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GENERATION OF NEW EXACTLY SOLVABLE POTENTIALS OF A NONSTATIONARY SCHRÖDINGER EQUATION

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A method for generating integrable potentials of a nonstationary Schrödinger equation (i.e., with time-dependent potential) is developed on the basis of the method of "dressing" of linear differential operators. Potentials that admit separation of the variables generate classes of nonseparating potentials for which the Schrödinger equation has nonlocal symmetry operators.

1. Introduction

The method of "dressing" of linear differential operators (known in mathematics literature as the method of transformation operators [1,2]) has traditionally been used to solve scattering problems, realizing a transition from an equation with constant coefficients to an equation with variable coefficients. In this respect, it was also found to be effective for the construction of nonlinear differential equations integrable by the inverse scattering method, giving moreover an explicit indication of a method for calculating exact solutions of such equations [3,4].

The integration of a linear differential equation is associated with a symmetry algebra formed by symmetry operators. By definition, these carry every solution of the equation to some other solution of it. In particular, commutative subalgebras of differential symmetry operators of a definite form lead to complete separation of the variables [5]. Extension of the classes of the employed symmetry operators permits in principle the finding of new approaches to the solution of equations. From this point of view, undoubted interest attaches to nonlocal symmetries (generators of one-parameter groups of symmetry operators).

Nonlocal symmetries of nonlinear differential equations have been considered in several studies (see, for example, [6,7]) as a way of extending the jet manifold and, accordingly, the domain of application of Lie–Bäcklund groups. However, significant progress in this direction has not been achieved [8].

Nonlocal symmetries of linear equations, in particular the Maxwell and Dirac equations, in the absence of external fields (i.e., equations with constant coefficients) were calculated by means of Fourier transformation in studies by Fushchich and collaborators. A bibliography of these studies can be found in the review [9]. However, equations with external fields have important applications. A review of studies on the problem of the separation of variables in the basic quantum-mechanical equations with an external electromagnetic field and also on the construction of exact solutions can be found in the book [10]. In these studies, differential symmetry operators of not higher than second order are used.

The "dressing" method makes it possible to integrate new classes of linear differential equations with external fields (i.e., with variable coefficients), and for such equations one can find an integrodifferential symmetry operator whose structure is consistent with the structure of the equation. Such operators give nontrivial examples of nonlocal symmetry operators.

In this paper, we use "dressing" to develop a method of generating exactly solvable

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potentials and a corresponding complete set of wave functions of the linear and, in the general case, nonstationary Schrödinger equation. Special cases of the proposed transformations are the Abraham—Moses method [11] and the modifications of it in [12,13].

2. Method for Generating Potentials

Following [4], we do not fix in advance the function space on which the operators act, emphasizing the algebraic nature of the results.

The transformations of the operators will be associated with one spatial variable x, and therefore we consider the one-dimensional Schrödinger equation

$$\hat{M}\psi(x,t)\equiv(i\partial_t-\hat{H})\psi(x,t)=0,\quad \hat{H}=-\partial_{xx}+a(x,t). \tag{2.1}$$

Here, t is the time, $\partial_t=\partial/\partial t$, $\partial_x=\partial/\partial x$, $\partial_{xx}=\partial_x\partial_x$, $a(x,t)$ is the potential, a real scalar function. The results can also be repeated in the multidimensional case, as we shall discuss below.

We give some necessary facts relating to the procedure of "dressing" the operator (2.1); they are taken from [3,4].

Let $\hat{F}$ be a Fredholm operator:

$$\hat{F}\psi(x)=\int_{-\infty}^{+\infty}F(x,x')\psi(x')dx', \tag{2.2}$$

and $\hat{K}^{\pm}$ its Volterra factors, i.e.,

$$\hat{K}^+\psi(x)\equiv\int_x^{\infty}K^+(x,x')\psi(x')dx',\quad \hat{K}^-\psi(x)\equiv\int_{-\infty}^{x}K^-(x,x')\psi(x')dx', \tag{2.3}$$

$$1+\hat{F}=(1+\hat{K}^+)^{-1}(1+\hat{K}^-). \tag{2.4}$$

Equation (2.4) is equivalent to the conditions

$$\begin{gathered}K^+(x,x')+F(x,x')+\int_x^{\infty}K^+(x,s)F(s,x')ds=0,\quad x'>x,\\ F(x,x')+\int_x^{\infty}K^+(x,s)F(s,x')ds=K^-(x,x'),\quad x>x'.\end{gathered} \tag{2.5}$$

The operators $(1+\hat{K}^{\pm})$ are invertible. An operator $\hat{M}$ of the form (2.1), transformed in accordance with the formula

$$\hat{\hat{M}}=(1+\hat{K}^+)\hat{M}(1+\hat{K}^+)^{-1},$$

remains a differential operator if, for example,

$$[\hat{M},\hat{F}]=0. \tag{2.6}$$

At the same time, Eq. (2.1) goes over into the equation

$$\hat{\hat{M}}\varphi(x,t)=(i\partial_t+\partial_{xx}-u(x,t))\varphi(x,t)=0, \tag{2.7}$$

in which

$$u(x,t)=a(x,t)-2\partial_xK^+(x,x;t), \tag{2.8}$$

$$\varphi(x,t)=(1+\hat{K}^+)\psi(x,t). \tag{2.9}$$

The condition (2.6) leads to the equation

$$(i\partial_t-\hat{H}(x,t))F(x,y;t)=-\hat{H}(y,t)F(x,y;t). \tag{2.10}$$

Here, $\hat{H}^+(x,t)$ is the operator that is the formal adjoint of $\hat{H}(x,t)$ (of the form (2.1); in our case $\hat{H}^+=\hat{H}$).

Remark. Here and in what follows we consider transformations with participation of the operator $\hat{K}^+$. One can construct in exactly the same way a theory with the operator $\hat{K}^-$. For this, it is sufficient in all the expressions to make the substitution

$$\int_x^{\infty} \to \int_{-\infty}^{x}, \quad K^+(x,y;t) \to -K^-(x,y;t).$$

We turn to a description of the method for generating potentials of Eq. (2.1). The basis of it is formula (2.8), and the problem consists of effective calculation of the kernel $K^+(x, y; t)$. First, we write the formulas for inverting the operators $(1 + \hat{K}^{\pm})$ as follows. We introduce $K(x, y; t)$ such that $K(x, y; t) = K^+(x, y; t)$ for $y > x$ and $K(x, y; t) = -K^-(x, y; t)$ for $x > y$. Then $K(x, y; t)$ is determined by Eq. (2.5) without the restriction $x' > x$, and by means of it one can readily invert the operators $(1 + \hat{K}^{\pm})$. For example, if

$$\varphi(x,t)=(1+\hat{K}^+)\psi(x,t)=\psi(x,t)+\int_x^{\infty} K(x,y;t)\psi(y,t)\,dy,$$

then the kernel $R(x, y; t)$ of the inverse transformation

$$\psi(x,t)=\varphi(x,t)-\int_0^{\infty} R(x,y;t)\varphi(y,t)\,dy \tag{2.11}$$

satisfies the condition

$$K(x,y;t)-R(x,y;t)-\int_x^{y} K(x,s;t)R(s,y;t)\,ds=0. \tag{2.12}$$

Let $\Psi(x, t)$ be a column (finite or infinite) whose elements are linearly independent solutions of the Schrödinger equation (2.1); $\Psi^+(x, t)$ is the row with the complex conjugate elements. Then it can be shown that functions $K(x, y; t)$ and $F(x, y; t)$ satisfying Eqs. (2.5) and (2.10) and ensuring reality of the potential (2.8) have the form

$$F(x,y;t)=\Psi^+(y,t)q^{-1}\Psi(x,t), \quad q^+=q, \quad K(x,y;t)=-\Psi^+(y,t)D^{-1}(x,t)\Psi(x,t), \tag{2.13}$$

where q is an arbitrary nonsingular constant matrix. The matrix $D(x, t)$ is determined by the relation

$$D(x,t)=q+\int_x^{\infty} \Psi(s,t)\Psi^+(s,t)\,ds, \tag{2.14}$$

and the unique restriction imposed on q is that $D(x, t)$ be nonsingular for all x. We note here that the validity of Eqs. (2.7)-(2.9), in which $K(x, y; t)$ has the form (2.13), can be verified directly in the general case. At the same time, the functions $\Psi(x, t)$ must ensure convergence of the integrals (2.14) at the upper limit, but there is no requirement at all of decrease of $\Psi(x, t)$ as $x \to -\infty$ and still less that $\Psi(x, t)$ be elements of L_2. Thus, we base our method on the expressions (2.13) and (2.14). From (2.12), taking into account (2.13) and (2.14), we find the kernel of the inverse transformation (2.11):

$$R(x,y;t)=-\Psi^+(y,t)D^{-1}(y,t)\Psi(x,t).$$

In the special case when $\Psi(x, t)$ consists of a single element $\psi(x, t)$, we obtain

$$K(x,y;t)=-\frac{\psi(x,t)\psi^*(y,t)}{q+\int_x^{\infty}\psi^*(s,t)\psi(s,t)\,ds}. \tag{2.15}$$

Here, ψ^* is the complex conjugate of ψ.

We consider in more detail the case when the original Schrödinger equation (2.1) has a time-independent potential: $a(x, t) = a(x)$. Choosing as elements of the column $\Psi(x, t)$ linearly independent solutions (even time-independent ones), we obtain in the general case the nonstationary equation (2.7). In the case when the column $\Psi(x, t)$ consists of a single nonvanishing element

$$\psi_s(x,t)=\exp(-iE_st)\psi_s(x),$$

where $\psi_s(x)$ is a stationary (corresponding to energy E_s) solution of Eq. (2.1), we obtain for $K(x, y; t)$

$$K(x,y;t)=K^{(s)}(x,y)=-\frac{\psi_s(x)\psi_s^*(y)}{q+\int\limits_x^\infty|\psi_s(z)|^2dz}. \quad (2.16)$$

Equation (2.7) is then also stationary, and its solutions have the form

$$\varphi_n^{(s)}(x)=\psi_n(x)+\int\limits_x^\infty K^{(s)}(x,y)\psi_n(y)dy, \quad u^{(s)}(x)=a(x)-2\partial_x K^{(s)}(x,x)$$

and correspond to the energy levels E_n. Since the system of functions $\{\psi_n(x)\}$ is complete and orthogonal, it is easy to show that $\{\varphi_n^{(s)}(x)\}$ is also a complete and orthogonal (with respect to the index n) system. Thus, the transformed Schrödinger equation (2.7) has the same spectrum. Applying our procedure repeatedly, and also choosing different s, we obtain classes of isospectral potentials. But if the energy $E \neq E_s$, i.e., does not belong to the spectrum of Eq. (2.1), but for the energy E there exists a solution of Eq. (2.1) that decreases as $x \to \infty$, then the transformed equation (2.7) can have for this E a solution in L_2, but the potential (2.8) of the transformed equation (2.7) is not equal to any of the potentials $u^{(s)}(x)$, and its spectrum cannot be determined in advance from Eq. (2.1).

Transformations with the kernel (2.16) were considered in [11-13] (see also the literature cited there). A more detailed exposition of the method of generating stationary potentials on the basis of the "dressing" procedure can be found in our [14,15], which also give appropriate examples.

For the multidimensional Schrödinger equation

$$\hat{M}\psi(x,t)=(i\partial_t-\hat{H})\psi(x,t)=0, \quad (2.17)$$

where $x=(x_1, \ldots, x_n)$, $\hat{H}=-\Delta+a(x,t)$, $\Delta=\sum\limits_{i=1}^n \partial_{x_i x_i}$, with potential a(x, t) of the form

$$a(x,t)=\sum_{i=1}^n a_i(x_i,t) \quad (2.18)$$

the scheme described above can be applied with respect to each variable x_i independently. As a result, for Eq. (2.17) we obtain expressions analogous to (2.7)-(2.9):

$$\hat{\hat{M}}\varphi(x,t)=Q\hat{M}Q^{-1}\varphi(x,t)=(i\partial_t+\Delta-u(x,t))\varphi(x,t)=0, \quad (2.19)$$

$$u(x,t)=\sum_i\{a_i(x_i,t)-2\partial_{x_i}K_i(x_i,x_i;t)\}, \quad (2.20)$$

$$\varphi(x,t)=Q\psi(x,t), \quad Q=\prod_{i=1}^n(1+\hat{K}_i^+). \quad (2.21)$$

Here, the kernel $K_i^+(x, y; t)$ of the Volterra operator $\hat{K}_i^+$ of the form (2.3) and $K_i(x, y; t)$ of the form (2.13) are specified by the column $\Psi_i(x, t)$, which is composed of linearly independent solutions of the one-dimensional Schrödinger equation (2.1) with a(x, t) replaced by the potential $a_i(x, t)$ from the expansion (2.18) and corresponding to the spatial variable x_i. Note also that $[\hat{K}_i^+, \hat{K}_j^+] = 0$.

The potential (2.20) is not trivial. The fact is that the standard method for solving the Schrödinger equation is separation of variables. A complete classification of potentials for a nonstationary Schrödinger equation is given in [16]. In accordance with the theorem on the necessary and sufficient conditions for complete separation of the variables [5], the variables are separated by means of a complete set of commuting differential symmetry operators of not higher than the second order. Note that the potential (2.18) does not admit complete separation, since the time t is not separated with the variables x_i due to the fact that the functions $a_i(t, x_i)$ are arbitrary. If as "bare" operators we now take the operator (2.17) with separating potential (2.18) together with the symmetry operators of the corresponding complete set, then the "dressed" symmetry operators become integrodifferential operators, and in the Schrödinger equation (2.19) the transformed potential (2.20) does not admit separation of the variables. Nevertheless, a solution of

Eq. (2.19) can be constructed in accordance with (2.21), where $\psi(x, t)$ is taken to be a solution of Eq. (2.17) in separated variables. Thus, we have presented for the first time classes of integrable potentials of a Schrödinger equation that in the general case do not admit separation of the variables. These considerations are also valid in the case of a single spatial variable.

3. Examples

We illustrate the general propositions by characteristic examples that have independent value.

We consider the harmonic oscillator

$$a(x)=\omega^2x^2.$$

As "bare" solutions we take a linear combination of the coherent states first found by Schrödinger [17] (see also [18]):

$$\psi_\lambda(x,t)=(\sin 2\omega t)^{-1/2}\exp i\left\{\left(\frac{\omega}{2}\operatorname{ctg}2\omega t\right)x^2-\frac{\lambda x}{\sin 2\omega t}+\frac{\lambda^2}{2\omega}\operatorname{ctg}2\omega t\right\}. \tag{3.1}$$

Here, λ is a real parameter. The state (3.1) is determined by the system

$$\hat{M}\psi_\lambda=0,\quad \hat{X}\psi_\lambda=\lambda\psi_\lambda,$$

where

$$\hat{X}=i\sin 2\omega t\partial_x+(\omega\cos 2\omega t)x \tag{3.2}$$

is a symmetry operator of Eq. (2.1), $[\hat{M}, \hat{X}] = 0$, and is a solution in the separated variables.

We construct the kernel (2.15) from the function $\psi(x, t)$, which is a Gaussian wave packet of the functions (3.1):

$$\psi(x,t)=\int_{-\infty}^{+\infty}\exp(-\sigma\lambda^2)\psi_\lambda(x,t)\,d\lambda,$$

where σ is a real parameter. The transformed potential (2.8) is nonstationary and takes the form

$$u(x,t)=\omega^2x^2+4\frac{\partial}{\partial x}\frac{\pi\Omega\exp(-2\sigma\Omega^2x^2)}{1+\frac{\pi^{1/2}}{\sqrt{2\sigma}}\operatorname{erfc}(\sqrt{2\sigma}\,\Omega x)},\quad \Omega=\omega(\cos^2 2\omega t+4\omega^2\sigma^2\sin^2 2\omega t)^{-1/2}. \tag{3.3}$$

Here, $\operatorname{erfc}(z)=\frac{2}{\sqrt{\pi}}\int_z^\infty\exp(-t^2)\,dt$ is the error function.

Any solution $\Phi(x, t)$ of Eq. (2.1) for the harmonic oscillator can be transformed to a solution $\varphi(x, t)$ of Eq. (2.1) with potential (3.3) in accordance with (2.9) by means of the formula

$$\varphi(x,t)=\Phi(x,t)-\frac{2\pi\Omega\exp(-Ax^2)}{1+\frac{\pi^{1/2}}{\sqrt{2\sigma}}\operatorname{erfc}(\sqrt{2\sigma}\,\Omega x)}\int_x^\infty\exp(-A^*z^2)\Phi(z,t)\,dz. \tag{3.4}$$

Here $A=\sigma\Omega^2-i\frac{\omega}{2}\operatorname{ctg}(2\omega t)\left(1-\frac{\Omega^2}{\omega^2}\right)$. For $\sigma = 1/\omega$, $\Omega = \omega$ the potential (3.3) becomes stationary:

$$u(x)=\omega^2x^2+4\frac{\partial}{\partial x}\frac{\pi\omega\exp(-\omega x^2)}{1+\pi^{1/2}\sqrt{\omega}\operatorname{erfc}(\sqrt{\omega}\,x)}.$$

We emphasize that the solution (3.4) is not a solution in the separated variables, and the "dressed" symmetry operator (3.2) is an integrodifferential operator.

A simple example of a Schrödinger equation with nonstationary integrable potential and maximal symmetry group can be obtained by "dressing" the free ($a(x, t) = 0$) Schrödinger equation (2.1). As a result of "dressing," the generators of the group of the free Schrödinger equation become integrodifferential operators. In (2.15) we set

$\psi(x,t)=t^{-1/2}\exp\left\{\frac{1}{2t}\left[\frac{i}{2}(x^2-1)-x\right]\right\}$, and then the potential (2.8) will have the form

$$u(x,t)=-\frac{1}{2t^2}\operatorname{ch}^{-2}\left(\frac{x}{2t}\right),$$

with, for example, the displacement operator $\partial/\partial x$ becoming the integrodifferential operator

$$\tilde{p}\varphi(x,t)\equiv(1+\hat{K})\frac{\partial}{\partial x}(1+\hat{K})^{-1}\varphi(x,t)=\frac{\partial\varphi(x,t)}{\partial x}+\int_x^\infty p(x,y;t)\varphi(y,t)\,dy,$$

where

$$p(x,y;t)=\psi(x,t)\psi^*(y,t)\left\{\frac{iy+1}{2}c(x,t)+\frac{ix-1}{2}c(y,t)+\Phi(x,y,t)+1\right\}\Big/ tc(x,t)c(y,t);$$

$$c(x,\ t)=1+\exp(-x/t),\quad \Phi(x,y,t)\equiv\frac{1}{2t}\int_y^x\frac{iz-1}{c(z,t)}dz.$$

The remaining generators of the group have a similar form.

In classical mechanics, such nonlocal symmetries apparently correspond to integrals of the motion that are rational (nonpolynomial) in the momenta. However, this question requires further study.

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