||y(x)|^2+\varepsilon^2|\sqrt{c^2-\varepsilon^2}|/2. \tag{A.4}$$

We show that the least upper bound x_2 of the set X is equal to infinity. Suppose otherwise, $x_2 < \infty$. Bearing in mind that $|y(x_1)| = \varepsilon/2$, we find from the relation (A.4) that $|y(x)|$ satisfies the inequality $|y(x)| < \varepsilon/2$ for all $x\in X$. But then $y(x_2) \leqq \varepsilon/2$, which contradicts the condition of the least upper bound. Therefore, $|y(x)| < \varepsilon$ for all $x\in(x_1, \infty)$. But this contradicts the assumption that there exists an arbitrarily large x for which $|y(x)| > \varepsilon$. The lemma is proved.

Proof of Proposition 1. The initial condition $v_0(x)$ is extended to the entire axis in such a way that the condition $v_0(x) = \bar{v}(-x)$ is satisfied. It remains to verify the smoothness at the origin of the obtained potential. Continuity is contained in the requirement $\bar{v}(0) = v_0(0)$. Indeed, we have $\bar{v}(0+) = v_0(0+)$, but, on the other hand, from the condition $\bar{v}(x) = v_0(-x)$ we find $\bar{v}(0+) = v_0(0-)$. Continuity of the derivative follows from formula (A.1): $\bar{v}_x(0+) = v_{0x}(0+) + 2cv_0(0)$, and the boundary condition $v_{0x}(0+) + cv_0(0+) = 0$. Bearing in mind that $\bar{v}_x(0+) = -v_{0x}(0-)$, we find from this $v_{0x}(0+) = v_{0x}(0-)$. The continuity of the second derivative can be verified similarly. The proposition is proved.

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INFRARED SINGULARITIES OF GLUON GREEN'S FUNCTIONS AND TWO-QUARK INTERACTION QUASIPOTENTIAL IN QUANTUM CHROMODYNAMICS

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A study is made of the influence of the infrared singularities of the gluon Green's functions on the behavior of the interaction quasipotential of two quarks in quantum chromodynamics.

Introduction

Many studies now exist in which one can find fairly weighty arguments for the singular infrared behavior $M^2/(k^2)^2$ for the gluon Green's functions in quantum chromodynamics (QCD) (see, for example, the review [1] and the bibliography given there). In particular, it is well known that a linearly rising quark–antiquark interaction potential, which is consistent with the experimental data on quarkonium spectroscopy, corresponds to the static limit of the diagram of exchange of one dressed gluon, the propagator of which has that infrared asymptotic behavior. Essentially, this correspondence was the first and main argument for introducing the assumption of such singular infrared behavior for the total single-particle gluon Green's function. Further investigations of the structure of QCD made it possible to draw the very important conclusion that the infrared behavior M^2/k^4 of the gluon propagator gives a self-consistent description of the infrared region of QCD. This result, and also the successes of the potential model in describing the spectroscopy of heavy quarkonium with an interaction potential between the quarks in the form of a sum of a Coulomb term and a term that rises linearly with the distance suggests that we shall obtain a fairly good approximation for the total single-particle gluon Green's function if we represent it in the form

$$D_{\mu\nu}(k)=D_{\mu\nu}^{(0)}(k)+D_{\mu\nu}^{(1)}(k), \tag{1}$$

where $D_{\mu\nu}^{(0)}(k)$ determines the ultraviolet behavior of the gluon propagator and is equal to the free gluon Green's function, while $D_{\mu\nu}^{(1)}(k)$ describes the above singular infrared behavior, so that

$$D_{\mu\nu}(k)=D_{\mu\nu}^{(0)}(k), \quad k^2\to\infty, \quad D_{\mu\nu}(k)=D_{\mu\nu}^{(1)}(k), \quad k^2\to 0.$$

On the other hand, the accumulation of many experimental data on the spectroscopy and decays of quarkonium systems and careful analysis of these data show that the naive nonrelativistic potential model is not fully adequate to describe the physical picture observed experimentally. The problem of describing the properties of quarkonium systems in the framework of the fundamental original QCD Lagrangian is still an important unresolved problem of QCD. Great hopes are here placed on calculations that use lattice methods, and great efforts are being made in this direction [2,3].

In our papers [4,5], we made an attempt to show that there exists a simpler and more consistent way of solving this problem based on the method of single-time reduction in quantum field theory. Using the method of single-time reduction of the Bethe–Salpeter formalism developed in [4] for two-fermion systems, we calculated the interaction potential of two quarks in QCD in the approximation of one-gluon exchange. We showed that a systematic relativistic treatment of the quark interaction problem leads to a nontrivial dependence of the quark interaction potential on the energy, this energy dependence giving the interaction potential very specific properties characterizing its behavior in the region of large and small distances; these properties would have been difficult to imagine on the basis of

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