classical, quantum, and optimal.

I am sincerely grateful to M. B. Menskii for assistance in this work.

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UNIQUENESS AND HALF-SPACE NONUNIQUENESS OF GIBBS STATES IN CZECH MODELS

S.B. Shlosman

A study is made of a new class of interactions for which there is uniqueness of a Gibbs state in a whole space together with nonuniqueness of the state in a half-space. The method of reflection positivity is used to study these interactions.

1. Introduction

In [1], a new criterion of uniqueness for a Gibbs state was found. In many cases, this criterion makes it possible to exhaust the entire uniqueness region. Moreover, it is constructive in the following sense: If in a certain model uniqueness holds at a certain temperature above the critical temperature, then this circumstance can be verified over a finite number of steps of a certain algorithm. For some time, the authors of [1] believed that their criterion is also a necessary condition of uniqueness away from the critical point. Later, however, it became clear that this is not so for models in which there is uniqueness of the Gibbs state in the complete space but not in a half-space (if such models exist). In the present paper, the question of the existence of such models is answered in the affirmative. We call them Czech models. The simplest model of such form was found by the Czechoslovak Mathematician J. Navrátil and was discussed at the Paris Seminar on Mathematical Physics.

One of the ways of describing Czech models is that in them there is precisely one ground-state configuration in the complete space (i.e., configuration with minimal energy) but several ground-state configurations in the half-space (with the same boundary condition), these possessing a certain stability property (Peierls condition).

In this paper, we study uniqueness problems for the simplest Czech models. We shall discuss the reflection of the unusual picture of Gibbs states on their analytic properties in a separate publication, in which it will be shown that the properties of complete analyticity [2] break down for Czech interactions.

Navrátil's example, contained in his diploma thesis, was constructed initially to prove the existence of a lacuna in the proof of the uniqueness theorem in [3]. Nevertheless, all the main propositions of [3] are correct, and the lacuna can be filled; in Sec. 4 of the present paper, we prove a "uniqueness theorem with logarithmic strip," which establishes the uniqueness of the low-temperature Gibbs states for models with one periodic ground-state configuration satisfying the Peierls condition. The words "logarithmic strip" mean that the region of instability in the container V which arises when unstable boundary conditions are imposed has width of order $\ln |V|$. Note that by uniqueness we here understand uniqueness in the class of all (and not only translationally invariant) Gibbs states. A similar uniqueness assertion has been announced by D. Martirosyan.

All-Union Scientific-Research Institute of the Organization, Control, and Economics of the Oil and Gas Industry. Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 66, No. 3, pp. 430-444, March, 1986. Original article submitted December 14, 1984.

 0040-5779/86/6603-0284$12.50

The paper is arranged as follows. In Sec. 2, we describe the simplest Czech model. In the following section we prove for it half-space nonuniqueness in the low-temperature region. The proof is based on the use of a variant of the method of reflection positivity. In the fifth section, we discuss the problem of the existence of phase transitions in Czech models.

In the process of writing this paper, the author drew much of value from discussions with R. L. Dobrushin, D. G. Martirosyan, E. A. Pecherskii, and S. A. Pirogov. In particular, a remark of Dobrushin permitted a simplification of the original proof of the uniqueness theorem of Sec. 4. I should like to take this opportunity of expressing my thanks to them.

2. The Simplest Czech Model

Let $\mathbb{Z}^\nu$ be a ν-dimensional integral lattice, $\mathbb{Z}^\nu=\{t=(t^{(1)}, \ldots, t^{(\nu)}), t^{(i)}\in\mathbb{Z}\}$. Suppose that at each site $t\in\mathbb{Z}^\nu$ there is a spin variable σ_t, which takes four values, $\sigma_t \in \{0, 1, 2, 3\} \equiv S$. Let Ω be the set of all configurations $\sigma:\mathbb{Z}^\nu\to S$, and $\mathfrak{B}$ be the usual σ algebra generated by cylindrical subsets. For $\Lambda\subset\mathbb{Z}^\nu$, the corresponding objects will be denoted by $\Omega_\Lambda, \mathfrak{B}_\Lambda$.

In our model, the interaction $U=\{U_A(\sigma), A\subset\mathbb{Z}^\nu, |A|<\infty\}$ is determined by the specification of two functions $U(a)$, $U(a, b)$, $a, b\in S$. Namely,

$$U_A(\sigma)=\begin{cases} U(\sigma_t), & A=\{t\}, \\ U(\sigma_t,\sigma_s), & A=\{s,t\}, \quad |s-t|=1, \\ 0 & \text{otherwise}. \end{cases} \tag{2.1}$$

Here, $|t|=\sqrt{(t^{(1)})^2+\ldots+(t^{(\nu)})^2}$. Below, we shall require a different norm: $\|t\|=\max|t^{(i)}|$. The formal Hamiltonian is determined by the expression $H(\sigma)=\sum_{A\subset\mathbb{Z}^\nu} U_A(\sigma)$. The interaction U is specified by the following tables:

$$U(a,b)=U(b,a)=\begin{cases} 0, & a=b, \\ x, & (a,b)=(0,1),(0,2). \\ X, & (a,b)=(0,3), \\ y, & (a,b)=(1,3),(2,3), \\ Y, & (a,b)=(1,2); \end{cases} \quad U(a)=\begin{cases} 0, & a=0, \\ z_a, & a=1,2, \\ Z & a=3. \end{cases} \tag{2.2}$$

In this way, the interaction is specified by seven numbers, and we assume that $X, Y, Z, x, y, z_1, z_2>0$; $X, Y, Z\gg x, y, z_1, z_2$. Below, we impose additional restrictions on them.

A specific feature of the model is the following. It has a unique ground-state configuration, i.e., a configuration σ that minimizes the Hamiltonian $H(\sigma)$, namely $\sigma \equiv 0$. The values $\sigma_t = 1$ and 2 are energetically less advantageous, and the value $\sigma_t = 3$ even more so. However, if the event $\sigma_{t_i}=3$ occurs along a certain contour $\Gamma = \{t_i\}$ as a result of thermal fluctuations, the energetically most advantageous method of returning to zero values within Γ consists of transition through the strip of intermediate values $\sigma_t \in \{1, 2\}$ (see Fig. 1). Two possibilities σ_1 and σ_2 are the best. The unusual properties of the Czech models are due to the fact that the number of such best possibilities is greater than one. More precisely, in accordance with the picture we have given there is in our model a "phase transition along a boundary." Of course, if such a phase transition is to take place the boundary must be at least two dimensional, and therefore the unusual properties of Czech models appear only if the dimension is at least three. Therefore, we obtain the natural hypothesis: the criterion of uniqueness mentioned above (the criterion of [1], see also Sec. 4 of the present paper) is a necessary and sufficient condition of uniqueness away from the critical temperature if the dimension is two.

3. Half-Space Nonuniqueness

The ground-state configuration of our model being unique, uniqueness of the low-temperature Gibbs states is a perfectly natural proposition. We shall consider it in the following section. The question of the uniqueness of Gibbs states at all temperatures is not so simple. In Sec. 5, we shall discuss a Czech model in which instead of two intermediate spin values ($\sigma = 1, 2$) there are $2q$, and we shall show that at large q a phase transition of the first kind with respect to the temperature takes place in the system. In particular, at a certain critical value $T_{cr}(q, \nu)$ of the temperature there exist in this model at least two Gibbs states. The case with $q = 1$ remains open.

We now turn to a new property of the models – the half-space nonuniqueness. We begin with some

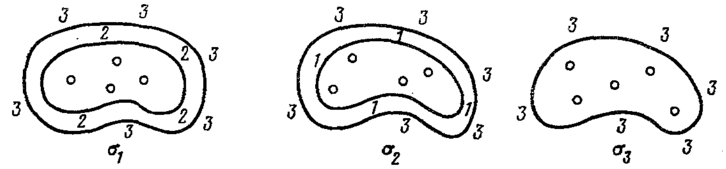

Fig. 1

definitions. Let

$$Q^\beta=\{Q_\Lambda{}^\beta(\cdot|\cdot),\quad \Lambda\subset\mathbb{Z}^\nu,\quad |\Lambda|<\infty\}$$

be the Gibbs specification for the interaction U and reciprocal temperature β (see [4]):

$$Q_\Lambda{}^\beta(\sigma_\Lambda|\bar\sigma)=Z^{-1}(\Lambda,\beta,\bar\sigma)\exp\{-\beta H_\Lambda(\sigma_\Lambda|\bar\sigma)\},\quad \sigma_\Lambda\in\Omega_\Lambda,\quad \bar\sigma\in\Omega,$$

with relative Hamiltonian

$$H_\Lambda(\sigma_\Lambda|\bar\sigma)=\sum_{A:A\cap\Lambda\neq\varnothing}U_A(\sigma_\Lambda\cup\bar\sigma_{\Lambda^c}),\quad \Lambda^c=\mathbb{Z}^\nu\setminus\Lambda,\quad Z(\Lambda,\beta,\bar\sigma)=\sum_{\sigma_\Lambda\in\Omega_\Lambda}\exp\{-\beta H_\Lambda(\sigma_\Lambda|\bar\sigma)\}.$$

Let $\mathbb{Z}_+{}^\nu=\{t\in\mathbb{Z}^\nu,\ t^{(\nu)}>0\}$, $\Omega_+=\Omega\,\mathbb{Z}_+^\nu$, $\mathfrak{B}_+=\mathfrak{B}\mathbb{Z}_+^\nu$, $\mathbb{Z}_0^\nu=\mathbb{Z}^\nu\setminus\mathbb{Z}_+{}^\nu$, $\Omega_0=\Omega\mathbb{Z}_0^\nu$ and $\bar\sigma_0\in\Omega_0$ be a fixed configuration (boundary condition in the half-space). We define the Gibbs specification in the half-space, $Q_{\bar\sigma_0}^\beta=\{Q_{\bar\sigma_0,\Lambda}^\beta(\cdot|\cdot);\ \Lambda\subset{}_+{}^\nu,$ $|\Lambda|<\infty\}$, as follows: for $\Lambda\subset\mathbb{Z}_+^\nu$, $\sigma_\Lambda\in\Omega_\Lambda$, $\bar\sigma_+\in\Omega_+$ we set $Q_{\bar\sigma_0,\Lambda}^\beta(\sigma_\Lambda|\bar\sigma_+)=Q_\Lambda{}^\beta(\sigma_\Lambda|\bar\sigma_+\cap\bar\sigma_0)$.

An arbitrary probability measure $\mathscr{P}$ on $(\Omega,\mathfrak{B})$ is called a state. We shall denote averaging with respect to this measure by $\langle\,\rangle_{\mathscr{P}}$ (or simply $\langle\,\rangle$). The state $\mathscr{P}$ is called a Gibbs state with interaction U and reciprocal temperature β (or (U,β) Gibbs state, see [5]) if its conditional distributions satisfy

$$\langle\{\sigma\in\Omega:\sigma_\Lambda=\hat\sigma_\Lambda\}|\bar\sigma\rangle_{\mathscr{P}}=Q_\Lambda{}^\beta(\hat\sigma_\Lambda|\bar\sigma),\quad \bar\sigma\in\Omega,\ \Lambda\subset\mathbb{Z}^\nu.$$

Similarly, the probability measure $\mathscr{P}_+$ on $(\Omega_+,\mathfrak{B}_+)$ is called a state in the half-space $\mathbb{Z}_+{}^\nu$. The state $\mathscr{P}_+$ is called a Gibbs state with interaction U, reciprocal temperature β, and boundary condition $\bar\sigma_0\in\Omega_0$ if for its conditional distributions we have

$$\langle\{\sigma\in\Omega_+:\sigma_\Lambda=\hat\sigma_\Lambda\}|\bar\sigma_+\rangle_{\mathscr{P}_+}=Q_{\bar\sigma_0,\Lambda}^\beta(\hat\sigma_\Lambda|\bar\sigma_+),\quad \bar\sigma_+\in\Omega_+,\Lambda\subset\mathbb{Z}_+{}^\nu.$$

We define similarly Gibbs states in the strip $\mathbb{Z}_{[1,N]}^\nu=\{t\in\mathbb{Z}^\nu,\ 1\leqslant t^{(\nu)}\leqslant N\}$ with boundary conditions $\sigma_0\in\Omega_0$, $\sigma_{N+1}\in\Omega_{N+1}$, where $\Omega_{N+1}=\Omega\,\mathbb{Z}_{N+1}^\nu$, $\mathbb{Z}_{N+1}^\nu=\{t\in\mathbb{Z}^\nu:t^{(\nu)}\geqslant N+1\}$.

To formulate our theorem, we introduce indicators $\mathrm{P}_t^i(\sigma)$ of events $\{\sigma\in\Omega;\ \sigma_t=i\}$, $t\in\mathbb{Z}^\nu$, $i=0,1,2,3$.

THEOREM 1. We assume that $\nu\geq 3$ and that the following restrictions hold: $z_1=z_2=z$, $X\geqslant x$, $Y\geqslant y$, $Z\geqslant z$,

$$X>2^{\nu-1}(2Y+z),\quad Z>2^{\nu-1}[(2\nu-1)Y+z],\quad Y>2^{\nu-2}(2x+y+z). \tag{3.1}$$

I. Let $\bar\sigma_0\equiv 3$. There exists $\beta_c=\beta_c(\mathrm{U},\nu)$ such that for all $\beta\geq\beta_c$ there exist at least two Gibbs states $\langle\cdot\rangle^{\beta,i}$, $i=1,2$, in the half-space with interaction U, reciprocal temperature β, and boundary condition $\bar\sigma_0$. Moreover,

$$\langle P_t{}^i\rangle^{\beta,i}>{}^1/_2, \tag{3.2}$$

$$\langle P_t{}^i\rangle^{\beta,i}\to 1\quad\text{as}\quad \beta\to\infty,\quad t^{(\nu)}=1. \tag{3.3}$$

II. Let $N\geqslant 2$, $\bar\sigma_0=3$, $\bar\sigma_{N+1}=0$. There exists $\beta_c=\beta_c(U,\nu)$ such that for all $\beta\geqslant\beta_c$ there exist at least two Gibbs states $\langle\cdot\rangle_N^{\beta,i}$, $i=1,2$, in the strip $\mathbb{Z}_{[1,N]}^\nu$ with interaction U, reciprocal temperature β, and boundary conditions $\bar\sigma_0\in\Omega_0$, $\bar\sigma_{N+1}\in\Omega_{N+1}$. Moreover,

$$\langle P_t^i\rangle_N^{\beta,i}>1/2, \tag{3.4}$$

$$\langle P_t^i\rangle_N^{\beta,i}\to 1\quad\text{as}\quad\beta\to\infty\quad\text{for}\quad t^{(\nu)}=1. \tag{3.5}$$

The convergence in (3.5) is uniform in N.

Remark 1. Theorem 1 is based on the following simple observation. We consider the conditional ground-state configurations of our model in the half-space $\mathbb{Z}_+{}^\nu$ with boundary condition $\bar\sigma_0\equiv 3$, i.e., configurations that minimize the formal relative Hamiltonian $H\mathbb{Z}_+^\nu(|\bar\sigma_0)$. It is easy to see that there are precisely two such configurations, namely, $\sigma_+^{(1)}$ and $\sigma_+^{(2)}$:

$$(\sigma_+^{(i)})_t=\begin{cases} i, & t^{(\nu)}=1,\\ 0, & t^{(\nu)}>1.\end{cases} \tag{3.6}$$

It is only necessary to prove their stability, i.e., that there are Gibbs states near them.

Remark 2. The configurations $\sigma_+^{(i)}$ differ only at the points $t\in\mathbb{Z}_+^\nu$ with $t^{(\nu)}=1$. Nevertheless, the states $\langle\cdot\rangle^{\beta,i}$ differ. Moreover, for all $t\in\mathbb{Z}_+^\nu$

$$\langle P_t^i\rangle^{\beta,i}>\langle P_t^{(3-i)}\rangle^{\beta,i},\quad i=1,2, \tag{3.7}$$

$$\langle P_t^i\rangle_N^{\beta,i}>\langle P_t^{(3-i)}\rangle_N^{\beta,i},\quad i=1,2;\quad N=3,4,\ldots . \tag{3.8}$$

However, we shall not prove this.

Proof. We note first of all that the studied interaction has the property of reflection positivity with respect to reflections in the coordinate hyperplanes of the lattice $\mathbb{Z}^\nu$ (the method of reflection positivity is presented in [6-9]). We recall that a state $\langle\cdot\rangle$ in $\Lambda\subset\mathbb{Z}^\nu$, $|\Lambda|<\infty$, is said to be a reflection-positive state with respect to the reflection ϑ in the hyperplane $L\subset\mathbb{R}^\nu$ if $\vartheta\Lambda=\Lambda$ and $\langle F\cdot\vartheta F\rangle\geqslant 0$ for any $\mathfrak{B}_{\Lambda^+}$-measurable function $F=F(\sigma_\Lambda)$. Here, $\Lambda^+\subset\Lambda$ is one of the halves of the container Λ cut off from it by the plane L, $\Lambda=\Lambda^+\cup\vartheta\Lambda^+$, $(\vartheta F)(\sigma_\Lambda)=F(\vartheta\sigma_\Lambda)$, and $(\vartheta\sigma_\Lambda)_t=(\sigma_\Lambda)_{\vartheta t}$. Now let $H_\Lambda(\sigma_\Lambda)$ be the relative Hamiltonian corresponding to the interaction U and certain ϑ-symmetric boundary conditions. These could be periodic boundary conditions, if Λ is a parallelepiped, or any boundary conditions $\bar\sigma\in\Omega$ such that $\bar\sigma_t=\bar\sigma_{\vartheta t}$ for all t. Then the relative Hamiltonian can be represented in the form

$$H_\Lambda(\sigma_\Lambda)=B(\sigma_\Lambda)+\vartheta B(\sigma_\Lambda), \tag{3.9}$$

where B is a certain $\mathfrak{B}_{\Lambda^+}$-measurable function, if, of course, $\Lambda=\vartheta\Lambda$. From (3.9) it follows (see [8]) that the corresponding state in Λ is reflection positive.

An important consequence of reflection positivity for us will be a chess-board estimate, which we shall describe in the form in which we need it. Let Λ be a cylindrical container, the product of a discrete torus $T^{\nu-1}$ and a ray $\mathbb{Z}_+=\{t\in\mathbb{Z},\ t\geqslant 0\}$. On the boundary of Λ we specify the boundary condition $\bar\sigma$ and denote the corresponding Gibbs state by $\langle\,\rangle_{\mathrm{per},\bar\sigma}$. We assume that all periods of the torus $T^{\nu-1}$ are even. Then the corresponding Gibbs state in Λ is reflection positive with respect to all reflections of the group $\Theta^{\nu-1}$, which is generated by the reflections in the hyperplanes orthogonal to the hyperplane $\{t:t^{(1)}=0\}$. We shall give the name elementary cube $\varkappa\subset T^{\nu-1}$ to a subset of $2^{\nu-1}$ different points of the torus $T^{\nu-1}$, $\varkappa=\{\tau_1,\tau_2,\ldots\}$, such that for any two of them $\|\tau_i-\tau_j\|=1$, $i\neq j$. Suppose that for every cube $\varkappa\subset T^{\nu-1}$ there is defined a function $F_\varkappa(\sigma_\Lambda)$ that depends only on the restriction $\sigma_\Lambda|_{\varkappa\times\mathbb{Z}_+}$. For any two cubes $\varkappa$, $\varkappa'$ we choose an element $\vartheta\in\Theta^{\nu-1}$ such that $\varkappa'=\vartheta\varkappa$. We define the functions

$$F_{\varkappa,\varkappa'}(\sigma_\Lambda)=F_\varkappa(\vartheta^{-1}\sigma_\Lambda),\qquad F_\varkappa^\Theta(\sigma^\Lambda)=\prod_{\varkappa'\subset T}F_{\varkappa,\varkappa'}(\sigma_\Lambda).$$

It is easy to see that these definitions are correct. The chess-board estimate is now the inequality

$$\left\langle\prod_\varkappa F_\varkappa(\sigma_\Lambda)\right\rangle_{\mathrm{per},\bar\sigma}\leqslant\prod_\varkappa\langle F_\varkappa^\Theta(\sigma_\Lambda)\rangle_{\mathrm{per},\bar\sigma}^{1/|T|}. \tag{3.10}$$

The estimate (3.10) follows from the fact that U is a reflection-positive interaction. Its derivation from reflection positivity can be found in [8]. With obvious modifications, all that has been said above can be extended to the case of states in the container $T^{\nu-1}\times[1,N]$.

We now prove proposition II of the theorem. We consider the container

$$V_{MN}=\{t\in\mathbb{Z}^\nu: -M\leqslant t^{(i)}<M,\ i=1,\ldots,\nu-1,\ 1\leqslant t^{(\nu)}\leqslant N\}$$

and denote the set of configurations $\Omega_{V_{MN}}$ by Ω_{MN}. We consider the configuration $\bar\sigma^{03}\in\Omega\mathbb{Z}^1$:

$$\bar\sigma_t^{03}=\begin{cases}3, & t\leqslant 0,\\ 0, & t>0.\end{cases}$$

In V_{MN} we consider a periodic boundary condition along the first $\nu-1$ directions and a boundary condition independent of them and corresponding to the configuration $\bar\sigma^{03}$ along the last direction. Our container is thereby identified with the product $T_M^{\nu-1}\times[1, N]$ of a $(\nu-1)$-dimensional torus with periods 2M and the segment [1, N]. We shall denote the corresponding Gibbs state $\langle\ \rangle^{\beta}_{\mathrm{per},\bar\sigma^{03}}$ by $\langle\rangle_{MN}{}^{\beta}$. Let $\langle\ \rangle_N{}^{\beta}$ be some limiting point of the family of states $\langle\ \rangle_{MN}{}^{\beta}$. It is clear that $\langle\ \rangle_N{}^{\beta}$ is a Gibbs state in the strip $\mathbb{Z}^{\nu}_{[1,N]}$. We show first that this state is not ergodic for $\beta\geqslant\beta_c$ and some $\beta_c<\infty$ (we mean ergodicity with respect to the group $\mathbb{Z}^{\nu-1}$). This is a consequence of the two following assertions: for all s, t $\in V_{MN}$ with $s^{(\nu)}=t^{(\nu)}=1$

$$\langle P_s{}^0\rangle_{MN}{}^{\beta}+\langle P_s{}^3\rangle_{MN}{}^{\beta}\to 0 \text{ as } \beta\to\infty \text{ uniformly in } M, N, s, \tag{3.11}$$

$$\langle P_s{}^1 P_t{}^2\rangle_{MN}{}^{\beta}\to 0 \text{ as } \beta\to\infty \text{ uniformly in } M, N, s, t. \tag{3.12}$$

Indeed, by virtue of the symmetry $\langle P_s{}^1\rangle_{MN}{}^{\beta}=\langle P_s{}^2\rangle_{MN}{}^{\beta}$, hence, $\langle P_s{}^1\rangle_{MN}{}^{\beta}$, $\langle P_s{}^2\rangle_{MN}{}^{\beta}\to{}^1/_2$ as $\beta\to\infty$ uniformly in M, N by virtue of (3.11). Therefore, $\langle P_s{}^1\rangle_N{}^{\beta}$, $\langle P_s{}^2\rangle_N{}^{\beta}\to{}^1/_2$, $\langle P_s{}^1P_t{}^2\rangle_N{}^{\beta}\to 0$ as $\beta\to\infty$, as follows from (3.11), (3.12) for s, t with $s^{(\nu)}=t^{(\nu)}=1$, and this proves the state $\langle\cdot\rangle_N{}^{\beta}$ is nonergodic.

The proofs of the assertions (3.11) and (3.12) consist of the usual application of the estimate (3.10) (see [6-9]). We prove (3.11). Let s = (0, ..., 0, 1). Then by virtue of (3.10),

$$\langle P_s{}^0\rangle_{MN}{}^{\beta}\leqslant[\langle P_V{}^0\rangle_{MN}{}^{\beta}]^{1/(2M)^{\nu-1}}, \tag{3.13}$$

where $P_V^i(\sigma)$ is the indicator of the event $\{\sigma\in\Omega_{MN}:\sigma_t=i$ for all t of the form $t=(2k^{(1)},\ldots,2k^{(\nu-1)}, 1)\in V_{MN}\}$, $i=0,\ldots,3$. For any configuration $\sigma\in\Omega_{MN}$ with $P_V^0(\sigma)=1$, we consider the configuration $\Pi_1\sigma\in\Omega_{MN}$:

$$(\Pi_1\sigma)_t=\begin{cases}\sigma_t, & t^{(\nu)}>1,\\ 1, & t^{(\nu)}=1.\end{cases}$$

The number of different σ with the same configuration $\Pi_1\sigma$ is $4^{M^{\nu-1}}$. The relative Hamiltonian $H_V^{\mathrm{per}}(\cdot|\bar\sigma^{03})$ corresponding to the considered cylindrical boundary conditions satisfies the estimate

$$H_V^{\mathrm{per}}(\sigma|\bar\sigma^{03})-H_V^{\mathrm{per}}(\Pi_1\sigma|\bar\sigma^{03})\geqslant M^{\nu-1}X-(2M)^{\nu-1}z-2(2M)^{\nu-1}Y=(2M)^{\nu-1}(X/2^{\nu-1}-2Y-z)\equiv(2M)^{\nu-1}c_1,$$

since $Y\geq y$. By virtue of (3.1), $c_1>0$. Therefore

$$\langle P_s{}^0\rangle_{MN}{}^{\beta}\leqslant[4^{M^{\nu-1}}\exp\{-\beta c_1(2M)^{\nu-1}\}]^{1/(2M)^{\nu-1}}=2^{1/2^{\nu-2}}\exp\{-\beta c_1\}. \tag{3.14}$$

Similarly $\langle P_s{}^3\rangle_{MN}{}^{\beta}\leqslant\langle P_V{}^3\rangle^{1/(2M)^{\nu-1}}$. For σ with $P_V^3(\sigma)=1$

$$H_V^{\mathrm{per}}(\sigma|\bar\sigma^{03})-H_V^{\mathrm{per}}(\Pi_1\sigma|\bar\sigma^{03})\geqslant(2M)^{\nu-1}(Z/2^{\nu-1}-z-2Y)\equiv(2M)^{\nu-1}c_2,$$

where $c_2>0$ by virtue of (3.1), and therefore

$$\langle P_s{}^3\rangle_{MN}{}^{\beta}\leqslant[4^{M^{\nu-1}}\exp\{-\beta c_2(2M)^{\nu-1}\}]^{1/(2M)^{\nu-1}}=2^{1/2^{\nu-2}}\exp\{-\beta c_2\}. \tag{3.15}$$

We obtain (3.11) from (3.14) and (3.15). To prove (3.12), we must define contours. Consider the indicators:

$$P_{st}{}^{i*}(\cdot) \quad \text{of the event } \{\sigma\in\Omega_{MN}:\sigma_s=i,\ \sigma_t\in\{0,3\}\},\ i=1,2,\ |s-t|=1,$$

$$P_{st}{}^{12}(\cdot) \quad \text{of the event } \{\sigma\in\Omega_{MN}:\sigma_s=1,\ \sigma_t=2\},\ |s-t|=1.$$

We shall say that the edge $l=\{\alpha,\gamma\}\subset V_{MN}$ is a boundary edge for the configuration σ if $P_{\alpha\gamma}{}^{1*}(\sigma)+P_{\alpha\gamma}{}^{12}(\sigma)=1$ and $\alpha^{(\nu)}=\gamma^{(\nu)}=1$. We shall denote the set of boundary edges of the configuration σ by $B(\sigma)$. We shall say that the subset $A\subset B(\sigma)$ is connected if the set A^* of dual $(\nu-1)$-cells is connected. The connected components of $B(\sigma)$ are called the supports of contours and are denoted by supp Γ, and the restrictions $\sigma|_{\mathrm{supp}\,\Gamma}$ are called contours of the configuration σ, $\Gamma=\Gamma(\sigma)$. Now suppose $s,t\in V_{MN}$, $s^{(\nu)}=t^{(\nu)}=1$ and for $\sigma\in\Omega_{MN}$, $P_s{}^1(\sigma)P_t{}^2(\sigma)=1$. Then it is easy to show that there exists a contour Γ of the configuration σ such that the surface Γ^* separates the points s and t (of course, only in the torus $T_M^{\nu-1}$). The edges of the contour Γ are divided into two subsets Δ_0 and Δ_1, so that

$$\prod_{(\alpha,\gamma)\in\Delta_0}P_{\alpha\gamma}{}^{12}(\sigma)\prod_{(\alpha,\gamma)\in\Delta_1}P_{\alpha\gamma}{}^{1*}(\sigma)=1, \tag{3.16}$$

and at the same time

$$\langle\Gamma\rangle_{MN}{}^{\beta}=\left\langle\prod_{(\alpha,\gamma)\in\Delta_0}P_{\alpha\gamma}{}^{12}\prod_{(\alpha,\gamma)\in\Delta_1}P_{\alpha\gamma}{}^{1*}\right\rangle^{\beta}_{MN}. \tag{3.17}$$

We go over from the sets Δ_0, Δ_1 to their subsets $\bar\Delta_0$, $\bar\Delta_1$, which satisfy the following additional property: For every edge $l\in\bar\Delta_0\cup\bar\Delta_1$ one can find a cube $\varkappa(l)\subset T_M^{\nu-1}$ such that $l\subset\varkappa(l)$ and $\varkappa(l_1)\neq\varkappa(l_2)$ for $l_1\neq l_2\in\bar\Delta_0\cup\bar\Delta_1$. It is easy to see that the subsets $\bar\Delta_0$, $\bar\Delta_1$ can be chosen in such a way that

$$|\bar\Delta_0|+|\bar\Delta_1|\geqslant\nu^{-1}|\mathrm{supp}\,\Gamma| \tag{3.18}$$

(for example, in $\tilde\Delta_0\cup\tilde\Delta_1$ we can include edges of only one direction). It is clear that

$$\langle\Gamma\rangle_{MN}^{\beta}\leqslant\Big\langle\prod_{(\alpha,\gamma)\in\tilde\Delta_0}P_{\alpha\gamma}^{12}\prod_{(\alpha,\gamma)\in\tilde\Delta_1}P_{\alpha\gamma}^{1*}\Big\rangle_{MN}^{\beta}. \tag{3.19}$$

Further, we can take a subset $\tilde\Delta_2=\tilde\Delta_2(\Gamma)\subset\tilde\Delta_1$ such that

$$\prod_{(\alpha,\gamma)\in\tilde\Delta_1}P_{\alpha\gamma}^{1*}(\sigma)\leqslant\prod_{(\alpha,\gamma)\in\tilde\Delta_2}P_{\gamma}^{0}(\sigma)\prod_{(\alpha,\gamma)\in\tilde\Delta_1\setminus\tilde\Delta_2}P_{\gamma}^{3}(\sigma). \tag{3.20}$$

It is easy to see that for the edge $(\alpha,\gamma)\in\tilde\Delta_0$

$$P_{\alpha\gamma}^{12}(\sigma)=P_{\alpha\gamma}^{12}(\sigma)\,[P_{\varkappa(\alpha\gamma),3}^{3}(\sigma)+P_{\varkappa(\alpha\gamma),3}^{\neq3}(\sigma)]\leqslant P_{\alpha\gamma}^{12}(\sigma)P_{\varkappa(\alpha\gamma),3}^{\neq3}(\sigma)+P_{\varkappa(\alpha\gamma),3}^{3}(\sigma), \tag{3.21}$$

where for the cube $\varkappa\subset T_M^{\nu-1}$ we denote by $P_{\varkappa,3}^{\neq3}$ the indicator of the event $\{\sigma\in\Omega_{MN},\ \sigma_r\neq3$ for all $r=(\tau,3)\in V_{MN},\ \tau\in\varkappa\}$, and $P_{\varkappa,3}^{3}=1-P_{\varkappa,3}^{\neq3}$. Substitution of (3.20) and (3.21) in (3.19) gives a sum of not more than $2^{|\operatorname{supp}\Gamma|}$ terms of the form

$$\Big\langle\prod_{(\alpha\gamma)\in\tilde\Delta_2}P_{\gamma}^{0}\prod_{(\alpha\gamma)\in\tilde\Delta_1\setminus\tilde\Delta_2}P_{\gamma}^{3}\prod_{(\alpha\gamma)\in\tilde\Delta_3}(P_{\alpha\gamma}^{12}P_{\varkappa(\alpha\gamma),3}^{\neq3})\prod_{(\alpha\gamma)\in\tilde\Delta_0\setminus\tilde\Delta_3}P_{\varkappa(\alpha\gamma),3}^{3}\Big\rangle_{MN}^{\beta}, \tag{3.22}$$

each of which corresponds to a certain subset $\tilde\Delta_3\subset\tilde\Delta_0$. The chess-board estimate makes it possible to bound the expression (3.22) above by

$$[\langle P_V^{0}\rangle^{|\tilde\Delta_2|/2}\langle P_V^{3}\rangle^{|\tilde\Delta_1\setminus\tilde\Delta_2|/2}\langle P_V^{12}P_{V,3}^{\neq3}\rangle^{|\tilde\Delta_3|}\langle P_{V,3}^{3}\rangle^{|\tilde\Delta_0\setminus\tilde\Delta_3|}]^{1/(2M)^{\nu-1}}, \tag{3.23}$$

where P_V^{12} is the indicator of the event $\{\sigma\in\Omega_{MN}:\sigma_r=2$ for $r=(2k^{(1)},\dots,2k^{(\nu-1)},1)$; $\sigma_r=1$ for $r=(2k^{(1)}+1,2k^{(2)},\dots,2k^{(\nu-1)},1)\}$, $P_{V,3}^{3}$ is the indicator of the event $\{\sigma\in\Omega_{MN}$: in every cube $\varkappa\subset T_M^{\nu-1}$ there exists a point $\tau=\tau(\sigma)\in\varkappa$ such that $\sigma_{(\tau,3)}=3\}$, $P_{V,3}^{\neq3}$ is the indicator of $\{\sigma\in\Omega_{MN}:\sigma_r\neq3$ for all $r=(\tau,3),\ \tau\in T_M^{\nu-1}\}$. The estimate of the mean value $\langle P_{V,3}^{3}\rangle_{MN}^{\beta}$ is identical to the estimate of $\langle P_V^{3}\rangle_{MN}^{\beta}$ above, and we shall not repeat the details. It remains to estimate $\langle P_V^{12}P_{V,3}^{\neq3}\rangle_{MN}^{\beta}$. We assume that $P_V^{12}(\sigma)P_{V,3}^{\neq3}(\sigma)=1$. We define the configuration $\Pi_{1,0}(\sigma)$ by

$$(\Pi_{1,0}(\sigma))_r=\begin{cases}1, & r=(\tau,1),\\ 0, & r=(\tau,2),\\ \sigma_r & \text{for the remaining}\end{cases}$$

We have

$$H_V^{\rm per}(\sigma\,|\,\bar\sigma^{03})-H_V^{\rm per}(\Pi_{1,0}(\sigma)\,|\,\bar\sigma^{03})\geqslant(2M)^{\nu-1}[Y/2^{\nu-2}-z-y-2x]\equiv(2M)^{\nu-1}c_3, \tag{3.24}$$

where $c_3>0$. Therefore

$$\langle P_V^{12}P_{V,3}^{\neq3}\rangle_{MN}^{\beta}\leqslant16\exp\{-\beta c_3\}.$$

From what was said above there follows the existence of a positive constant c such that for any set of edges $R\subset T_M^{\nu-1}$ the probability that R is the support of a contour of a configuration is bounded above by $\exp\{-c\beta|R|\}$. The assertion (3.12) then follows in the standard manner.

We now explain briefly how to construct the states $\langle\cdot\rangle_N^{\beta,i}$, $i=1,2$. For the details of the construction, see [10]. Consider the limits

$$\lim_{n\to\infty}\frac{1}{(2n)^{\nu-1}}\sum_{\|\tau\|\leqslant n}P_{(\tau,1)}^{i}(\sigma)=\pi^{(i)}(\sigma),\qquad i=1,2.$$

One can show that $\langle\pi^{(1)}+\pi^{(2)}\rangle_N^{\beta}\to1$, $\langle\pi^{(1)}\pi^{(2)}\rangle_N^{\beta}\to0$ as $\beta\to\infty$. It follows from this by symmetry considerations that for sufficiently large values of β the events $A_i=\{\sigma:\pi^{(i)}(\sigma)\geqslant\gamma>1/2\}$ have nonzero probabilities. We now assume

$$\langle B\rangle_N^{\beta,i}=\langle B\cap A_i\rangle_N^{\beta}/\langle A_i\rangle_V^{\beta},\quad B\in\mathfrak{B}_+,\ i=1,2.$$

Since the events A_i are translationally invariant, the states $\langle\cdot\rangle_N^{i}$ are Gibbs states in the strip $\mathbb{Z}_{[1,N]}^{\nu}$. The relation (3.4) follows from the fact that $\langle P_i^{i}\rangle_N^{\beta,i}=\langle\pi^{(i)}\rangle_N^{\beta,i}\geqslant\gamma>1/2$. The relation (3.5) is as readily proved. This completes part II of the theorem. The proof of part I follows from the fact that all our estimates were uniform in N. Therefore, as states $\langle\cdot\rangle^{\beta,i}$ we can take the limit points of the states $\langle\cdot\rangle_N^{\beta,i}$ as $N\to\infty$.

4. Uniqueness and Logarithmic Strip Theorem

In this section, we prove a theorem on uniqueness of the Gibbs states in the low-temperature region for a general class of interactions including Czech interactions. In this section, U is an arbitrary finite-range interaction with finite values, S is a finite set, and we shall assume that $S = \{0, 1, \ldots, n\}$.

Let $d \geq 1$, $V_d = \{t \in \mathbb{Z}^\nu, \|t\| \leq d\}$, and $\bar{\sigma} \in \Omega$. We denote by $D_d(\bar{\sigma}) \subset \Omega_{V_d}$ the set of relative ground-state configurations in V_d: for all $\sigma \in D_d(\bar{\sigma})$, $\sigma' \in \Omega_{V_d}$ $H_{V_d}(\sigma|\bar{\sigma}) \leq H_{V_d}(\sigma'|\bar{\sigma})$. The interaction U is called a B(d) interaction if for all $\bar{\sigma} \in \Omega$, $\sigma \in D_d(\bar{\sigma})$ we have $\sigma_{(0,\ldots,0)} = 0$.

This property was introduced in [3], in which a theorem was proved which showed that low-temperature uniqueness follows from fulfillment of the condition B(d) for the interaction U. However, the proof contained an error, Proposition 1 being incorrect. Navrátil's example was constructed as a counterexample to this proposition (it is easy to see that the interaction (2.1)-(2.2) possesses the B(1) property). Important here is the following property of Czech interactions, which we describe for the example of the model obtained by replacing in (2.1) the norm $|\cdot|$ by $\|\cdot\|$. All the results of the previous section remain valid in this case too. Thus, we consider a container V_d and suppose that $t \notin V_d$, $\text{dist}(t, V) = 1$. We determine a pair of boundary conditions $\bar{\sigma}^{(1)}$, $\bar{\sigma}^{(2)}$ by the formula

$$(\bar{\sigma}^i)_s = \begin{cases} i, & s = t, \\ 3, & s \neq t. \end{cases}$$

Then both sets $D_d(\bar{\sigma}^{(i)})$ consist of one configuration $\sigma^{(i)}$, and

$$(\sigma^{(i)})_s = \begin{cases} i, & \text{dist}(s, V_d^c) = 1, \\ 0 & \text{for the remaining } s \in V. \end{cases}$$

We see that change of the boundary condition at one point leads to a change along the complete boundary V_d however large the value of d.

This same feature makes it impossible to obtain a uniqueness proposition for Czech models as a consequence of the general criterion of [1]. In accordance with it, for given U and β, one must consider the pair of conditional Gibbs distributions $q_V^\beta(\cdot|\bar{\sigma}')$, $q_V^\beta(\cdot|\bar{\sigma}'')$ determined by a pair of boundary conditions $\bar{\sigma}'$, $\bar{\sigma}''$ that differ at a single point $t \in \mathbb{Z}^\nu$, calculate the Kantorovich–Rubinstein–Ornstein–Wasserstein distance $R(q_V^\beta(\cdot|\bar{\sigma}'), q_V^\beta(\cdot|\bar{\sigma}''))$ between them, and consider the function $k_t^V = \sup_{\bar{\sigma}', \bar{\sigma}''} R(q_V^\beta(\cdot|\bar{\sigma}'), q_V^\beta(\cdot|\bar{\sigma}''))$, where sup is taken over all pairs $\bar{\sigma}'$, $\bar{\sigma}''$ that are identical outside the point $t \in \mathbb{Z}^\nu$. Then for the interaction U and reciprocal temperature β there is uniqueness if for some finite container $V \subset \mathbb{Z}^\nu$

$$\sum_{t \in \mathbb{Z}^\nu} k_t^V < |V|. \tag{4.1}$$

But as we have just seen in the given example, for points $t \in V^c$ with $\text{dist}(t, V) = 1$, $k_t^V \sim |\partial V|$, and therefore

$$\sum_{t \in \mathbb{Z}^\nu} k_t^V \sim |\partial V|^2, \tag{4.2}$$

and for $\nu \geq 3$ this contradicts (4.1). The relation (4.2) also holds for the original Czech model (2.1), but only for containers having the shape of a parallelepiped. Moreover, for it one can choose the container in such a way that the relation (4.1) holds at a sufficiently low temperature. Namely, as V one must consider a polyhedron having all of its faces parallel to one of the planes $\{t^{(1)} \pm t^{(2)} \pm \ldots = 0\}$. In this way, we see that for some Czech models low-temperature uniqueness follows from the given criterion, but for others it does not. Fortunately, uniqueness follows directly from B(d).

THEOREM 2. Let U be a B(d) interaction, $d \geq 1$. There exists a $\tilde{\beta} = \tilde{\beta}(U)$ such that for $\beta \geq \tilde{\beta}$ there exists precisely one (U, β) Gibbs state.

We assume that the reader is familiar with the condition of stability of periodic ground-state configurations – the Peierls condition, also called the Gertsik–Pirogov–Sinai condition (see [11,12]).

PROPOSITION 1. The following two conditions are equivalent: 1) the interaction U is a B(d) interaction with a certain $d \geq 1$; 2) the interaction U has precisely one periodic ground-state configuration. For it, the Peierls condition is satisfied.

Proof of the Proposition. The implication $2 \Rightarrow 1$ is the content of Proposition 2.III of [3]. To prove the opposite assertion, we define the "gap" $\delta_d(U)$ by

$$\delta_d(U) = \inf_{\bar{\sigma} \in \Omega} \min_{\substack{\sigma' \in \Omega_{V_d}, \\ \sigma_{(0)} \neq 0}} \min_{\sigma'' \in D_d(\bar{\sigma})} (H_{V_d}(\sigma' \mid \bar{\sigma}) - H_{V_d}(\sigma'' \mid \bar{\sigma})).$$

By virtue of the condition B(d), the configuration $\sigma' \notin D_d(\bar{\sigma})$ for any $\bar{\sigma}$, and therefore $\delta_d(U) > 0$. For any configuration $\sigma' \in \Omega$, $\sigma'_{(0)} \neq 0$, we now set

$$(\bar{\sigma}')_t = \begin{cases} \sigma_t', & t \notin V_d, \\ \eta_t, & t \in V_d, \end{cases}$$

where η_t is an arbitrary configuration in $D_d(\sigma')$. By definition,

$$H(\sigma') - H(\bar{\sigma}') \geqslant \delta_d(U) > 0.$$

Therefore, if σ is a ground-state configuration, $\sigma = \bar{\sigma}^0 = 0$. Moreover, if for $\sigma' \in \Omega$, $|\{t \in \mathbb{Z}^\nu,\ \sigma_t' \neq 0\}| = k$, then it can be proved by induction that

$$H(\sigma') - H(\bar{\sigma}^0) \geqslant k |V_d|^{-1} \delta_d(U).$$

But this is the Peierls condition in our case.

COROLLARY. In the Czech models considered above there is precisely one low-temperature Gibbs state.

The assertion of Theorem 2 can be augmented by a "logarithmic strip" assertion, which means the following. For any configuration $\sigma \in \Omega_{V_n}$ we call the union of the connected components of the set $\{t \in V_n;\ \sigma_t \neq 0\}$ adjacent to the boundary ∂V_n the strip $R(\sigma)$. We set $\xi_n'(\sigma) = \max_{t \in R(\sigma)} \mathrm{dist}(t, \partial V_n)$. Let $f(n) \geqslant 0$ be some function. We shall say that the interaction U has the property of an f strip at the reciprocal temperature β if the probability $q_{V_n}{}^\beta(\{\sigma \in \Omega_{V_n} : \xi_n'(\sigma) > f(n)\} | \bar{\sigma}) \to 0$ as $n \to \infty$ uniformly with respect to $\bar{\sigma} \in \Omega$.

THEOREM 3. Under the assumptions of Theorem 2, the interaction U has the log-strip property for $\beta \geq \tilde{\beta}$.

Proofs of Theorems 2 and 3. It is easy to see that for any container $W \subset$, $|W| < \infty$ the functions $q_W{}^\beta(\sigma_W | \bar{\sigma})$ are uniformly continuous with respect to $\beta \in [0, \infty]$. Therefore, for any $\varepsilon > 0$ we can find a $\beta(\varepsilon) < \infty$ such that for all $\beta \geq \beta(\varepsilon)$

$$q_{V_d}{}^\beta(\{\sigma \in \Omega_{V_d} : \sigma_{(0)} \neq 0\} | \bar{\sigma}) < \varepsilon \text{ uniformly w.r.t. } \bar{\sigma} \in \Omega. \tag{4.3}$$

We shall say that the point $t \in W$ is irregular for the configuration σ_W if $\sigma_t \neq 0$. Let $\tilde{R}(\sigma)$ be the set of irregular points of the configuration $\sigma \in \Omega_{V_n}$, $\tilde{R}^d(\sigma) = \{t \in \mathbb{Z}^\nu,\ \mathrm{dist}(t, \tilde{R}(\sigma)) \leqslant d/2+1\}$, and $R(\sigma)$ be the maximal connected component of the union $\tilde{R}^d(\sigma) \cup [V_n \setminus V_{n-d}]$ that contains the second term. We shall study the random variable $\xi_n(\sigma) = \max_{t \in R(\sigma)} \mathrm{dist}(t, \partial V_n)$. Suppose we are given an increasing function $f(n) \to \infty$ as $n \to \infty$. The connected component T of the set $R(\sigma) \setminus [V_n \setminus V_{n-d}]$ is called a tongue if $\max_{t \in T} \mathrm{dist}(t, \partial V_n) \geqslant f(n)$. If T_1 and T_2 are two tongues of the configuration σ and t_1 and t_2 are irregular points, $t_i \in T_i$, $i = 1, 2$, then $\|t_1 - t_2\| > d + 1$. Every tongue has a nonempty intersection with the layer $F_n = V_{(n-d)} \setminus V_{(n-d-1)}$. We label its points and call the point of intersection of $T \cap F_n$ with the minimal number the root of the tongue T. We denote the tongue T with root at the point $s \in T \cap F_n$ by T_s. If the point s is not a root of the tongue, we shall assume that $T_s = \varnothing$. We obtain in this manner an ensemble of tongues $\mathcal{T} = \{T_s,\ s \in F_n\}$ with probability distribution $\langle \cdot \rangle^\beta_{n,\bar{\sigma}}$ on it; if by $T(\sigma)$ we denote the set of tongues of the configuration σ, then

$$\langle T \rangle^\beta_{n,\bar{\sigma}} = q^\beta_{V_n}(\{\sigma' \in \Omega_{V_n} : T(\sigma') = T\} | \bar{\sigma}).$$

We estimate the probability of the configuration of tongues $T = \{T_s\}$. If r is the number of irregular points in the tongue T_s, then $|T_s| \leqslant r(d+2)^\nu$. On the other hand, for any set of r points one can find a subset of at least $[r(2d+1)^{-\nu}]$ points for which the distances pairwise between them are greater than d. Therefore, in the tongue T_s there are $|T_s|[2d(d+2)]^{-\nu}$ irregular points separated pairwise by not less than $d + 1$. Hence and from (4.3) it follows that

$$\langle T \rangle^\beta_{n,\bar{\sigma}} \leqslant \exp\left\{ -c_\beta^{(1)} \sum_s |T_s| \right\}, \tag{4.4}$$

where $c_\beta^{(1)} \to \infty$ as $\beta \to \infty$ (like the constants $c_\beta^{(j)}$, which are introduced below). Indeed, let $t_1, t_2, \ldots \in V_n$ be a sequence of points, $\|t_i - t_j\| \geqslant d$ for $i \neq j$, $\mathrm{dist}(t_i, \partial V_n) > d$. Then for $\beta \geqslant \beta(\varepsilon)$

$$\begin{gathered} q^\beta_{V_n}(\{\sigma_{t_i} \neq 0, i = 1, 2, \ldots\} | \bar{\sigma}) = q^\beta_{V_n}(\{\sigma_{t_i} \neq 0, i = 2, 3, \ldots\} | \bar{\sigma}) q^\beta_{V_n}(\{\sigma_{t_1} \neq 0\} | \bar{\sigma}, \{\sigma_{t_i} \neq 0, \\ i = 2, \ldots\}) \leqslant q_{V_n}(\{\sigma_{t_i} \neq 0, i = 2, 3, \ldots\} | \bar{\sigma}) \sup_{\bar{\sigma}} q^\beta_{V_n}(\{\sigma \in \Omega_{V_d}, \sigma_{(0)} \neq 0\} | \bar{\sigma}) < \varepsilon q^\beta_{V_n}(\{\sigma_{t_i} \neq 0, i = 2, 3, \ldots\} | \bar{\sigma}). \end{gathered} \tag{4.5}$$

Iterating (4.5), we obtain (4.4). We can now estimate the probability of the event $\xi_n(\sigma)>f(n)$. By definition, if $\xi_n(\sigma)>f(n)$, then for at least one point $s\in F_n$, $T_s\neq\varnothing$. Therefore

$$\langle\{\sigma:\xi_n(\sigma)>f(n)\}\rangle^{\beta}_{n,\bar\sigma}\leqslant\sum_{\{T_s\}}\exp\left\{-c_\beta^{(1)}\sum_s|T_s|\right\}-1, \tag{4.6}$$

where the summation is extended to all sets of tongues, including the empty one; it corresponds to the subtracted unity. The last expression does not, in its turn, exceed

$$\left[\max_{s\in F_n}\left(1+\sum_{T_s:|T_s|>f(n)}\exp\{-c_\beta^{(1)}|T_s|\}\right)\right]^{|\partial V_n|}-1. \tag{4.7}$$

On the other hand, the number of (d + 2)-connected sets of irregular points to which a tongue with fixed root and with N points corresponds does not exceed $[(2d+4)^{\nu}]^{2N}$. Therefore

$$\sum_{T:|T|\geqslant f(n)}\exp\{-c_\beta^{(1)}|T|\}\leqslant\sum_{k=f(n)}^{\infty}(2d+4)^{2k\nu}\exp\{-c_\beta^{(1)}k\}\leqslant\exp\{-c_\beta^{(2)}f(n)\},$$

and, hence,

$$\langle\xi_n>f(n)\rangle^{\beta}_{n,\bar\sigma}\leqslant(1+\exp\{-c_\beta^{(2)}f(n)\})^{(2n)^{\nu-1}2\nu}-1.$$

For example, setting $f(n)=\ln n$, we have

$$\langle\xi_n>\ln n\rangle^{\beta}_{n,\bar\sigma}\leqslant(1+n^{-c_\beta^{(2)}})^{\nu 2^{\nu}n^{\nu-1}}-1\leqslant\exp\{2^{\nu}\nu n^{\nu-1-c_\beta^{(2)}}\}-1\to 0$$

as $n\to\infty$ for sufficiently large β. Thus, the logarithmic strip theorem is proved. Similarly, for $f(n)=\lambda n$, $\lambda<1$, we have for large n

$$\langle\xi_n>\lambda n\rangle^{\beta}_{n,\bar\sigma}\leqslant\exp\{-c_\beta^{(3)}n\},$$

where $c_\beta^{(3)}=c_\beta^{(3)}(\lambda)\to\infty$ as $\beta\to\infty$ for any λ. In conjunction with the standard assertion of the exponentially weak dependence of the correlation functions on the shape of the container at low temperatures we can conclude from this that

$$\left|\frac{\langle\sigma_A\rangle^{\beta}_{n,\bar\sigma_1}}{\langle\sigma_A\rangle^{\beta}_{n,\bar\sigma_2}}-1\right|\leqslant C_{A,\beta}\exp\{-c_\beta^{(4)}n\},\quad C_{A,\beta}<\infty,$$

uniformly with respect to $\bar\sigma_1,\bar\sigma_2\in\Omega$. With this we complete the proof.

Remarks. 1. From Theorem 2 and Proposition 1 there follows low-temperature uniqueness for models with unique periodic ground-state configuration. This result has been independently announced by D. Martirosyan. 2. Similar ideas have already been applied to different situations, in [13, 14]. 3. Outside the low-temperature region, the assertion of uniqueness for models with unique ground state is no longer true (see [10] for the case of continuous spins, and the following section for discrete spins). 4. In the case of continuous spins, uniqueness of the ground-state configuration does not entail uniqueness of the Gibbs state even in the low-temperature region. A corresponding example is constructed in [15].

5. Phase Transitions in Czech Models

We do not know whether phase transitions take place in the examples of the Czech models described above. Therefore, in the present section we consider one further Czech model, in which the existence of a phase transition can be proved.

In the new model, the phase space is $S=\{0,1,\ldots,2q+1\}$ (in the original model, $q=1$). As before, the interaction $\bar U$ is given by two functions $\bar U(a)$, $\bar U(a,b)$, the relation (2.1) holds, and the table (2.2) must be replaced by

$$\bar U(a,b)=\bar U(b,a)=\begin{cases}0, & a=b,\\ 0, & 1\leqslant a,b\leqslant q,\quad q+1\leqslant a,b\leqslant 2q,\\ x, & a=0,b=1,\ldots,2q,\\ X, & a=0,b=2q+1,\\ y, & a=1,\ldots,2q,b=2q+1,\\ Y, & a=1,\ldots,q,b=q+1,\ldots,2q,\end{cases}\qquad \bar U(a)=\begin{cases}0, & a=0,\\ z, & a=1,\ldots,2q,\\ Z, & a=2q+1.\end{cases} \tag{5.1}$$

It is clear that in the model considered all the above properties of the Czech models still remain. However, it also has new features.

THEOREM 4. Suppose $\nu \geq 2$ and that the parameters of the interaction (5.1) satisfy the conditions of Theorem 1. Then one can find a function $x(q, \nu)$ and a number $q_0 = q_0(\nu)$ such that for all $q \geq q_0$, $x \geq x(q, \nu)$ there exists a value $\beta(q)$ of the reciprocal temperature for which there exist at least two different translationally invariant Gibbs states $\langle\cdot\rangle^0$, $\langle\cdot\rangle^1$. At the same time, $\langle P_t^0\rangle^0 > 1/2$, $\langle P_t^0\rangle^1 < 1/2$ and $\beta \to \infty$ as $q \to \infty$.

The proof of the theorem is based on the technique of [10,16], and we therefore limit ourselves to sketching it. It is necessary to consider a state in the volume V_n with periodic boundary conditions, $\langle\cdot\rangle_n^\beta$, and prove for it the validity of the three following propositions:

$$\langle P_t^0\rangle_n^\beta \to 1 \text{ as } \beta\to\infty; \quad \langle P_t^0\rangle_n^{\beta_0} < 1/3 \text{ for some } \beta_0<\infty; \quad \langle P_t^0(1-P_s^0)\rangle_n^\beta < 1/5 \text{ for all } \beta\geqslant\beta_0 \text{ and } s\neq t. \tag{5.2}$$

These propositions must hold uniformly in n. From this one deduces that the limit point $\langle\cdot\rangle^\beta$ of the sequence $\langle\cdot\rangle_n^\beta$ is nonergodic for a certain $\beta = \beta(q) > \beta_0$, and the theorem is established by applying to $\langle\cdot\rangle^{\beta(q)}$ the analog of the ergodic expansion (see [10]).

We show only that $\langle P_t^0\rangle_n^\beta \to 0$ as $q \to \infty$ uniformly in n, this being equivalent to (5.2). For this, we note first that the partition function can be estimated below as follows:

$$Z(V_n, \beta) \geqslant q^{|V_n|}\exp\{-\beta z|V_n|\}. \tag{5.3}$$

By virtue of the chess-board estimate,

$$\langle P_t^0\rangle_n^\beta \leqslant [\langle P_V^0\rangle_n^\beta]^{2^\nu/|V_n|},$$

where P_V^0 is the indicator of $\{\sigma\in\Omega_{V_n} : \sigma_t=0 \text{ for } t=(2k^{(1)},\ldots,2k^{(\nu)})\}$. Therefore

$$\langle P_t^0\rangle_n^\beta \leqslant \left[\frac{(2q+1)^{|V_n|(2^\nu-1)/2^\nu}}{q^{|V_n|}\exp\{-\beta z|V_n|\}}\right]^{2^\nu/|V_n|} \leqslant 3^{(2^\nu-1)}q^{-1}\exp\{2^\nu\beta z\},$$

and this proves our assertion. The remaining assertions are verified as in [16]. With this we complete the outline of the proof of Theorem 4.

It would be very interesting to understand the behavior of the original Czech model at the point T_{cr}, where half-space nonuniqueness commences: What is the rate of decrease of the correlations, does the free energy have a singularity, and so forth? However, we are not yet able to answer these questions.

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