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Fluctuations in the Curie–Weiss Version of the Random Field Ising Model

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The fluctuations of the order parameter in the Curie–Weiss version of the Ising model with random magnetic field are computed. Away from criticality or at first-order critical points they have a Gaussian distribution with random (i.e., *sample-dependent*) mean, thermal fluctuations contributing in same order as the fluctuations of the field; at second- or higher-order critical points, non-Gaussian sample-dependent distributions appear, and the fluctuations of the fields are enhanced, dominating over the thermal ones.

KEY WORDS: Random field Ising model; nontrivial fluctuations.

1. INTRODUCTION

An important problem in equilibrium statistical mechanics is the determination of how macroscopic observables fluctuate around their mean values for systems which are close to or at their critical temperatures. Renormalization group ideas produced great advance both in the heuristic understanding and in the numerical computation of critical indices. Not much, however, is rigorously controlled for nontrivial systems, notable exceptions being explicitly solvable models. In this category fall the so-called Curie–Weiss models, whose critical behavior has been studied in a series of papers by Ellis and Newman.[1–3] They computed the statistics of large spin-block variables, showing in particular nontrivial (i.e., non-Gaussian) fluctuations at second-order critical temperatures.

This paper is part of a program where we propose to investigate how quenched randomness affects the Ellis and Newman picture. We consider

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the Curie–Weiss version of the random field Ising model, whose thermodynamics and phase diagram have been determined by Salinas and Wreszinski[4] (see also ref. 5), and we compute the fluctuations of the spin-block variable. These calculations show some remarkable new features as compared to the usual nonrandom model. At criticalities of second or higher order the contribution to fluctuations coming from the random field are enhanced so as to completely offset the contribution coming from thermal fluctuations; in particular the probability distributions obtained are not of the type described in refs. 1–3. Away from criticality both fluctuations contribute in the same order, implying that the fluctuations variable is non-self-averaging. The results concerning critical fluctuations correct erroneous statement made in a previous letter[5] announcing some of our results.

This paper is organized as follows. In the remainder of this section we define the model. In Section 2 we state the limit theorems for thermodynamics and fluctuations, proofs being deferred to Section 3. The notation and proofs follow as closely as possible those of refs. 2 and 3, such that the improvements required by this more complex problem with the presence of randomness become clearer.

1.1. The Model

We consider a system of n sites. For each $i = 1, 2,..., n$ we define a (spin) random variable $\sigma_i = \pm 1$ and a (local) random field h_i. These fields are supposed to be independent and identically distributed random variables, according to a measure in $\mathbf{R}$, $dv(h)$, which we shall denote by $h_i \sim dv(h)$. A spin configuration σ is defined as a function

$$\sigma: \mathbf{Z}_+ \to \{-1, +1\}$$
$$i \in \mathbf{Z}_+ \to \sigma_i \in \{-1, +1\}$$

where $\{-1, +1\}^n$ denotes the Cartesian product of n copies of $\{-1, +\}$.

Of course, one defines a configuration of fields $\mathbf{h}$ in $\mathbf{R}^\infty$ in the same way.

For each $n = 1, 2,...$ and fixed $\mathbf{h}$, we state the probability of a configuration σ restricted to $\{1,..., n\}$ as

$$\mu_n(\sigma, \mathbf{h}) = \frac{1}{Z_n(\mathbf{h})} e^{-\beta H_n(\sigma, \mathbf{h})} \tag{1.1}$$

where β is the inverse of the physical temperature and H_n is the so-called Hamiltonian function which characterizes the model in whose probability properties one is interested. $Z_n(\mathbf{h})$ is just a normalization factor, the partition function, given by

$$Z_n(\mathbf{h}) = \sum_{\sigma} e^{-\beta H_n(\sigma, \mathbf{h})} \tag{1.2}$$

In this paper we shall study the model with Hamiltonian

$$H_n = -\frac{J}{2n}\left(\sum_{i=1}^{n} \sigma_i\right)^2 - \sum_{i=1}^{n} h_i \sigma_i \tag{1.3}$$

with $J > 0$ describing the (ferromagnetic) interaction between each pair of spins.

2. MAIN RESULTS

In this section we state our main results concerning both the thermodynamics and the fluctuations of the system.

Let us first denote by $(\Omega, \mathscr{B}(\Omega), \mu)$ the probability space on which $\mathbf{h}$ is defined. Here, μ is the product measure induced by $d\nu(h)$.

To describe the thermodynamics, we must evaluate the free energy

$$f = \lim_{n \to \infty} \left\{ -\frac{1}{n\beta} \log Z_n(\mathbf{h}) \right\} \tag{2.1}$$

where Z_n may be written as

$$\begin{aligned} Z_n(\mathbf{h}) &= \sum_{\sigma} \exp[-\beta H_n(\sigma, \mathbf{h})] \\ &= \frac{1}{(2\pi)^{1/2}} \sum_{\sigma} \int dx \exp\left[-\frac{x^2}{2} + \beta \sum_{i=1}^{n} h_i \sigma_i + x \left(\frac{\beta J}{n}\right)^{1/2} \sum_{i=1}^{n} \sigma_i \right] \\ &= 2^n \left(\frac{n}{2\pi}\right)^{1/2} \int dx \exp[-n G_n(\mathbf{h}, x)] \end{aligned} \tag{2.2}$$

Here

$$G_n(\mathbf{h}, x) = \frac{x^2}{2} - \frac{1}{n} \sum_{i=1}^{n} \ln \cosh[(\beta J)^{1/2} x + \beta h_i] \tag{2.3}$$

and use has been made of the identity

$$\exp \frac{\eta^2}{2} = \frac{1}{(2\pi)^{1/2}} \int dx \exp\left(-\frac{x^2}{2} + \eta x\right)$$

It is important to stress that the function $G_n(\mathbf{h}, x)$ is a different random variable defined on Ω for each n. Therefore, the proof that (2.1) may be obtained from (2.2) through the Laplace asymptotic method is somehow tricky and requires a careful procedure. This result may be stated as follows.

Theorem 2.1. If

$$\int |h| \, dv(h) < \infty \tag{2.4}$$

then

$$\beta f = \log 2 + \inf_{x \in \mathbf{R}} \{ \lim_{n \to \infty} G_n(\mathbf{h}, x) \} \quad \text{ae}[\mu] \tag{2.5}$$

For rest of this paper, we shall omit the factor log 2 in (2.5), since it plays no role.

Remark 2.2. By the strong law of large numbers, $G_n(\mathbf{h}, x)$ converges ae$[\mu]$ to

$$G(x) = \frac{x^2}{2} - \int dv(h) \ln \cosh[(\beta J)^{1/2} x + \beta h_i] \tag{2.6}$$

for every fixed x. There *may* then exist a set $B_x \subset \Omega$, possibly depending on x and with $\mu(B_x) = 0$, such that for some $\mathbf{h} \in B_x$ the convergence would not hold. Therefore, to ensure that the free energy is obtained ae$[\mu]$ in (2.5), one must also prove that $\mu(\bigcup_x B_x) = 0$. A related result for the limiting Gibbs states exists,[6] but here we need a stronger control on the randomness in order to analyze fluctuations.

Given a function $G(x)$, let us characterize some properties of its global minima. Let $\{x_i^*\}_{i=1}^{\alpha}$ be the set of the α global minima of $G(x)$. If a Taylor expansion holds around each minimum such that

$$G(x) = G(x_i^*) + \theta_i \frac{(x - x_i^*)^{2k_i}}{(2k_i)!} + o[(x - x_i^*)^{2k_i}] \qquad \text{as} \quad x \to x_i^*$$

we shall call $k_i \equiv k_i(x_i^*)$ the "type" and $\theta_i \equiv \theta_i(x_i^*)$ the "strength" of the minimum x_i^*.

The number α of global minima, and their coordinates, types, and strengths, will be functions of the physical parameters β and J, and also of the measure $dv(h)$ (depending, for instance, on its variance).

Typical cases[4] for the structure of global minima are:

(a) Paramagnetic phase (with one global quadratic minimum): $\alpha = 1$, $k = 1$.

(b) Ferromagnetic phase (two global quadratic minima): $\alpha = 2$, $k_i = 1$, $i = 1, 2$.

(c) First-order phase transition (several global quadratic minima): $\alpha > 1$, $k_i = 1$, $i = 1, 2, \ldots, \alpha$.

(d) Second-order phase transition (one global minimum): $\alpha = 1$, $k = 2$.

(e) Tricritical point (one global, also nonquadratic minimum): $\alpha = 1$, $k = 3$.

After these general remarks, we may turn to the limiting theorems. Let us define the random variable

$$A_n(a, \gamma) = \frac{S_n - na}{n^{1-\gamma}} \tag{2.7}$$

for any a and γ real.

The main ingredient of our central theorems will be the following lemma, which relates the probability density of $A_n(a, \gamma)$ with the function $G_n(\mathbf{h}, x)$.

Lemma 2.3. Let W be a random variable independent of S_n, distributed according to a Gaussian of mean zero and variance 1, $N(0, 1)$, which we denote by $W \sim N(0, 1)$. Then

$$\frac{W}{(\beta J)^{1/2} n^{1/2-\gamma}} + A_n(a, \gamma) \sim \frac{ds \exp\{-nG_n[\mathbf{h}, (\beta J)^{1/2}(s/n^\gamma + a)]\}}{\int ds \exp\{-nG_n[\mathbf{h}, (\beta J)^{1/2}(s/n^\gamma + a)]\}} \tag{2.8}$$

Remark 2.4. The random variable will not contribute to the right-hand side of (2.8) if $\gamma < 1/2$.

What we will do is study whether the right-hand side of (2.8) converges weakly to some probability measure as $n \to \infty$. This implies the control of the asymptotic convergence of integrals of the type

$$\int ds\, t(s) \exp\{-nG_n[\mathbf{h}, (\beta J)^{1/2}(s/n^\gamma + a)]\} \tag{2.9}$$

where $t(s)$ is some arbitrary bounded function. Because the Laplace asymptotic method is shown to be valid, roughly speaking, one must control the expansion of $G_n(\mathbf{h}, x)$ in the neighborhood of each of its global minima together with the limit $n \to \infty$.

Let $\{x_i^*\}_{i=1}^{\alpha}$ be the set of the global minima of $G(x)$. It is easy to verify that the physical magnetization m_i will be given by

$$m_i = \frac{x_i^*}{(\beta J)^{1/2}} \tag{2.10}$$

Now the study of fluctuations is very simple. Just taking $a = x^*/(\beta J)^{1/2}$ in Lemma 2.3, we should be able to see for what values of γ this leads to a probability density for the fluctuation variable.

Doing so, some subtle technical difficulties will arise in the control of (2.9) related to the fact that the value of s around which the function $G_n(\mathbf{h}, x)$ must be expanded has a nontrivial dependence on n.

To avoid these difficulties, we consider a sequence $\{x_i^{(n)}\}_{i=1}^{\alpha}$ of minima of $G_n(\mathbf{h}, x)$, such that each $x_i^{(n)} \to x_i^*$ as $n \to \infty$, for every $i = 1, 2,..., \alpha$. Therefore, (2.8) may well give us the asymptotic probability measure of $A_n(x_i^{(n)}/(\beta J)^{1/2}, \gamma)$ by expanding $G_n[\mathbf{h}, (\beta J)^{1/2}(s/n^\gamma + x_i^{(n)}/(\beta J)^{1/2})]$ around $s = 0$.

This is indeed the case, and a Gaussian will always arise as stated in the following result.

Lemma 2.5. For each x_i^*, $i = 1,..., \alpha$, suppose there is a sequence $\{x_i^{(n)}\}_{i=1}^{\alpha}$ of minima of $G_n(\mathbf{h}, x)$ such that each $x_i^{(n)} \to x_i^*$ as $n \to \infty$. Also, let k_i be the type of x_i^*.

Then, with $\gamma_i = k_i/[2(2k_i - 1)]$, the following limiting behavior holds:

(a) $\alpha = 1$,

$$\lim_{n \to \infty} A_n\left(\frac{x^{(n)}}{(\beta J)^{1/2}}, \gamma\right) \sim \exp\left\{-\beta J\left[\frac{(2k-2)!}{G^{(2k)}(x^*)[v_k(\mathbf{h})]^{2k-2}} - \delta_{k,1}\right]^{-1} \frac{s^2}{2}\right\} ds$$

(b) $\alpha > 1$. There exists an $A \equiv A(x_i^*)$ such that for every a $(0 < a < A)$

$$\lim_{n \to \infty} A_n\left(\frac{x_i^{(n)}}{(\beta J)^{1/2}}, \gamma_i\right) : \left|\frac{S_n}{n} - \frac{x_i^{(n)}}{(\beta J)^{1/2}}\right| < a$$

$$\sim \exp\left\{-\beta J\left[\frac{(2k_i-2)!}{G^{(2k_i)}(x_i^*)[v_{k_i}(\mathbf{h})]^{2k_i-2}} - \delta_{k_i,1}\right]^{-1} \frac{s^2}{2}\right\} ds$$

Here $v_{k_i}(\mathbf{h})$ is a random variable defined on Ω, distributed as

$$v_{k_i}(\mathbf{h}) \sim s^{2k_i-2} \exp\left\{-\left[\frac{G^{(2k_i)}(x_i^*)}{(2k_i-1)!\,\sigma(1)}\right]^2 \frac{s^{2(2k_i-1)}}{2}\right\} ds \tag{2.11}$$

with

$$\sigma^2(1)=\beta J\left(\int d\nu(h)\; tgh^2[(\beta J)^{1/2}x_i^* + \beta h]\right.$$
$$\left. -\left\{\int d\nu(h)\;\mathrm{tgh}[(\beta J)^{1/2}x_i^* + \beta h]\right\}^2\right) \tag{2.12}$$

Remark 2.6. The $\delta_{k_i,1}$ in the variance of the Gaussian distribution is due to the random variable W in (2.8).

Of course, the asymptotic probability measure of $A_n(x^{(n)}/(\beta J)^{1/2}, \gamma)$ is not our aim. We would like to know that of $A_n(x^*/(\beta J)^{1/2}, \gamma)$. The way to get it is now clear. We must control the asymptotic convergence of $x^{(n)}$ to x^* as a random variable.

Indeed, $x^{(n)}$ satisfies the equation

$$G_n'(\mathbf{h}, x^{(n)})=0$$

From (2.3) this means that $x^{(n)}$ may be written as the sum of clearly *dependent* random variables

$$x^{(n)}=\frac{(\beta J)^{1/2}}{n}\sum_{i=1}^{n}\mathrm{tgh}[(\beta J)^{1/2}x^{(n)}+\beta h_i]$$

Now, for sequences like those in Lemma 2.5, it is possible to prove that $x_i^{(n)}$ converges to x_i^* in the following way:

Lemma 2.7. Let $\delta_i=[2(2k_i-1)]^{-1}$. Then, in distribution

$$n^{\delta_i}(x_i^{(n)}-x_i^*)=v_{k_i}(\mathbf{h})+o(1) \qquad \text{as} \quad n\to\infty \tag{2.13}$$

with $v_{k_i}(\mathbf{h})$ distributed as in (2.11) and (2.12).

Lemma 2.7 describes the sample driven fluctuations of the order parameter: they are stronger than (if $k>1$) or of the same order as (if $k=1$) the purely thermal fluctuations computed in Lemma 2.5. In fact, with m_i given by (2.10) and $\gamma_i=1/2(2k_i-1)$, i.e., $\gamma_i=\delta_i$ (as given by Lemma 2.7!), we compute the limiting distribution of $A_n(m_i, \gamma_i)$:

Theorem 2.8. Let α be the number of global minima of $G(x)$. Then:

(a) If $\alpha=1$,

$$\lim_{n\to\infty} A_n(m_i,\gamma_i)=\begin{cases} v_k(\mathbf{h}) & \text{if} \quad k>1 \\ u\sim N\left(\dfrac{u_1(\mathbf{h})}{G^{(2)}(x^*)}, \dfrac{1}{\beta J}\left(\dfrac{1}{G^{(2)}(x^*)}-1\right)\right) & \text{if} \quad k=1 \end{cases}$$

where $u_1\sim N(0,\sigma^2(1))$, with $\sigma^2(1)$ and $v_k(\mathbf{h})$ defined in (2.11) and (2.12).

(b) If $\alpha > 1$, there is an $A = A(x_i^*) > 0$ such that for every $a \in [0, A]$,

$$\lim_{n \to \infty} A_n \left(\frac{x_i^*}{(\beta J)^{1/2}}, \gamma_i \right) : \left| \frac{S_n}{n} - \frac{x_i^*}{(\beta J)^{1/2}} \right| < a$$

$$\sim N \left(\frac{u_1(\mathbf{h})}{H^{(2)}(x_i^*)}, \frac{1}{\beta J} \left(\frac{1}{G^{(2)}(x_i^*)} - 1 \right) \right)$$

with $k_i = 1$, $i = 1, 2,..., \alpha$.

One should notice that the two parts of the theorem refer, respectively, to situations where the minimum is nondegenerate (pure phase) and degenerate (coexisting phases). The result in (b) reproduces the result in (a) just by conditioning the finite magnetization $m_n = S_n/n$ to a neighborhood of one of the α equilibrium values.

In both cases the fluctuations depend on $\mathbf{h}$, i.e., on the configuration of the fields. Fluctuations are therefore said to be non-self-averaging.

It is also interesting to compare our scaling $\gamma = 1/2(2k-1)$ with that of Ellis and Newman, $\gamma = 1/2k$. They only coincide at $k = 1$, or, in other words, in situations where the spin variables are weakly dependent,[7] giving rise to $\gamma = 1/2$ and Gaussian probability distributions in the spirit of the central limit theorem. For strong dependence ($k > 1$), our value of γ is less than theirs, meaning that strong dependence is in this case stronger than in deterministic models.

3. PROOFS

The strategy of the proofs is simple, though the proofs themselves are somehow long. We first establish the Laplace asymptotic method for the partition function (Theorem 2.1). In order to do so, we state in Lemma 3.1 some conditions to have sufficient control on a general random function of the kind of $G_n(\mathbf{h}, x)$.

Lemma 3.2 will assure the existence of at least one subsequence of minima of these functions converging to each of the minima of the limiting function, just in the spirit of the hypothesis of Lemma 2.5. On the other hand, if one defines a partition function $Z_n(\mathbf{h})$ like (2.2) and a free energy f like (2.1) for these general random functions of the kind of $G_n(\mathbf{h}, x)$, Lemma 3.3 shows that in the integral defining $Z_n(\mathbf{h})$ only the neighborhood of those minima described in Lemma 3.2 will contribute in the limiting process to obtain the free energy. After that, we finally can prove the Laplace formula for these general functions in Lemma 3.4.

The proof of Theorem 2.1 will then only require to show that $G_n(\mathbf{h}, x)$ satisfies the conditions in Lemma 3.1. We shall then need the result of

Lemma 2.7 describing how the minima of $G_n(\mathbf{h}, x)$ converge to the minima of $G(x)$, which therefore must be proved before Theorem 2.1. And this will do for the thermodynamics.

With respect to the fluctuations, we will first prove Lemma 2.3, which relates the function $G_n(\mathbf{h}, x)$ with the probability distribution of the fluctuations. This result, together with the Laplace asymptotic method derived before, leads to the proof of Lemma 2.5, which describes the asymptotic probability distribution of fluctuations around the mean magnetization of a finite-size system. Finally, our main results as stated in Theorem 2.8 come as an immediate consequence of the combination of Lemmas 2.5 and 2.7, and our proofs will be finished.

Lemma 3.1. Suppose we are given $\mathbf{h} \in \Omega$, $s \in \mathbf{R}$, and $\{\Gamma_n(\mathbf{h}, s)\}$, a family of random variables in Ω with Γ_n infinitely differentiable in relation to s. Furthermore, let us assume that:

(1) There is an $A \subset \Omega$, with A independent of s and $\mu(A) = 1$, such that

$$\Gamma_n^{(j)}(\mathbf{h}, s) \equiv \frac{\partial^j \Gamma_n(\mathbf{h}, s)}{\partial s^j} \to \Gamma^{(j)}(s) \qquad \forall \mathbf{h} \in A$$

uniformly on compacta of $\mathbf{R}$, for $j = 0, 1, 2,\ldots$, with $\Gamma^{(0)} = \Gamma$.

(2) There exists $C(\mathbf{h}) > 0$, independent of n, and real τ such that

$$\exp[-\Gamma_n(\mathbf{h}, s)] \leqslant C(\mathbf{h}) \exp(-s^2/2 + \tau |s|) \tag{3.1}$$

(3) We have

$$\int ds\, e^{\quad \Gamma(\mathbf{h}, s)} < \infty \qquad \text{for} \quad \forall \mathbf{h} \in A \tag{3.2}$$

Then there exists an $\varepsilon > 0$ such that

$$e^{ng} \int_{\mathbf{V}} ds\, e^{-n\Gamma_n(\mathbf{h}, s)} = O(e^{-n\varepsilon}) \qquad \text{as} \quad n \to \infty \tag{3.3}$$

where $g = \inf\{\Gamma(s) : s \in \mathbf{R}\}$ with $\mathbf{V}$ being any closed, possibly unbounded subset of $\mathbf{R}$ containing no global minima on Γ.

Suppose furthermore that there is an integer l, real, positive ρ, and q_i, $i = 1, 2,\ldots$, a sequence $\lambda_1(\mathbf{h}), \lambda_2(\mathbf{h}),\ldots, \lambda_{2l}(\mathbf{h})$ of random variables in Ω with $\lambda_{2l}(\mathbf{h}) > 0$, and a sequence $\{s_n\}_{n=1}^{\infty}$ satisfying, for each $j = 1, 2,\ldots$

$$\Gamma_n^{(j)}(\mathbf{h}, s_n) = \frac{\lambda_j(\mathbf{h})}{n^{q_j}} + o(n^{-q_j}) \qquad \text{as} \quad n \to \infty \tag{3.4}$$

such that

$$q_j = 1 - j\rho \qquad \text{for} \quad j \leqslant 2l \tag{3.5a}$$

$$q_j > 1 - j\rho \qquad \text{for} \quad j > 2l \tag{3.5b}$$

Defining

$$B_n(\mathbf{h}, s, s_n) = \Gamma_n(\mathbf{h}, s + s_n) - \Gamma_n(\mathbf{h}, s_n)$$

then there exists $\delta > 0$ sufficiently small, such that, as $n \to \infty$,

$$nB_n\left(\mathbf{h}, \frac{s}{n^\rho}, s_n\right) = \sum_{j=1}^{2l} \lambda_j(\mathbf{h}) \frac{s^j}{j!} + O\left(\frac{|s|^{2l+1}}{n^{\rho(2l+1)}}\right) + o(1)\, P_1(s) \qquad \text{for} \quad |s| < \delta n^\rho \tag{3.6}$$

and

$$nB_n\left(\mathbf{h}, \frac{s}{n^\rho}, s_n\right) \geqslant \frac{1}{2} \lambda_{2l}(\mathbf{h}) \frac{s^{2l}}{(2l)!} + P_2(s) \qquad \text{for} \quad |s| < \delta n^\rho \tag{3.7}$$

where P_1 and P_2 are polynomials of degree $2l$ and $2l - 1$, respectively.

Proof. From the uniformity of the convergence of Γ_n, for n sufficiently large, there exists $\varepsilon > 0$ such that

$$\inf\{\Gamma_n(\mathbf{h}, s) : s \in \mathbf{V}\} \geqslant \inf\{\Gamma(s) : s \in \mathbf{R}\} + \varepsilon = g + \varepsilon$$

Therefore,

$$e^{ng} \int_{\mathbf{V}} ds\, e^{-n\Gamma_n(\mathbf{h}, s)} < e^{ng} e^{-(n-1)(g+\varepsilon)} \int_{\mathbf{V}} ds\, e^{-\Gamma_n(\mathbf{h}, s)} = O(e^{-n\varepsilon}) \qquad \text{as} \quad n \to \infty$$

just by using (3.1), (3.2), and the dominated convergence theorem. And this suffices to prove (3.3).

Once again, because of the uniformity of the convergence of $\Gamma_n^{(j)}(\mathbf{h}, s_n)$, there exists $\delta_1 > 0$ such that for sufficiently large n

$$\left| B_n(\mathbf{h}, s, s_n) - \sum_{j=1}^{2l} \Gamma_n^{(j)}(\mathbf{h}, s_n) \frac{s^j}{j!} \right| = O(|s|^{2l+1}) \qquad \text{for} \quad |s| < \delta_1 \tag{3.8}$$

Now (3.6) follows from (3.4) and (3.8).

Moreover, from the identity

$$B_n(\mathbf{h}, s, s_n) = \Gamma_n^{(2l)}(\mathbf{h}, s_n) \frac{s^{2l}}{(2l)!} + \left[B_n(\mathbf{h}, s, s_n) - \sum_{j=1}^{2l} \Gamma_n^{(j)}(\mathbf{h}, s_n) \frac{s^j}{j!} \right]$$
$$+ \sum_{j=1}^{2l-1} \Gamma_n^{(j)}(\mathbf{h}, s_n) \frac{s^j}{j!}$$

one immediately obtains

$$B_n(\mathbf{h}, s, s_n) \geqslant \Gamma_n^{(2l)}(\mathbf{h}, s_n)\frac{s^{2l}}{(2l)!} + \left| B_n(\mathbf{h}, s, s_n) - \sum_{j=1}^{2l} \Gamma_n^{(j)}(\mathbf{h}, s_n)\frac{s^j}{j!}\right| + \sum_{j=1}^{2l-1} \Gamma_n^{(j)}(\mathbf{h}, s_n)\frac{s^j}{j!} \tag{3.9}$$

while for sufficiently small $\delta_2 > 0$

$$O(|s|^{2l+1}) \leqslant \frac{1}{2}\lambda_{2l}(\mathbf{h})\frac{s^{2l}}{(2l)!} \qquad \text{for} \quad |s| < \delta_2$$

Now inequality (3.7) follows from (3.4), (3.8), and (3.9) with $\delta = \min(\delta_1, \delta_2)$. ∎

The next lemma assures that it is possible to approach each global minimum of $\Gamma(s)$ by an infinite subsequence of minima of $\Gamma_n(\mathbf{h}, s)$ with increasing values of n. On the other hand, Lemma 3.3 states that the result (3.3) will not change if one considers a sequence of sets $\mathbf{V}_n$ containing none of these minima of $\Gamma_n(\mathbf{h}, s)$, instead of the set $\mathbf{V}$. Therefore, the only relevant contribution to an integral over $\mathbf{R}$ as (3.3) will come from the sequence $\mathbf{R} - \mathbf{V}_n$, or, in other words, from integration in the neighborhood of the minima of $\Gamma_n(\mathbf{h}, s)$ converging to the global minima of $\Gamma(s)$. This is the key ingredient to prove in Lemma 3.4 the Laplace method for these integrals.

Let $\{s_i^*\}_{i=1}^{\alpha}$ be the set of the α global minima of $\Gamma(s)$ and $\{s^{(n,k)}\}_{k=1}^{\Delta}$ the set of the Δ global minima of $\Gamma_n(\mathbf{h}, s)$. Then we get the following result.

Lemma 3.2. Given hypothesis (1) in Lemma 3.1 and $\xi > 0$, for each $i = 1,..., \alpha$, theree is an infinite subsequence $\{s_i^{(n_j)}\}_{j=1}^{\infty}$ (with $n_j < n_{j+1}$) such that $s_i^{(n_j)} \in \{s^{(n_j, k)}\}$ and $s^{(n_j)} \in B_\xi(s_i^*) \equiv [s_i^* - \xi, s_i^* + \xi]$, converging therefore to s_i^*.

Proof. If there is no such subsequence, then one can define $\bar{n}$ as the largest n such that there is an $s^{(n)}$ in $B_\xi(s_i^*)$. Then, for every $n > \bar{n}$,

$$\inf\{|\Gamma_n^{(1)}(\mathbf{h}, s)| : s \in B_\xi(s_i^*)\} \equiv g_{n,\xi}(s_i^*) > 0 \tag{3.10}$$

But since $\Gamma_n^{(1)}(\mathbf{h}, s_i^*)$ converges uniformly to $\Gamma^{(1)}(s_i^*) = 0$, for arbitrary $\varepsilon > 0$ there is n_0 such that for every $n > n_0$

$$|\Gamma_n^{(1)}(\mathbf{h}, s_i^*)| < \varepsilon \tag{3.11}$$

Since ε is arbitrary, it may be chosen such that $\varepsilon < g_{n,\xi}(s_i^*)$ and $n_0(\varepsilon) > \bar{n}$. Then (3.11) contradicts (3.10), since $s_i^* \in B_\xi(s_i^*)$ and so the subsequence in the lemma must exist. Since this subsequence is contained in the compact set $B_\xi(s_i^*)$, its convergence follows. ▮

Given a subsequence like $\{s_i^{(n_j)}\}_{j=1}^{\infty}$, we may build a sequence $\{s_{n,i}\}$ such that $s_{n,i} = s^{(n,k)}$, k being chosen such as to minimize $|s_i^{(n_j)} - s^{(n,k)}|$ for fixed i and n, with $n_j \leqslant n < n_{j+1}$. Of course, by definition, $s_{n_j,i} = s_i^{(n_j)}$.

Given θ satisfying

$$0 < \theta < \Theta \equiv \min\{\delta, |s_i^* - s_j^*| \, i \neq j\} \tag{3.12}$$

where δ is the same as in (3.6) and (3.7), we define the n-dependent set

$$\mathbf{V}_n = \mathbf{R} - \{s_{n,i} - \theta, s_{n,i} + \theta\} \tag{3.13}$$

and state the following result.

Lemma 3.3. With the hypotheses (1)–(3) on $\Gamma_n(\mathbf{h}, s)$ in Lemma 3.1, there exists an $\varepsilon > 0$ such that

$$e^{ng} \int_{\mathbf{V}_n} ds\, e^{-n\Gamma_n(\mathbf{h},s)} = O(e^{-n\varepsilon}) \qquad \text{as} \quad n \to \infty \tag{3.14}$$

Proof. Since $s_{n,i}$ converges to s_i^*, given $\omega > 0$, there exists n_0 such that $|s_i^* - s_{n,i}| < \omega$ for every $n > n_0$. Therefore, taking ω so small that $\theta' \equiv \theta - \omega > 0$ and $\theta'' \equiv \theta + \omega < \Theta$, where Θ is defined in (3.12), we set

$$\mathbf{V}' = \mathbf{R} - \{s_i^* - \theta', s_i^* + \theta'\}$$

and

$$\mathbf{V}'' = \mathbf{R} - \{s_i^* - \theta'', s_i^* + \theta''\}$$

By construction, $\mathbf{V}'' \subset \mathbf{V}_n \subset \mathbf{V}'$. The use of (3.3) for $\mathbf{V}'$ and $\mathbf{V}''$ proves the lemma. ▮

Now we are ready to state the main ingredient in the proof of Theorem 2.1.

Lemma 3.4. Under the hypothesis of Lemma 3.1 and supposing also that $\Gamma(s)$ has a finite number of minima (say α) and that the sequences $\{s_{n,i}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, \alpha$, satisfy (3.4), we then have

$$\lim_{n\to\infty} -\frac{1}{n} \ln \int_{\mathbf{R}} ds\, e^{-n\Gamma_n(\mathbf{h},s)} = g \qquad \mathrm{ae}[\mu] \tag{3.15}$$

Proof. Using θ and $\mathbf{V}_n$ as in (3.13),

$$\begin{aligned} e^{ng} \int_{\mathbf{R}} ds\, e^{-n\Gamma_n(\mathbf{h},s)} &= e^{ng} \left[\sum_{i=1}^{\alpha} \int_{s_{n,i}-\theta}^{s_{n,i}+\theta} ds\, e^{-n\Gamma_n(\mathbf{h},s)} + \int_{\mathbf{V}_n} ds\, e^{-n\Gamma_n(\mathbf{h},s)} \right] \\ &= e^{q(n)} \left\{ \sum_{i=1}^{\alpha} n^{\rho_i} \int_{-\theta n^{\rho_i}}^{\theta n^{\rho_i}} ds \exp\left(- \sum_{=1}^{2l_i} \lambda_j^{(i)}(\mathbf{h}) \frac{s^j}{j!} \right) + O(n^{-\rho_i}) \right\} \\ &\quad + O(e^{-n\varepsilon}) \qquad \text{as} \quad n \to \infty \end{aligned}$$

where $q(n) = n[g - \Gamma_n(\mathbf{h}, s_n)] = o(n)$, and ρ_i, l_i, and $\lambda_j^{(i)}$ are numbers which, according to (3.4), we can associate with each sequence $\{s_{n,i}\}_{n=1}^{\infty}$, $i = 1, 2,..., \alpha$. In obtaining the last equality we have used (3.3), (3.6), and the dominated convergence theorem, valid due to (3.7) and Lemma 3.3.

Multiplying the expression by e^{-ng}, taking its logarithm, and reversing its sign, one obtains (3.15). Since the only dependences in $\mathbf{h}$ are in the $\lambda_j^{(i)}(\mathbf{h})$, which do not contribute in the limiting process, the result is obtained ae$[\mu]$. ∎

Proving that the hypotheses in Lemma 3.1 are satisfied by $G_n(\mathbf{h}, x)$, one obtains (2.5) directly from (3.15) because of the definition of the free energy in (2.1). But first, as we explained in the beginning of this section, we must prove Lemma 2.7. To do so, we shall apply the result of Lemma 3.2 to $G_n(\mathbf{h}, x)$. Therefore, we must first prove that the hypothesis of this last lemma holds for such a function. This is stated in the following result.

Lemma 3.5. The hypothesis (1) in Lemma 3.1 holds for $G_n(\mathbf{h}, x)$.

Proof. We first verify the convergence ae$[\mu]$ for $j = 0$ and its uniformity on compacta of $\mathbf{R}$. Then we will generalize this result for $j > 0$. To begin with, let us consider the simple probability measure

$$dv(h) = \tfrac{1}{2}[\delta(h - H) + \delta(h + H)]$$

In this case $\varphi_n(\mathbf{h}, x)$ may be rewritten as

$$\begin{aligned} \varphi_{nb}(\mathbf{h}, x) &= \ln \cosh[(\beta J)^{1/2} x + \beta H] \frac{1}{n} \sum_{i=1}^{n} \left(\frac{H + h_i}{2H} \right) \\ &\quad + \ln \cosh[(\beta J)^{1/2} x - \beta H] \frac{1}{n} \sum_{i=1}^{n} \left(\frac{H - h_i}{2H} \right) \end{aligned}$$

The law of large numbers may now be applied to each of the sums, independently of x, and therefore $\mu(\bigcup_x B_x) = 0$ trivially. This argument may be used whenever h_i are discrete random variables assuming a finite number of values. In the case of continuous random variables, however, it does not work, and the way out is the following:

Since

$$|\varphi_n(\mathbf{h}, x) - \varphi_n(\mathbf{h}, y)| \leqslant |x - y| \sup_{z \in \mathbf{R}} |\varphi_n(\mathbf{h}, z)| \leqslant (\beta J)^{1/2} |x - y|, \quad \forall n \tag{3.16}$$

the $\{\varphi_n(\mathbf{h}, x)\}$ form an equicontinuous sequence and therefore converge uniformly in compacta of $\mathbf{R}$. Since the countable intersection of sets of measure 1 (in a probability space) has also measure 1, it is possible to choose $A \subset \Omega$ with $\mu(A) = 1$ such that if $\mathbf{h} \in A$, $\varphi_n(\mathbf{h}, x) \to \varphi(x)$ for every $x \in \mathbf{Q}$.

Since $\mathbf{Q}$ is dense in $\mathbf{R}$, it is now easy to prove that for every $\mathbf{h} \in A$, $\varphi_n(\mathbf{h}, x) \to \varphi(x)$ for every $x \in \mathbf{R}$. From (3.16), given $\varepsilon > 0$, there exists $\delta > 0$ $[\delta < \varepsilon/3(\beta J)^{1/2}]$, so that $|x - y| \leqslant \delta$ implies $|\varphi_n(\mathbf{h}, x) - \varphi_n(\mathbf{h}, y)| \leqslant \varepsilon/3$ for every n. On the other hand, since $\mathbf{Q}$ is dense in $\mathbf{R}$, there exists a $y \in \mathbf{Q}$ such that $|x - y| \leqslant \delta$ with $x \in \mathbf{R}$; and also there exists an integer n_0 such that for every m and n larger than n_0, $|\varphi_m(\mathbf{h}, y) - \varphi_n(\mathbf{h}, y)| \leqslant \varepsilon/3$. Then

$$\begin{aligned} &|\varphi_m(\mathbf{h}, x) - \varphi_n(\mathbf{h}, x)| \\ &\quad \leqslant |\varphi_m(\mathbf{h}, x) - \varphi_m(\mathbf{h}, y)| + |\varphi_m(\mathbf{h}, y) - \varphi_n(\mathbf{h}, y)| + |\varphi_n(\mathbf{h}, y) - \varphi_n(\mathbf{h}, x)| \leqslant \varepsilon \end{aligned}$$

and therefore the sequence converges for every $x \in \mathbf{R}$ and $\mathbf{h} \in A$.

To generalize this argument to any jth-order derivative of φ_n, one must simply show that $\varphi_n^{(j)}(\mathbf{h}, x)$ is bounded as $\varphi_n(\mathbf{h}, x)$ in (3.16), and therefore $\{\varphi_n^{(j-1)}(\mathbf{h}, x)\}$ is an equicontinuous family of functions. Then the argument is straightforward.

To check this is quite easy, since from simple properties of the function $\cosh x$, for each $j \in \{1, 2,...\}$, there are real positive numbers $A(j)$, $a_k(j)$, and $b_k(j)$ with $k \in \{1, 2,..., 2^{j-1}\}$ such that

$$\varphi_n^{(j)}(\mathbf{h}, x) = \frac{1}{n} \sum_{i=1}^{n} \frac{(\beta J)^{j/2} p_j(x, h_i)}{\{\cosh(\beta J)^{1/2} x + \beta h_i]\}^{2^{j-1}}} \qquad \text{for every } n \text{ and } j \in \{1, 2,...\}$$

where

$$\begin{aligned} p_j(x, h_i) = A(j) + \sum_{k=1}^{2^{j-1}} \{&a_k(j) \sinh^k[(\beta J) x + h_i] + b_k(j) \cosh[(\beta J)^{1/2} x + h_i] \\ &\times \sinh^{k-1}[(\beta J)^{1/2} x + h_i]\} \end{aligned}$$

Since

$$0 < \frac{p_j(x, h_i)}{\{\cosh[(\beta J)^{1/2}x + \beta h_i]\}^{2^{j-1}}} \leqslant A(j) + \sum_{k=1}^{2^{j-1}} [a_k(j) + b_k(j)]$$

$\varphi_n^{(j)}(\mathbf{h}, x)$ is bounded and the hypothesis is verified. ∎

Proof of Lemma 2.7. Defining

$$\varphi(x) = \int dv(h) \ln \cosh[(\beta J)^{1/2}x + \beta h_i] \tag{3.17}$$

x_i^* must be a solution of

$$x_i^* = \varphi^{(1)}(x_i^*) \tag{3.18}$$

Because of Lemma 3.5, we can use the results of Lemma 3.2. Accordingly, there exists a sequence $\{n_j\}$, $j = 1,..., \infty$, such that for every element n in this sequence and every $i = 1, 2,..., \alpha$ there is an $x_i^{(n)}$ converging to x_i^* and satisfying

$$x_i^{(n)} = \varphi_n^{(1)}(\mathbf{h}, x_i^{(n)}) \tag{3.19}$$

with $\varphi_n(\mathbf{h}, x) = x^2/2 - G_n(\mathbf{h}, x)$.

Therefore, given $\varepsilon > 0$, there exists an n_0 such that $|x_i^{(n)} - x_i^*| < \varepsilon$ for every $n > n_0$. We can therefore expand $\varphi_n^{(1)}(\mathbf{h}, x)$ around x_i^* and then make $x = x_i^{(n)}$, for each $n > n_0$, which gives

$$\varphi_n^{(1)}(\mathbf{h}, x_i^{(n)}) = \sum_{j=0}^{2k_i-1} \varphi_n^{(1+j)}(\mathbf{h}, x_i^*) \frac{(x_i^{(n)} - x_i^*)^j}{j!} + O(|x_i^{(n)} - x_i^*|^{2k_i})$$

$$\text{for} \quad |x_i^{(n)} - x_i^*| < \varepsilon \tag{3.20}$$

Since the central limit theorem ensures that

$$\varphi_n^{(j)}(\mathbf{h}, x_i^*) = \varphi^{(j)}(x_i^*) + \frac{u_j(\mathbf{h})}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right) \qquad \text{as} \quad n \to \infty \tag{3.21}$$

with $u_j(\mathbf{h}) \sim N(0, \sigma^2(j))$, where $\sigma^2(j) > 0$ is a well-defined variance, the use of (3.20) and (3.21) give us a simple equation in $(x_i^{(n)} - x_i^*)$ that reads

$$G^{(2k_i)}(x_i^*) \frac{(x_i^{(n)} - x_i^*)^{2k_i-1}}{(2k_i - 1)!} = \frac{u_1(\mathbf{h})}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right) \qquad \text{as} \quad n \to \infty \tag{3.22}$$

Since from (3.21) $u_1(\mathbf{h})$ is nothing but

$$u_1(\mathbf{h}) = \lim_{n \to \infty} (\beta n J)^{1/2} \left\{ \frac{1}{n} \sum_{j=1}^{n} \operatorname{tgh}[(\beta J)^{1/2} x_i^* + \beta h_j] - \int d\nu(h) \operatorname{tgh}[(\beta J)^{1/2} x_i^* + \beta h] \right\}$$

$\sigma(1)$ will be given by (2.12). Defining now the random variable $v_{k_i}(\mathbf{h})$ by

$$[v_{k_i}(\mathbf{h})]^{2k_i - 1} = \frac{(2k_i - 1)!\, u_1(\mathbf{h})}{G^{(2k_i)}(x_i^*)}$$

one gets immediately (2.13) from (3.22), proving the lemma. ∎

We now turn to the proof that $G_n(\mathbf{h}, x)$ satisfies the hypotheses in Lemma 3.1, proving therefore Theorem 2.1.

Proof of Theorem 2.1. The proof will be split in two parts. Since hypothesis (1) in Lemma 3.1 was already verified in Lemma 3.5, we must check only the following hypotheses for $\Gamma_n(\mathbf{h}, s) = G_n(\mathbf{h}, s)$:

(h1) Equations (3.1) and (3.2) hold.

(h2) Also (3.4) and (3.5) are satisfied.

Once these two points are proved, Lemma 3.2 immediately assures Theorem 2.1.

Proof of (h1). This proof is rather simple. Noticing that $\ln \cosh[f(x)] \leqslant |f(x)|$, one has

$$\frac{1}{n} \sum_{i=1}^{n} \ln \cosh[(\beta J)^{1/2} x + \beta h_i] \leqslant \frac{1}{n} \sum_{i=1}^{n} |(\beta J)^{1/2} x + \beta h_i| \leqslant (\beta J)^{1/2} |x| + \frac{\beta}{n} \sum_{i=1}^{n} |h_i| \tag{3.23}$$

It is then straightforward to see that

$$G_n(\mathbf{h}, x) \geqslant \frac{x^2}{2} - (\beta J)^{1/2} |x| - \beta \sup\{|h_i|\} \tag{3.24}$$

which agrees with (3.1) if $\mathbf{C}(\mathbf{h}) = \exp[\beta \sup\{|h_i|\}]$ and $\tau = (\beta J)^{1/2}$.

To check the validity of (3.2), we write

$$\int ds \exp[-G(s)] = \lim_{n\to\infty} \int ds \exp[-G_n(h, s)]$$
$$\leqslant \left[\beta \int |h|\, d\nu(h)\right] \int dx \exp\left[-\frac{x^2}{2} + (\beta J)^{1/2}|x|\right]$$

which is finite because of (2.4). In the last expression the equality comes from the dominated convergence theorem valid because of (3.24), and the inequality comes from (3.23) and the law of large numbers. And so (h1) is proved.

Proof of (h2). Because of Lemma 3.5, we can use Lemma 3.2 to pick up $x_i^{(n)}$ obeying (3.19), for each fixed i. Considering the s_n in (3.4) equal to $x_i^{(n)}$, we shall prove that (3.4) and (3.5) hold with $l_i = 1$,

$$\rho \equiv \rho(k_i) = \frac{k_i}{2(2k_i - 1)} \tag{3.25}$$

$$\lambda_1^i(\mathbf{h}) = 0 \qquad \text{and} \qquad \lambda_2^i(\mathbf{h}) = \frac{G^{(2k_i)}(x_i^*)[v_{k_i}(\mathbf{h})]^{2k_i - 2}}{2k_i - 2} \tag{3.26}$$

with $v_{k_i}(\mathbf{h})$ defined in (2.11) for each i.

This comes from the analysis of the convergence of $G_n^{(j)}(\mathbf{h}, x_i^{(n)})$, now easy to do with the help of Lemma 2.7. For $j = 1$, $G_n^{(1)}(\mathbf{h}, x_i^{(n)}) = 0$, by definition. For $j > 1$ we may proceed as in the proof of Lemma 2.7, expanding $G_n^{(j)}(\mathbf{h}, x)$ around x_i^* and then, for sufficiently large n, calculating the function at $x = x_i^{(n)}$. We get, as $n \to \infty$,

$$G_n^{(j)}(\mathbf{h}, x_i^{(n)}) = \sum_{p=0}^{\infty} G_n^{(j+p)}(\mathbf{h}, x_i^{(n)}) \frac{(x_i^{(n)} - x_i^*)^p}{p!}$$
$$\to \begin{cases} \dfrac{G^{(2k_i)}(x_i^*)[v_{k_i}(\mathbf{h})]^{2k_i - j}}{(2k_i - j)!\, n^{\delta_i(2k_i - j)}} + \dfrac{u_j}{\sqrt{n}} + o\left(\dfrac{1}{\sqrt{n}}\right) & \text{if } j < 2k_i \\ G^{(j)}(x_i^*) + \dfrac{u_j}{\sqrt{n}} + o\left(\dfrac{1}{\sqrt{n}}\right) & \text{if } j \geqslant 2k_i \end{cases} \tag{3.27}$$

at least for a certain infinite subsequence of values of n. Here we applied Lemma 3.2, the law of large numbers, and the central limit theorem, as in the proof of Lemma 2.7.

From (3.27) one sees immediately that (3.4) and (3.5) are satisfied if $l_i = 1$ for every i and the other parameters are chosen according to (3.25) and (3.26), which proves (h2) and therefore Theorem 2.1. ∎

Now that we know how to obtain the free energy of the system, we shall turn to the problem of describing the fluctuations. As was said in the beginning of this section, the first step in that direction is to prove Theorem 2.3.

Proof of Lemma 2.3. Let $K_n(t)$ be the characteristic function of the random variable $A_n(a, g)$ defined in (2.7). Then

$$\begin{aligned} K_n(t) &= \langle \exp[iA_n(a, \gamma)t] \rangle \\ &= \frac{1}{Z_n} \sum_{\sigma} \exp\left(i \frac{S_n - na}{n^{1-\gamma}} t \right) \exp[-\beta H_n(h, \sigma)] \\ &= \frac{1}{Z_n} \left(\frac{n}{2\pi} \right)^{1/2} \int dx \\ &\quad \times \exp\left(\frac{x^2}{2} - \frac{1}{n} \sum_{i=1}^{n} \ln\cosh\left[(\beta J)^{1/2} x + \beta h_i + \frac{it}{n^{1-\gamma}} \right] + \frac{iat}{n^{1-\gamma}} \right) \end{aligned}$$

where use has been made of the Gaussian transformation as in (2.2). Making the substitution

$$x = -\frac{it}{(\beta J)^{1/2} n^{1-\gamma}} + \left(\frac{s}{n^\gamma} + a \right) (\beta J)^{1/2}$$

both in the numerator and in the denominator of the expression above, one gets

$$\begin{aligned} K_n(t) \exp\left(-\frac{t^2}{2\beta J n^{1-2\gamma}} \right) \\ = \frac{\int ds \exp(its) \exp\{-nG_n[\mathbf{h}, (\beta J)^{1/2}(s/n^\gamma + a)]\}}{\int ds \exp\{-nG_n[\mathbf{h}, (\beta J)^{1/2}(s/n^\gamma + a)]\}} \end{aligned}$$

which may be read as (2.8), and that proves the lemma. ∎

As said in Section 2, the study of the fluctuations must be made in two steps. The first is the proof of Lemma 2.5, which comes out quite simply with the help of Lemma 2.3 and 3.1. Actually, in the case of a single minimum ($\alpha = 1$), what has to be proved is that for any bounded continuous function $t(s)$ the following limiting behavior holds:

$$\frac{\int ds\, t(s) \exp\{-nG_n[\mathbf{h}, (\beta J)^{1/2}(s/n^\gamma + x^{(n)}/(\beta J)^{1/2})]\}}{\int ds \exp\{-nG_n[\mathbf{h}, (\beta J)^{1/2}(s/n^\gamma + x^{(n)}/(\beta J)^{1/2})]\}} \to \int t(s)\, d\mu(s)$$

where $d\mu(s)$ is the suitably normalized Gaussian measure given in Lemma 2.5,

$$d\mu(s)=\frac{\exp\{-\beta J[(2k-2)!/G^{(2k)}(x^*)[v_k(\mathbf{h})]^{2k-2}-\delta_{k,1}]^{-1}s^2/2\}\,ds}{\int\exp\{-\beta J[(2k-2)!/G^{(2k)}(x^*)[v_k(\mathbf{h})]^{2k-2}-\delta_{k,1}]^{-1}s^2/2\}\,ds}$$

and with $\gamma=[2(2k-1)]^{-1}$.

According to Lemma 2.3, that is enough to prove that $d\mu(s)$ is the probability measure of $A_n(x^{(n)}/(\beta J)^{1/2},\gamma)$, if $\alpha=1$. If there are more minima, the situation becomes slightly more complicated, as we shall see.

Proof of Lemma 2.5. (a) $\alpha=1$. Taking θ such that $0<\theta=\delta$, with δ defined for G_n just as it was defined for Γ_n in (3.6) and (3.7), one has

$$\int_{\mathbf{R}} ds\, t(s)\exp\left\{-nG_n\left[\mathbf{h},\,(\beta J)^{1/2}\left(\frac{s}{n^\gamma}+\frac{x^{(n)}}{(\beta J)^{1/2}}\right)\right]\right\}$$
$$=\int_{|s|<\theta n^\gamma/(\beta J)^{1/2}} ds\, t(s)\exp\left\{-nG_n\left[\mathbf{h},\,(\beta J)^{1/2}\left(\frac{s}{n^\gamma}+\frac{x^{(n)}}{(\beta J)^{1/2}}\right]\right\}$$
$$+\int_{|s|>\theta n^\gamma/(\beta J)^{1/2}} ds\, t(s)\exp\left\{-nG_n\left[\mathbf{h},\,(\beta J)^{1/2}\left(\frac{s}{n^\gamma}+\frac{x^{(n)}}{(\beta J)^{1/2}}\right)\right]\right\}$$

By virtue of Lemma 3.3, there exists an $\varepsilon>0$ such that

$$e^{nf}\int_{|s|>\theta n^\gamma/(\beta J)^{1/2}} ds\, t(s)\exp\left\{-nG_n\left[\mathbf{h},\,(\beta J)^{1/2}\left(\frac{s}{n^\gamma}+\frac{x^{(n)}}{(\beta J)^{1/2}}\right]\right\}=O(n^\gamma e^{-n\varepsilon})\quad\text{as}\quad n\to\infty$$

Now,

$$e^{nf}\int_{|s|<\theta n^\gamma/(\beta J)^{1/2}} ds\, t(s)\exp\left\{-nG_n\left[\mathbf{h},\,(\beta J)^{1/2}\left(\frac{s}{n^\gamma}+\frac{x^{(n)}}{(\beta J)^{1/2}}\right)\right]\right\}$$
$$=e^{q(n)}\int_{|s|<\theta n^\gamma/(\beta J)^{1/2}} ds\, t(s)\exp\left[-nB_n\left(\mathbf{h},\,(\beta J)^{1/2}\frac{s}{n^\gamma},\,x^{(n)}\right)\right]$$
$$=e^{q(n)}\left\{\int ds\, t(s)\exp\left[-\lambda_2(\mathbf{h})\frac{s^2}{2}\right]+o(1)\right\}\qquad\text{as}\quad n\to\infty$$

Here $q(n)=n[f-G_n(\mathbf{h},x^{(n)})]=o(n)$, and in the last step we used (3.6), (3.7), and the dominated convergence theorem. The term $\lambda_2(\mathbf{h})$ is given in (3.26) and the extra $\delta_{k,1}$ appearing in the result of Lemma 2.5 is due to the trivial contribution of the random variable W in Lemma 2.3, which happens only if $k=1$ (see Remark 2.4). The last expression proves therefore the statement of the first part of the lemma.

(b) $\alpha > 1$. Let $\sum'_\sigma$ denote the sum over all configurations σ such that $|S_n/n - x^{(n)}/(\beta J)^{1/2}| < a$, for each fixed $j = 1, 2,..., \alpha$.

We then have to prove that there exists an $A > 0$ such that for each real r and any $a \in (0, A)$

$$\frac{\sum'_\sigma \exp\{i[(S_n - nx_j^{(n)})/n^{\gamma_j} t\} \exp[-\beta H_n(\sigma, h)]}{\sum'_\sigma \exp[-\beta H_n(\sigma, h)]} \to \int \exp(its)\, d\mu_j(s) \quad \text{as} \quad n \to \infty \tag{3.28}$$

with γ_j given as in Lemma 2.5 and $d\mu_j(s)$ being the probability measure

$$d\mu_j(s) = \exp\left\{-\beta J\left[\frac{(2k_j-2)!}{G^{(2k_j)}(x_j^*)[v_{k_j}(\mathbf{h})]^{2k_j-2}} - \delta_{k_j,1}\right]^{-1}\frac{s^2}{2}\right\} ds$$
$$\times\left(\int \exp\left\{-\beta J\left[\frac{(2k_j-2)!}{G^{(2k_j)}(x_j^*)[v_{k_j}(\mathbf{h})]^{2k_j-2}} - \delta_{k_j,1}\right]^{-1}\frac{s^2}{2}\right\} ds\right)^{-1}$$

Proceeding as in the proof of Lemma 2.3, the left-hand side of the above limit becomes

$$\sum_\sigma{}' \int dx \exp\left(-\frac{nx^2}{2}\right) \exp\left\{\sum_{l=1}^n \left[\frac{it}{n^{1-\gamma_l}} + \beta h_l + x(\beta J)^{1/2}\right]\sigma_l\right\}$$
$$\times \exp\left[-i\frac{n^{\gamma_j}x_j^{(n)}}{(\beta J)^{1/2}} t\right]$$
$$\times\left(\sum_\sigma{}' \int dx \exp\left(-\frac{nx^2}{2}\right) \exp\left\{\sum_{l=1}^n [\beta h_1 + x(\beta J)^{1/2}]\,\sigma_1\right\}\right)^{-1}$$

With the substitution

$$x = -\frac{it}{(\beta J)^{1/2} n^{1-\gamma_j}} + (\beta J)^{1/2}\frac{s}{n^{\gamma_j}} + x_j^{(n)}$$

this reads

$$\exp\left(\frac{t^2}{2\beta J n^{1-2\gamma_j}}\right) \int ds \exp\left[-n\frac{\beta J}{2}\left(\frac{s}{n^{\gamma_j}} + \frac{x_j^{(n)}}{(\beta J)^{1/2}}\right)\right]$$
$$\times\left(\sum_\sigma{}' \exp\left\{\sum_{l=1}^n \left[\frac{\beta J s}{n^{\gamma_j}} + \beta h_l + x_j^{(n)}(\beta J)^{1/2}\right]\sigma_l\right\}\right) \exp(its)$$
$$\times\left[\int ds \exp\left[-n\frac{\beta J}{2}\left(\frac{s}{n^{\gamma_j}} + \frac{x_j^{(n)}}{(\beta J)^{1/2}}\right)\right]\right.$$
$$\left.\times \sum_\sigma{}' \exp\left\{\sum_{l=1}^n \left[\frac{\beta J s}{n^{\gamma_j}} + \beta h_l + x_j^{(n)}(\beta J)^{1/2}\right]\sigma_1\right\}\right)\right]^{-1} \tag{3.29}$$

The next step to go ahead with the calculation is to eliminate the restriction on the sum over configurations. This is possible at the cost of a new restriction in the integration with respect to s. To this end, we shall use the transfer principle, for the proof of which we refer to the paper of Ellis *et al.*[3] It states the following:

Transfer Principle. There exists a number $\hat{B} > 0$ such that for every $B \in (0, \hat{B})$, $a \in (0, B/2)$, and each real t, there is a $\delta(a, B) > 0$ such that as $n \to \infty$,

$$\int ds \exp\left[-n\frac{\beta J}{2}\left(\frac{s}{n^{\gamma_j}} + \frac{x_j^{(n)}}{(\beta)^{1/2}}\right)\right]$$

$$\times\left(\sum_\sigma{}' \exp\left\{\sum_{l=1}^{n}\left[\frac{\beta Js}{n^{\gamma_j}} + \beta h_l + x_j^{(n)}(\beta J)^{1/2}\right]\sigma_l\right\}\right)\exp(its)$$

$$= \int_{|s| < Bn^{\gamma_j}} ds \exp(its) \exp\left[-n\frac{\beta J}{2}\left(\frac{s}{n^{\gamma_j}} + \frac{x_j^{(n)}}{(\beta J)^{1/2}}\right)\right]$$

$$\times\left(\sum_\sigma \exp\left\{\sum_{l=1}^{n}\left[\frac{\beta Js}{n^{\gamma_j}} + \beta h_l + x_j^{(n)}(\beta J)^{1/2}\right]\sigma_l\right\}\right) + O(\exp(-n\delta_n\gamma_j))$$

Now, without the restriction, the sum over configurations may immediately be rewritten as

$$\exp\left[\sum_{l=1}^{n} \ln\cosh\left(\frac{\beta Js}{n^{\gamma_j}} + \beta h_l + x_j^{(n)}(\beta J)^{1/2}\right)\right]$$

and the right-hand side of the above expression becomes simply

$$\int_{|s| < Bn^{\gamma_j}} ds \exp(its) \exp\left\{-nG_n\left[(\beta J)^{1/2}\left(\frac{s}{n^{\gamma_j}} + \frac{x_j^{(n)}}{(\beta J)^{1/2}}\right)\right]\right\}$$

$$+ O(\exp(-n\delta_n\gamma_j))$$

If one chooses $B < \Theta$ [with Θ defined as in (3.12)], the convergence of this integral is as straightforward as the one in the first part of the proof of this lemma. Using the result from the transfer principle both in the numerator and in the denominator of (3.29), and noticing that $\exp(t^2/2\beta Jn^{1-2\gamma_j})$ is nothing but the characteristic function of W in Lemma 2.3, one easily gets (3.28) and therefore the lemma is proved by taking $A = \Theta/2$. ∎

Now we can finally prove our main result in a very simple way.

Proof of Theorem 2.8. Lemma 2.5 ensures that

$$\frac{S_n}{n} = \frac{x_i^{(n)}}{(\beta J)^{1/2}} + \frac{\bar{u}}{n^{\gamma_i}} + o(n^{-\gamma_i}) \qquad \text{for each} \quad i = 1, 2, \ldots, \alpha \quad \text{as } n \to \infty \tag{3.30}$$

with

$$\bar{u} \sim N\left(0, \frac{1}{\beta J}\left[(\lambda_2^i(h))^{-1} - \delta_{k_i,1}\right]\right)$$

One the other hand, from Lemma 2.7 one has

$$\frac{x_i^{(n)}}{(\beta J)^{1/2}} = \frac{x_i^*}{(\beta J)^{1/2}} + \frac{v_{k_i}}{n^{[2(2k_i-1)]^{-1}}} + o(n^{-[2(2k_i-1)]^{-1}})$$
$$\text{for each } i = 1, 2, \ldots, \alpha \quad \text{as } n \to \infty \tag{3.31}$$

Using (3.31) in (3.30), and remembering that γ_i is given by (3.25), we get our fluctuation variable:

$$\frac{S_n}{n} - \frac{x_i^*}{(\beta J)^{1/2}} = \frac{\bar{u}}{n^{k_i/[2(2k_i-1)]}} + \frac{v_{k_i}}{n^{[2(2k_i-1)]^{-1}}} + o(n^{-[2(2k_i-1)]^{-1}}) \qquad \text{as} \quad n \to \infty$$

Since for $\alpha > 1$ (3.30) holds for S_n/n conditioned to a neighborhood of $x_i^{(n)}(\beta J)^{1/2}$, this establishes the theorem. ∎

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