NON-ABELIAN STOKES FORMULA

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A representation is formulated and proved for parallel transport along a closed contour in a principal fiber bundle with connection; it takes the form of an ordered integral over a surface spanned by this contour.

1. The aim of this note is to formulate and prove a representation for parallel transport along a closed contour in a principal fibre bundle with connection (see, for example, [1]) in the form of an ordered integral over a surface spanned by this contour.

It is sufficient to consider trivial fiber bundles. Parallel transport in a trivial fiber bundle over R^n with connection form $A = A_\mu dx^\mu$ and structure group that is a group of matrices G (in physics terminology, a phase factor [2]) is determined by a P-ordered exponential of the integral along the curve $\gamma : P\exp\int_\gamma A$. The analytic expression has the form

$$P\exp\int_\gamma A=1+\sum_{n=1}^{\infty}\int_0^1 ds_1\int_0^{s_1} ds_2\ldots\int_0^{s_{n-1}} ds_n (Af')(s_1)\ldots(Af')(s_n),\quad (Af')(s)=A_\mu(f(s))f_\mu'(s), \tag{1}$$

where the curve is defined by continuously differentiable functions $x_\mu=f_\mu(s)$, $\mu=1,\ldots,n$, $0\leqslant s\leqslant 1$, $A_\mu(x)$ is a matrix-valued, continuously differentiable function, and $f_\mu'=\partial f_\mu/\partial s$. It is well known that under the above assumptions the series (1) converges absolutely.

Suppose a two-dimensional surface Σ in R^n is defined by continuously differentiable functions with respect to each argument $x_\mu=f_\mu(s,t)$, $0\leqslant s\leqslant 1$, $0\leqslant t\leqslant 1$. The boundary $\partial\Sigma$ of the surface Σ is defined by the segments $x_\mu=f_\mu(s,0)$, $0\leqslant s\leqslant 1$, $x_\mu=f_\mu(1,t)$, $0\leqslant t\leqslant 1$, $x_\mu=f_\mu(s,1)$, $0\leqslant s\leqslant 1$, $x_\mu=f_\mu(0,t)$, $0\leqslant t\leqslant 1$.

We introduce the concept of the ordered exponential of the integral over the surface of a function $\Phi_{\mu\nu}=\Phi_{\mu\nu}(s,t)$ defined on Σ. We define

$$\mathscr{P}\exp\int_\Sigma \Phi_{\mu\nu}\, dx_\mu\wedge dx_\nu \overset{\text{def}}{=} 1+\sum_{n=1}^{\infty}\frac{1}{n!}\mathscr{P}_s\int_0^1 ds_1\int_0^1 dt_1\int_0^1 ds_2\int_0^1 dt_2\ldots\int_0^1 ds_n\int_0^1 dt_n\times$$

$$(\Phi f'\dot f)(s_1,t_1)\ldots(\Phi f'\dot f)(s_n,t_n)\overset{\text{def}}{=}1+\sum_{n=1}^{\infty}\int_0^1 ds_1\int_0^1 dt_1\int_0^{s_1} ds_2\int_0^1 dt_2\ldots\int_0^{s_{n-1}} ds_n\int_0^1 dt_n\,(\Phi f'\dot f)(s_n,t_n)\ldots(\Phi f'\dot f)(s_1,t_1),$$

$$(\Phi f'\dot f)(s,t)=\Phi_{\mu\nu}(s,t)f_\mu'(s,t)\dot f_\nu(s,t),\quad f_\mu'(s,t)=\partial f_\mu(s,t)/\partial s,\quad \dot f_\nu(s,t)=\partial f_\nu(s,t)/\partial t. \tag{2}$$

It is readily seen that the series (2) converges absolutely. In general, the expression on the right-hand side of (2) depends on the parametrization of Σ.

2. We have

<u>THEOREM.</u> Suppose that on Σ there is defined a matrix-valued, continuously differentiable function $A_\mu(x)$, $\mu = 1, 2, \ldots, n$. Then

$$P\exp\int_{\partial\Sigma} A_\mu\, dx^\mu=\mathscr{P}\exp\int_0^1\int_0^1(\mathscr{F}f'\dot f)(s,t)\, ds\, dt, \tag{3}$$

where $\mathscr{F}_{\mu\nu}(s,t)=h^{-1}(s,0)g^{-1}(s,t)F_{\mu\nu}(f(s,t))g(s,t)h(s,0)$,

$$h(s,t)=P\exp\int_0^s A_\mu(f(s',t))f_\mu'(s',t)ds', \tag{4}$$

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$$g(s,t)=P\exp\int_{t_0}^{t} A_\mu(f(s,t'))\dot f_\mu(s,t')\,dt',\quad F_{\mu\nu}(x)=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]. \tag{5}$$

Remarks. 1. In the Abelian case, formula (3) goes over into the exponentiated ordinary Stokes (Green) formula, and we therefore call it a non-Abelian Stokes formula.

2. We note a more compact expression of formula (3):

$$P\exp\int_{\partial\Sigma} A=\mathcal{P}\exp\int_{\Sigma}\mathcal{D}A, \tag{3'}$$

where $\mathcal{D}A=h^{-1}g^{-1}DAgh$, $DA=F_{\mu\nu}dx^\mu\wedge dx^\nu$. The geometrical meaning of our formulas is obvious: if A is the form of the connection, then DA is the curvature of the connection, and h and g implement parallel transport along the corresponding curves.

3. For a different, nonlexicographical ordering of Σ (see below), the form of formulas (3) and (3′) remains the same and only the definitions of the $\mathcal{P}$ ordering and the functions h and g are changed in accordance with the new ordering.

Proof. We divide the square [0, 1] × [0, 1] into nm rectangles with sides 1/n and 1/m. Then

$$P\exp\int_{\partial\Sigma} A_\mu dx^\mu=\lim_{n,m\to\infty} g_{n,m}(\partial\Sigma),\quad g_{n,m}(\partial\Sigma)=g_m^{-1}(0,1)h_m^{-1}(1,1)g_m(1,1)h_n(1,0), \tag{6}$$

$$h_n\left(\frac{k}{n},t\right)=U\left(\frac{k}{n},t\right)U\left(\frac{k-1}{n},t\right)\dots U\left(\frac{1}{n},t\right)\qquad 1\leqslant k\leqslant n; \tag{7}$$

$$g_m\left(s,\frac{l}{m}\right)=V\left(s,\frac{l}{m}\right)V\left(s,\frac{l-1}{m}\right)\dots V\left(s,\frac{1}{m}\right),\qquad 1\leqslant l\leqslant m;$$

$$U\left(\frac{k}{n},t\right)=\exp\left[A_\mu\left(f\left(\frac{k}{n},t\right)\right)f_\mu'\left(\frac{k}{n},t\right)\frac{1}{n}\right],\quad V\left(s,\frac{l}{m}\right)=\exp\left[A_\mu\left(f\left(s,\frac{l}{m}\right)\right)\dot f_\mu\left(s,\frac{l}{m}\right)\frac{1}{m}\right]. \tag{8}$$

The proof of formula (6) (cf [3]) follows from the representation of g(s, t) and h(s, t) [formulas (4) and (5)] in the form of the series (1) and the relations

$$\int_0^s A_\mu dx_\mu=\lim_{n\to\infty}\sum_{k=1}^{[sn]} A\left(\frac{k}{n}\right),\quad A\left(\frac{k}{n}\right)=A_\mu\left(f\left(\frac{k}{n}\right)\right)f_\mu'\left(\frac{k}{n}\right).$$

Indeed, each term of the series (1) can be approximated by the sum

$$\sum_{k_1=1}^{n}\frac{1}{n}A\left(\frac{k_1}{n}\right)\sum_{k_2=1}^{k_1}\frac{1}{n}A\left(\frac{k_2}{n}\right)\dots\sum_{k_l=1}^{k_{l-1}}\frac{1}{n}A\left(\frac{k_l}{n}\right),$$

which we can, using the symbol of P ordering (a matrix with larger argument stands to the left of a matrix with smaller argument), represent by the sum

$$\frac{1}{l!}P\left\{\sum_{k_1=1}^{n}\frac{1}{n}A\left(\frac{k_1}{n}\right)\sum_{k_2=1}^{n}\frac{1}{n}A\left(\frac{k_2}{n}\right)\dots\sum_{k_l=1}^{n}\frac{1}{n}A\left(\frac{k_l}{n}\right)\right\},$$

and, by the absolute convergence, the complete series can be regrouped as follows:

$$1+\sum_{l=1}^{\infty}\frac{1}{l!}P\left\{\sum_{k_1=1}^{n}\frac{1}{n}A\left(\frac{k_1}{n}\right)\sum_{k_2=1}^{n}\frac{1}{n}A\left(\frac{k_2}{n}\right)\dots\sum_{k_l=1}^{n}\frac{1}{n}A\left(\frac{k_l}{n}\right)\right\}=1+\sum_{l=1}^{\infty}\sum_{l_1,\dots l_{n-1}}\times$$

$$\frac{1}{l_1!\dots l_{n-1}!(l-l_1-\dots-l_{n-1})!}P\left\{\left(\frac{1}{n}A\left(\frac{1}{n}\right)\right)^{l_1}\dots\left(\frac{1}{n}A\left(\frac{n-1}{n}\right)\right)^{l_{n-1}}\left(\frac{1}{n}A(1)\right)^{l-l_1-\dots l_{n-1}}\right\}=$$

$$e^{A(1)/n}e^{A\left(\frac{n-1}{n}\right)/n}\dots e^{A\left(\frac{1}{n}\right)/n},$$

which proves formula (6).

For the pre-limit relation $g_{n,m}(\partial\Sigma)$, we have the representation (cf the factorization lemma in [1] and Fig. 1)

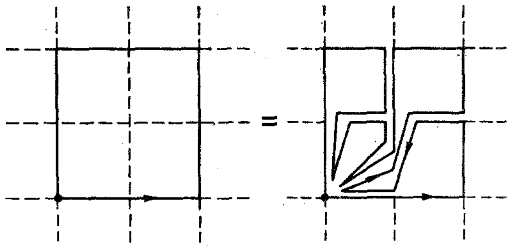

Fig. 1

$$g_{n,m}(\partial\Sigma)=\mathcal{P}_{s,t}\prod_{l,k=1}^{n,m} h^{-1}\left(\frac{l}{n},\,0\right)g^{-1}\left(\frac{l}{n},\,\frac{k}{m}\right)V^{-1}\left(\frac{l}{n},\,\frac{k}{m}\right)\times$$

$$U^{-1}\left(\frac{l}{n},\,\frac{k+1}{m}\right)V\left(\frac{l+1}{n},\,\frac{k}{m}\right)U\left(\frac{l}{n},\,\frac{k}{m}\right)g\left(\frac{l}{n},\,\frac{k}{m}\right)h\left(\frac{l}{n},\,0\right), \tag{9}$$

where $g(l/n, k/m)$, $h(l/n, 0)$ are defined by (7) and (8).

The product in (9) is ordered (the symbol $\mathcal{P}_{s,t}$) as follows: for any two matrices, the one with the larger value of the first argument stands on the right (the relationship of the second argument is immaterial), and if the first arguments of two matrices are equal, then the matrix with the smaller value of the second argument stands on the right (lexicographical ordering). A simple calculation shows (cf [3, 2]) that

$$V^{-1}\left(\frac{l}{n},\,\frac{k}{m}\right)U^{-1}\left(\frac{l}{n},\,\frac{k+1}{m}\right)V\left(\frac{l+1}{n},\,\frac{k}{m}\right)U\left(\frac{l}{n},\,\frac{k}{m}\right)=\exp\left\{F\left(\frac{l}{n},\,\frac{k}{m}\right)\frac{1}{nm}+o\left(\frac{1}{nm}\right)\right\},$$

where

$$F\left(\frac{l}{n},\,\frac{k}{m}\right)=F_{\mu\nu}\left(f\left(\frac{l}{n},\,\frac{k}{m}\right)\right)f_{\mu}'\left(\frac{l}{n},\,\frac{k}{m}\right)\dot{f}_{\nu}\left(\frac{l}{n},\,\frac{k}{m}\right). \tag{10}$$

Introducing the notation

$$\mathcal{F}\left(\frac{l}{n},\,\frac{k}{m}\right)=h^{-1}\left(\frac{l}{n},\,0\right)g^{-1}\left(\frac{l}{n},\,\frac{k}{m}\right)F\left(\frac{l}{n},\,\frac{k}{m}\right)g\left(\frac{l}{n},\,\frac{k}{m}\right)h\left(\frac{l}{n},\,0\right), \tag{11}$$

we have

$$g_{n,m}(\partial\Sigma)=\mathcal{P}_{s,t}\prod_{l,k=1}^{n,m}\exp\left\{\mathcal{F}\left(\frac{l}{n},\,\frac{k}{m}\right)\frac{1}{nm}+o\left(\frac{1}{nm}\right)\right\}. \tag{12}$$

Using the definition (2), we approximate the right-hand side of (3) as follows: we replace the integrals over s_i and t_i from 0 to s_{i+1} and from 0 to 1, respectively, in formula (2) with Φ equal to $\mathcal{F}$ by the sum $\sum_{l=1}^{[s_i n]}\sum_{k=1}^{m}\mathcal{F}(l/n,\,k/m)/nm$, where $\mathcal{F}(l/n,\,k/m)$ is calculated by means of the approximations of the matrices g and h by the matrices $g_m(k/n,\,l/m)$ and $h_n(k/n,\,0)$ defined in (7) and (8), i.e., it is given by formulas (10) and (11):

$$\mathcal{P}\exp\int_0^1\int_0^1 ds\,dt\,(\mathcal{F}f'\dot{f})(s,t)=\lim_{n,m\to\infty}\mathcal{G}_{n,m},\quad \mathcal{G}_{n,m}=1+\sum_{l=1}^{\infty}\sum_{k_1=1}^{n}\sum_{j_1=1}^{m}\cdots\sum_{k_l=1}^{k_{l-1}}\sum_{j_l=1}^{m}\mathcal{F}\left(\frac{k_l}{n},\,\frac{j_l}{m}\right)\frac{1}{nm}\cdots\mathcal{F}\left(\frac{k_1}{n},\,\frac{j_1}{m}\right)\frac{1}{nm}.$$

The convergence of this approximation follows from the absolute convergence of the series (2) with $\Phi=\mathcal{F}$, which holds by virtue of the continuity of the function $\mathcal{F}(s,\,t)$.

Arguing as in the proof of formula (6), we have

$$\mathcal{G}_{n,m}=1+\sum_{l=1}^{\infty}\frac{1}{l!}\mathcal{P}_s\left(\sum_{k_1=1}^{n}\sum_{j_1=1}^{m}\cdots\sum_{k_l=1}^{n}\sum_{j_l=1}^{m}\mathcal{F}\left(\frac{k_1}{n},\,\frac{j_1}{m}\right)\frac{1}{nm}\cdots\mathcal{F}\left(\frac{k_l}{n},\,\frac{j_l}{m}\right)\right)\frac{1}{nm}=$$

$$\mathcal{P}_s\prod_{k=1,j=1}^{n,m}\exp\left\{\mathcal{F}\left(\frac{k}{n},\,\frac{j}{m}\right)\frac{1}{nm}\right\},$$

where $\mathscr{P}_s$ is the following ordering: for different first arguments, the matrices are arranged such that the matrix with the larger first argument stands on the right of a matrix with smaller argument, while in the case of equal first arguments it is necessary to take into account in the product all possible ways of arranging the matrices. It is clear that we have

$$\mathscr{P}_s \prod_{k=1,j=1}^{n,m} \exp\left\{ \mathscr{F}\left(\frac{k}{n}, \frac{j}{m}\right)\frac{1}{nm}\right\} = \mathscr{P}_{s,t} \prod_{k=1,j=1}^{n,m} \exp\left\{ \mathscr{F}\left(\frac{k}{n}, \frac{j}{m}\right)\frac{1}{nm} + o\left(\frac{1}{nm}\right)\right\}.$$

It can be seen from this that the approximating expressions for the right- and left-hand sides of (3) are equal up to infinitesimally small quantities. This completes the proof of the theorem.

We note that formula (3) has not only a purely mathematical interest but also finds application in problems in physics. It can be used in the construction of an algebra of non-Abelian dual variables, in the derivation of the equation of a membrane from gauge theory [4], and is also helpful for studying monopoles in both the classical and the quantum case.

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ON A METHOD OF CONSTRUCTING FACTORIZED S MATRICES IN ELEMENTARY FUNCTIONS

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New factorized relativistic S matrices are found in elementary functions for two and more different particle species.

Zamolodchikov (see [1, 2]) has proposed the following interpretation of the relativistic factorized S matrix for a certain finite set X of particle species. We factorize the tensor algebra of a vector space over C with linearly independent generators $A_x(\theta)$ that depend on $x \in X$, $\theta \in C$ in accordance with the relations

$$A_x(\theta_1)A_y(\theta_2) = \sum_{a,b\in X} S_{xy}^{ab}(\theta_{12})A_a(\theta_2)A_b(\theta_1)$$

($\theta_{ij}=\theta_i-\theta_j$, S_{xy}^{ab} are complex-valued functions). If the monomials $A_{x_1}(\theta_1)\ldots A_{x_s}(\theta_s)$ remain linearly independent for fixed $\theta_1,\ldots,\theta_s$ and different sets $\{x_1,\ldots,x_s\}\subset X$, the factor is called a Zamolodchikov algebra, and S a factorized (two-particle) S matrix.* The definition reduces to the following functional equations for S:

$$\sum_{a,b\in X} S_{xy}^{ab}(\theta_{12})S_{ab}^{tu}(\theta_{21}) = \delta_{xt}\delta_{yu}, \quad \sum_{a,b,c\in X} S_{xy}^{ab}(\theta_{12})S_{bz}^{cv}(\theta_{13})S_{ac}^{tu}(\theta_{23}) = \sum_{a,b,c\in X} S_{yz}^{ab}(\theta_{23})S_{xa}^{tc}(\theta_{13})S_{cb}^{uv}(\theta_{12}), \tag{1}$$

where x, y, z, t, u, v $\in$ X, and δ is the Kronecker delta. The equations (1) (of "unitarity" and "factorization") have been investigated in various forms earlier (see [3, 4, 5]).

As was shown by Zamolodchikov (see [7]), matrices $S(\theta)$ satisfying (1) can be constructed for $X=\mathbf{Z}_2=\{0,1\}$ directly on the basis of the results of Baxter (see [6]), who solved the problem of finding transfer matrices with definite commutation properties. The corresponding formulas for the coefficients of S

* Definite properties of analyticity and crossing symmetry are also usually required of the S matrix. For the S matrices considered below, these properties can be readily investigated.

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