Russian], Énergoatomizdat, Moscow (1985).
23. M. V. Fedoryuk, Asymptotics: Integrals and Series [in Russian], Nauka, Moscow (1987).

MEAN-FIELD MODELS IN THE THEORY OF RANDOM MEDIA. I

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This paper commences a cycle devoted to analysis of problems in the theory of random media by means of the mean-field (nonlocal) diffusion approximation with corresponding operator $\bar{\Delta}_V$, $V\subset\mathbf{Z}^d$. This paper contains an introduction for the complete cycle with a brief review of problems in the theory of random media. Also considered is the problem of localization for the operator $H_V=\bar{\Delta}_V+\xi(x)$, where $\{\xi(x)\}$ are independent identically distributed continuous random variables, $|V|\to\infty$. It is shown that there is V-uniform localization on the average.

Introduction

The central problem in the theory of disordered (random) media, or structures, is the interplay of the two main mechanisms of their evolution: the dissipation mechanism (diffusion, heat conduction, etc.), which arises because of internal fluctuations ("noise") of the medium, and the consolidation mechanism (self-organization), due to external factors (potential fields, flows, etc.).

Such media are described mathematically by means of the equations of mathematical physics with random coefficients. It is very important whether or not a medium is stationary, i.e., whether or not its physical properties depend on the time.

We give the most important examples of the problems that arise and will be considered in detail later in an approximation that we shall call the mean-field approximation.

1. A classical example in stationary theory is Anderson's strong coupling model [1], in which one studies the energy levels and wave functions of the Hamiltonian

$$H=-\Delta+\sigma\xi(\mathbf{x},\omega), \tag{1}$$

where Δ is the lattice Laplacian,

$$\Delta f(\mathbf{x})=\frac{1}{2d}\sum_{|\mathbf{x}-\mathbf{y}|=1}(f(\mathbf{y})-f(\mathbf{x}))=\bar{f}(\mathbf{x})-f(\mathbf{x}),\quad \mathbf{x}\in\mathbf{Z}^d, \tag{2}$$

and $\xi(\mathbf{x},\omega)$, $\mathbf{x}\in\mathbf{Z}^d$, is the lattice potential, a collection of independent identically distributed random variables (say with standard normal distribution $N(0,1)$); $\sigma>0$ is the coupling constant.

The one-dimensional case ($d = 1$) has been well studied. In that case, the operator (1) for any σ has a purely point spectrum that is everywhere dense on the support of the distribution of the potential ξ (on $\mathbf{R}^1$ for a Gaussian potential), and the corresponding states are exponentially localized.

The central hypothesis of the multidimensional Mott–Anderson theory ($d \geqq 3$) is that there is a phase transition with respect to the parameter σ: for $\sigma \geqq \sigma_0$ the spectrum $S(H)$ is a purely point spectrum, whereas for $\sigma < \sigma_0$ two components — point and continuous — coexist in the spectrum, and they have nonintersecting supports. This hypothesis has been partly proved (the pure point nature of the spectrum at large σ) in a series of recent studies by Fröhlich, Spencer, and others. However, the appearance of a continuous component of the spectrum at small σ (and this is the main point of the Mott–Anderson philosophy) has not yet been proved. A detailed bibliography on these problems (one-dimensional and multidimensional) can be found in the reviews of [2–4].

In the language of quantum mechanics, complete localization of the eigenfunctions of

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the operator (1) is equivalent to absence of dissipation in the solutions to the time-dependent Schrödinger problem

$$i\frac{\partial\psi}{\partial t}=H\psi,\quad \psi(\mathbf{x},0)=\psi_0(\mathbf{x})\in \mathbf{L}^2(\mathbf{Z}^d). \tag{3}$$

This last means for any $\varepsilon > 0$ one can find a volume $V=V(\varepsilon)\subset\mathbf{Z}^d$ such that

$$\sum_{\mathbf{x}\in \mathbf{Z}^d\setminus V} |\psi(\mathbf{x},t)|^2<\varepsilon$$

for all $t \geqq 0$. This readily follows from the Fourier representation of the solution $\psi(\mathbf{x}, t)$ to the problem (3):

$$\psi(\mathbf{x},t)=\sum_{\lambda_n\in S(H)} \exp(i\lambda_n t)(\psi_0,\varphi_n)\varphi_n(\mathbf{x}),$$

where λ_n, $\varphi_n(\mathbf{x})$ are the energy levels and localized states of the operator H.

2. Another interesting group of problems in the theory of stationary media is associated with parabolic equations of the form*

$$\frac{\partial\psi}{\partial t}=\varkappa\Delta\psi+\xi(\mathbf{x},\omega)\psi,\quad \psi(\mathbf{x},0)=\psi_0(\mathbf{x})\geqslant 0, \tag{4}$$

which arise in linearized schemes of chemical kinetics and in the theory of branching processes.

For an initial function $\psi_0(\mathbf{x})$ one usually considers one of two types of initial condition of physical interest.† In the first case, it is assumed that $\psi_0=\psi_0(\mathbf{x},\omega)$ is a spatially homogeneous random field that does not depend on the potential ξ (in particular, ψ_0 = const). The homogeneity property is preserved under the time evolution of the solution $\psi(\mathbf{x}, t)$. In the second case, it is assumed that the function $\psi_0(\mathbf{x})$ is localized, for example, $\psi_0(\mathbf{x})=\delta(\mathbf{x}-\mathbf{x}_0)$. Such a formulation is natural when one is studying the effect of population of the medium.

From the expansion of the solution $\psi(\mathbf{x}, t)$ with respect to the eigenfunctions of the operator H on the right-hand side of Eq. (4), it is clear that the main role in the asymptotic behavior of $\psi(\mathbf{x}, t)$ as $t \to \infty$ is played by the structure of the spectrum S(H) near its right-hand boundary. It was shown in [5] for independent $\xi(\mathbf{x}, \omega)$ with normal distribution $N(0, \sigma^2)$ that with unit probability for $\varkappa>0$ there exists the limit

$$\lim_{t\to\infty} t^{-1}(\ln t)^{-1/2}\ln\psi(\mathbf{x},t)=(2d\sigma^2)^{1/2}. \tag{5}$$

It is important that the limit (5) does not depend on the diffusion coefficient $\varkappa$. Physical arguments (see [6]) suggest that there must exist a more accurate asymptotic behavior of the form

$$t^{-1}\ln\psi(\mathbf{x},t)=(2d\sigma^2\ln t)^{1/2}-C_1\varkappa+C_2+o(1), \tag{6}$$

where $C_1 > 0$ depends only on the dimension d, while the constant C_2 is related to the percolation properties of the field $\xi(\mathbf{x}, \omega)$. The proof of the asymptotic behavior (6) is an important unsolved problem.

The same applies to investigation of the asymptotic behavior of the statistical moments of the field $m_p=m_p(\mathbf{x},t)=\langle\psi^p(\mathbf{x},t)\rangle$, $p=1, 2, \ldots$. It was shown in [5] that

$$\lim_{t\to\infty} t^{-2}\ln m_p(\mathbf{x},t)={}^1/_2 p^2\sigma^2. \tag{7}$$

The growth of the moments m_p, progressive in the number p and superexponential in time, indicates a clearly expressed intermittency of the field $\psi(\mathbf{x}, t)$ [5–9]. This observation brings localization theory and intermittency theory closer together and gives a regular orientation of nonstationary theory, which will be discussed below. In the framework of

*The operator on the right-hand side of Eq. (4) is essentially identical to the Hamiltonian (1) for $\sigma=\varkappa^{-1}$.

†The remark about the initial condition also applies to the evolution problems of §§3 and 4.

the development of moment theory of stationary media, it is important to make the asymptotic behavior (7) more accurate in the spirit of (6) by establishing the part played by the diffusion $\varkappa$ in the generation of the moments.

3. In nonstationary theory (see the reviews of [6–9] and the paper [5]) the moment approach comes to the fore. It is particularly effective when the random properties of the medium are δ correlated in time.

Let us consider, for example, a nonstationary analog of Eq. (4):

$$\frac{\partial\psi}{\partial t}=\varkappa\Delta\psi+\xi(\mathbf{x},t,\omega)\psi,\quad \psi(\mathbf{x},0)=\psi_0(\mathbf{x})\geqslant 0, \tag{8}$$

where $\xi(\mathbf{x}, t, \omega)$ are independent (for different $\mathbf{x}$) "white Gaussian noise" processes with correlation function

$$\langle\xi(\mathbf{x},t,\omega)\xi(\mathbf{x}',t',\omega)\rangle=\sigma^2\delta(\mathbf{x}-\mathbf{x}')\delta(t-t').$$

Equation (8) is to be understood in the sense of Itô (although a Stratonovich interpretation [10] is also possible; see the detailed study of [11] for a discussion of the relationship between Itô and Stratonovich integrals). It can be shown (for the details, see [6]) that the moment functions $m_p(\mathbf{x}_1,\dots,\mathbf{x}_p,t)=\langle\psi(x_1,t)\dots\psi(\mathbf{x}_p,t)\rangle$, $p=1,2,\dots,$ satisfy many-particle equations of the Schrödinger type

$$\frac{\partial m_p}{\partial t}=\Big(\varkappa\sum_{i=1}^{p}\Delta_{\mathbf{x}_i}+\sigma^2\sum_{i\neq j}\delta(\mathbf{x}_i-\mathbf{x}_j)\Big)m_p,\quad m_p(\mathbf{x}_1,\dots,\mathbf{x}_p,0)=\langle\psi_0(\mathbf{x}_1)\dots\psi_0(\mathbf{x}_p)\rangle, \tag{9}$$

and that there exists the limit

$$\lim_{t\to\infty}t^{-1}\ln m_p(\mathbf{x}_1,\dots,\mathbf{x}_p,t)=\gamma_p(\varkappa)\geqslant 0,$$

with $\gamma_p(\varkappa)$ the upper bound of the spectrum of the operator on the right-hand side of Eq. (9).

One can also show (see [6]) that with unit probability there exists the nonvanishing limit $\lim\limits_{t\to\infty}t^{-1}\ln\psi(\mathbf{x},t)=\tilde{\gamma}(\varkappa)$.

The calculation of the functions $\gamma_p(\varkappa)$, $\tilde{\gamma}(\varkappa)$ and the study of their properties are very difficult problems. Hitherto, their behavior has been studied in detail only for small $\varkappa$ [5–7]. A plausible picture (though in many details not properly founded) is shown in Fig. 1 (for the case $d \geqq 3$). An important additional question is associated with the calculation of the fractional moments m_p, particularly in the intervals $p\in(0,1)$, $p\in(1,2)$.

4. The final group of questions that we wish to mention in the theory of nonstationary random media concerns phase transitions in heteropolymers. In Lifshitz's well-known studies (see, for example, [12]) the problem was reduced, in essence, to a problem of Lyapunov exponents in a nonstationary scattering scheme. Mathematically, we are concerned (in the simplest model) with calculation of the limit $\lim\limits_{t\to\infty}t^{-1}\ln\psi(\mathbf{x},t)=\gamma(\varkappa)$ for the solution $\psi(\mathbf{x},t)$ to the problem

$$\frac{\partial\psi}{\partial t}=\varkappa\Delta\psi+\delta_0(\mathbf{x})\xi(t,\omega)\psi,\quad \psi(\mathbf{x},0)=\psi_0(\mathbf{x})\geqslant 0,$$

where $\xi(t, \omega)$ is a random process that is homogeneous in time and has rapidly decreasing correlations. A phase transition is here understood as the existence of a critical value $\varkappa=\varkappa_0$ such that $\gamma(\varkappa)>0$ for $\varkappa<\varkappa_0$ (globule state) and $\gamma(\varkappa)\equiv 0$ for $\varkappa\geqslant\varkappa_0$ (coil state).

Of particular interest is the behavior of the exponent $\gamma(\varkappa)$ (which is the free energy of the heteropolymer) in the neighborhood of the critical point $\varkappa_0$, and also the geometrical structure of the solution $\psi(\mathbf{x}, t)$ for $\varkappa<\varkappa_0$, $\varkappa_0-\varkappa\ll 1$. The physical essence of the globule-coil phase transition has been completely clarified (see, for example, [13]), but as yet there are no mathematically rigorous results, and still less models with explicitly calculated free energy.

The aim of this paper is to solve the problems of §§1–4 for a model that we call a

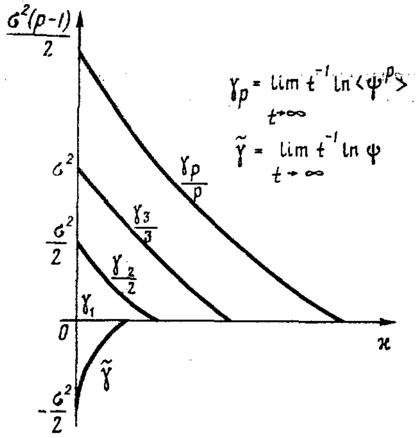

Fig. 1

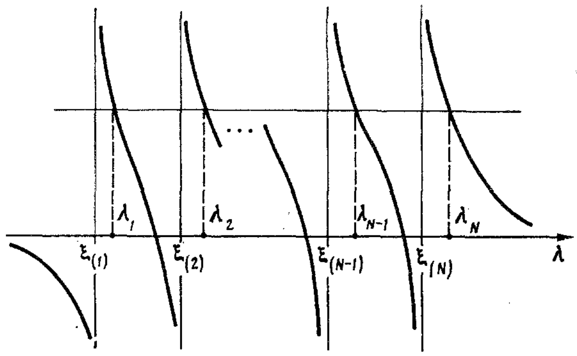

Fig. 2

random medium with mean-field diffusion.* In this model, the diffusion operator (2), which has the significance of local averaging, is replaced by an operator of averaging over a large volume $V\subset\mathbf{Z}^d$:

$$\bar{\Delta}_V f(\mathbf{x})=\frac{1}{|V|}\sum_{\mathbf{y}\in V}(f(\mathbf{y})-f(\mathbf{x}))=\bar{f}_V-f(\mathbf{x}). \tag{10}$$

One can here drawn an analogy with the well-known Curie–Weiss model (see [14]) in which a local interaction potential (of Ising type) in the Hamiltonian

$$\mathcal{H}=J\sum_{|\mathbf{y}-\mathbf{x}|=1}s_\mathbf{x}s_\mathbf{y},\quad s_\mathbf{x}=\pm1,\quad \mathbf{x}\in\mathbf{Z}^d,$$

is replaced by a weak long-range potential in the mean-field Hamiltonian

$$\mathcal{H}_V=\frac{J}{|V|}\sum_{\mathbf{x},\mathbf{y}\in V}s_\mathbf{x}s_\mathbf{y},\quad V\subset\mathbf{Z}^d.$$

Although the limit operator $\bar{\Delta}$ (as $|V|\to\infty$) is not defined, one can speak of a limiting behavior of the solutions to the evolution problems of §§1–4 in the limit $t\to\infty$, $|V|\to\infty$. The effects associated with the dimension of space $\mathbf{Z}^d$ are to a certain degree captured by the consistency condition

$$|V|\sim t^{d/2},\quad t\to\infty. \tag{11}$$

This condition is motivated by the fact that the typical path of a random walk associated with the lattice Laplacian Δ travels during a large time t to a distance of order $\sqrt{t}$ from the initial point.

An additional justification of the model with mean-field diffusion is the circumstance that the problems of §§1–4 themselves arose as the result of the single-particle (mean-field) approximation in many-particle problems (see, for example, [15]).

The paper consists of three parts. Part I is devoted to the localization problem in the spirit of §1, Part II to the asymptotic behavior of its solution and of its moments for a stationary medium (§2), and Part III is devoted to nonstationary media (§§3 and 4).

In the later publications we intend to consider more realistic models with nonlocal diffusion. In particular, we shall be very interested in the so-called hierarchical model in which the diffusion operator has a hierarchical structure defined in an infinite system of multiply embedded volumes by means of the mean-field approximation. In this model, the limit operator is defined, and the presence of a symmetry group of high order permits reduction of the problem to nonlinear integral equations. Continuing the analogy with statistical physics, we note that the hierarchical model plays in the theory of random

*Some of our results were announced in the review of [7].

media the same role as Dyson's well-known model in the theory of phase transitions in ferromagnetic systems (see [16]).

1. Localization Problem

Let $V=\{\mathbf{x}\}$ be a finite set,* and N be the number of its elements. Let $\xi(\mathbf{x})$, $\mathbf{x}\in V$, be independent identically distributed random variables with continuous distribution. We study the spectral properties of the operator† $H_V=\bar{\Delta}_V+\xi(\mathbf{x})$, which acts on the space of functions $f(\mathbf{x})\in\mathbf{L}^2(V)$, where $\bar{\Delta}_V$ is defined by Eq. (10). We shall seek the eigenvalues of the form $\lambda - 1$. From the equation $H_V\varphi=(\lambda-1)\varphi$ we obtain

$$\varphi(\mathbf{x})=\bar{\varphi}_V\cdot(\lambda-\xi(\mathbf{x}))^{-1}. \tag{12}$$

Averaging (12) over $\mathbf{x}\in V$ and dividing by $\bar{\varphi}_V$,** we arrive at the equation

$$1=\frac{1}{N}\sum_{\mathbf{x}\in V}\frac{1}{\lambda-\xi(\mathbf{x})}. \tag{13}$$

We arrange the random variables $\xi(\mathbf{x})$, $\mathbf{x}\in V$, in ascending order and denote by $\xi_{(1)}<\ldots<\xi_{(N)}$ their order statistics (with probability 1, all inequalities are strict). It is obvious (see Fig. 2) that Eq. (13) has precisely N real roots $\lambda_1 < \ldots < \lambda_N$, and

$$\xi_{(1)}<\lambda_1<\xi_{(2)}<\lambda_2<\ldots<\lambda_{N-1}<\xi_{(N)}<\lambda_N. \tag{14}$$

Remark. From (14), using the law of large numbers, we readily deduce that there exists the integrated density of states

$$\mathcal{N}(\lambda)=\lim_{N\to\infty}N^{-1}\sum_{\lambda_i\leqslant\lambda}1,$$

with $\mathcal{N}(\lambda)$ equal to the distribution function $F(\lambda)$ of the random variable ξ. If the distribution ξ has density $p(\lambda)$, then there also exists the density of states $d\mathcal{N}(\lambda)/d\lambda=p(\lambda)$.

Using formula (12) for $\lambda = \lambda_i$, we find the form of the eigenfunction $\varphi_i(\mathbf{x})$, normalized by the condition $\|\varphi_i\|=1$:

$$\varphi_i(\mathbf{x})=(\lambda_i-\xi(\mathbf{x}))^{-1}\Big[\sum_{\mathbf{y}\in V}(\lambda_i-\xi(\mathbf{y}))^{-2}\Big]^{-1/2}. \tag{15}$$

We fix the point $\mathbf{x}_0\in V$ and consider the spectral measure $d\mu^{(N)}$, which is concentrated at the points λ_i, i = 1, ..., N, with mass $\mu_i=(\delta_{\mathbf{x}_0},\varphi_i)^2$, where $\delta_{\mathbf{x}_0}(\mathbf{x})=\delta(\mathbf{x}-\mathbf{x}_0)$. Substituting (15), we find

$$\mu_i=\varphi_i^2(\mathbf{x}_0)=(\lambda_i-\xi(\mathbf{x}_0))^{-2}\Big[\sum_{\mathbf{y}\in V}(\lambda_i-\xi(\mathbf{y}))^{-2}\Big]^{-1}. \tag{16}$$

Note that by virtue of Parseval's equation

$$\mu_1+\ldots+\mu_N=\|\delta_{\mathbf{x}_0}\|^2=1, \tag{17}$$

i.e., the total variation of the measure $d\mu^{(N)}$ is 1.

The central problem of localization theory is the behavior of the spectral measure $d\mu^{(N)}$ as $N \to \infty$. In realistic models, there exists the weak limit $\lim d\mu^{(N)}=d\mu$, and localization means that the limit measure $d\mu$ is a purely point measure, i.e., consists of atoms distributed everywhere densely on the spectrum of the limit operator $H = \lim H_V$. In our case, as we have already noted, there exists neither the limit operator nor the limit spectral measure. Nevertheless, here too we can pose the localization problem, understanding localization as absence of dissipation of the measure $d\mu^{(N)}$ as the volume V becomes larger.

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*It can be assumed that $V\subset Z^d$, though the topology of the lattice is unimportant for what follows.

†Without loss of generality, we have set $\sigma = 1$.

**With unit probability, $\bar{\varphi}_V\neq0$. For if we had $\bar{\varphi}_V=0$, then by virtue of (12) $\varphi(\mathbf{x})\cdot(\lambda-\xi(\mathbf{x}))\equiv0$, $\mathbf{x}\in V$. But the factor $\lambda-\xi(\mathbf{x})$ vanishes at not more than one point $\mathbf{x}_0$, since $\xi(\mathbf{x})\neq\xi(\mathbf{y})$ with probability 1 for $\mathbf{x}\neq\mathbf{y}$. Then either $\varphi(\mathbf{x})\equiv0$, or $\varphi(\mathbf{x})\neq0$ at precisely one point $\mathbf{x}_0$, and then $\bar{\varphi}_V\neq0$.

This last means that the measure $d\mu^{(N)}$ in the limit $N \to \infty$ remains basically concentrated (in some sense) in a finite number of leading atoms.

The concept of volume-uniform localization is made more precise by the following definition.

DEFINITION. We denote

$$\sigma_{k,N} = \sum_{i=1}^{k} \mu_{i,N},$$

where $\mu_{i,N}$ are the atoms (16), arranged in decreasing order: $\mu_{1,N} \geqq \ldots \geqq \mu_{N,N}$. We shall say that there is (uniform in N) localization:

a) on the average, if

$$\lim_{k\to\infty} \liminf_{N\to\infty} \langle \sigma_{k,N} \rangle = 1 \tag{18}$$

(here, <...> is the symbol of averaging over the distribution ξ);

b) in probability, if for any $\varepsilon > 0$

$$\lim_{k\to\infty} \liminf_{N\to\infty} P\{\sigma_{k,N} > 1-\varepsilon\} = 1; \tag{19}$$

c) almost certainly, if

$$P\{\lim_{k\to\infty} \liminf_{N\to\infty} \sigma_{k,N} = 1\} = 1. \tag{20}$$

It is easy to show that (18) and (19) are equivalent to each other and, in turn, follow from (20).

LEMMA 1.1. The mathematical expectation of any atom (16), $i = 1, \ldots, N$, does not depend on i, and $\langle \mu_i \rangle = N^{-1}$.

Proof. We fix i and denote the atom μ_i by $\mu_i(\mathbf{x}_0)$ to emphasize the dependence on $\mathbf{x}_0$. Note that by virtue of (16)

$$\sum_{\mathbf{x}\in V} \mu_i(\mathbf{x}) = 1$$

and that the distribution of the random variable $\mu_i(\mathbf{x})$, and, hence, the mean $\langle \mu_i(\mathbf{x}) \rangle$, does not depend on $\mathbf{x} \in V$. Therefore

$$\sum_{\mathbf{x}\in V} \langle \mu_i(\mathbf{x}) \rangle = N \langle \mu_i(\mathbf{x}_0) \rangle = 1,$$

whence $\langle \mu_i(\mathbf{x}_0) \rangle = N^{-1}$.

Note that Lemma 1.1 in no way indicates dissipation of the measure $d\mu^{(N)}$. It merely shows that in the case of localization the main mass must be concentrated not at fixed but at leading (randomly distributed) atoms.

We denote by ν the (random) number that the random variable $\xi(\mathbf{x}_0)$ obtains when the random variables $\{\xi(\mathbf{x})\}$ are arranged in ascending order:

$$\xi(\mathbf{x}_0) = \xi_{(\nu)}, \quad \nu = \sum_{\mathbf{x}\in V} I\{\xi(\mathbf{x}_0) \geqslant \xi(\mathbf{x})\}.$$

It is obvious by symmetry that the random variable ν does not depend on the order statistics of $\{\xi_{(i)}\}$ and

$$P\{\nu = n\} = \frac{1}{N}, \quad n = 1, \ldots, N. \tag{21}$$

By means of (16) we readily understand that the leading atoms are grouped around μ_ν.

PROPOSITION 1.1. Let F(x) be the distribution function of the random variable ξ with density p(x), and $0 < C_1 \leqq p(x) \leqq C_2$, where C_1 and C_2 are constants. Then

$$\lim_{k\to\infty} \limsup_{N\to\infty} \left\langle \sum_{|i-\nu|\geqslant k} \mu_i \right\rangle = 0. \tag{22}$$

Proof. We consider the "right tail" of the sum in (22) and write it in accordance with the formula for the total mathematical expectation, using (21), in the form

$$\left\langle \sum_{i \geqslant \nu+k} \mu_i \right\rangle = \frac{1}{N} \sum_{n=1}^{N-k} \sum_{i \geqslant n+k} \langle \mu_i \rangle_{\nu=n}, \tag{23}$$

where $\langle \ldots \rangle_{\nu=n}$ is the conditional mathematical expectation under the condition $\{\nu = n\}$. Substituting here (16) and taking into account the fact that ν does not depend on the order statistics of $\xi_{(1)}, \ldots, \xi_{(N)}$ (which enables us to go over from the conditional mathematical expectation to the unconditional expectation), we obtain, using (14),

$$\langle \mu_i \rangle_{\nu=n} \leqslant \left\langle \left(\frac{\lambda_i - \xi_{(i)}}{\lambda_i - \xi_{(n)}} \right)^2 \right\rangle \leqslant \left\langle \left(\frac{\xi_{(i+1)} - \xi_{(i)}}{\xi_{(i)} - \xi_{(n)}} \right)^2 \right\rangle. \tag{24}$$

We consider random variables u_i with distribution uniform on [0, 1], setting $u_i = F(\xi_i)$, $i = 1, \ldots, N$ (then $u_{(i)} = F(\xi_{(i)})$). By hypothesis, the derivative $F'(x) = p(x)$ is bounded and separated from zero, and therefore

$$\left\langle \left(\frac{\xi_{(i+1)} - \xi_{(i)}}{\xi_{(i)} - \xi_{(n)}} \right)^2 \right\rangle \leqslant \left(\frac{C_2}{C_1} \right)^2 \cdot \left\langle \left(\frac{u_{(i+1)} - u_{(i)}}{u_{(i)} - u_{(n)}} \right)^2 \right\rangle. \tag{25}$$

To calculate the mathematical expectation on the right-hand side of (25) we use the well-known fact (see, for example, [17], Chap. 3) on the identity of the distributions of the random vectors $(u_{(1)}, \ldots, u_{(N)})$ and $(S_1/S_{N+1}, \ldots, S_N/S_{N+1})$, where $S_i = \eta_1 + \ldots + \eta_i$, $\eta_1, \ldots, \eta_{N+1}$ are independent identically distributed random variables with exponential distribution. Bearing in mind that S_i has a gamma distribution with density $p_i(x) = x^{i-1}e^{-x}/(i-1)!$, we obtain

$$\left\langle \left(\frac{u_{(i+1)} - u_{(i)}}{u_{(i)} - u_{(n)}} \right)^2 \right\rangle = \left\langle \left(\frac{\eta_{i+1}}{\eta_{n+1} + \ldots + \eta_i} \right)^2 \right\rangle = \langle \eta^2 \rangle \cdot \langle S_{i-n}^{-2} \rangle = \frac{2}{(i-n-1)(i-n-2)}. \tag{26}$$

Thus, by virtue of (24)-(26)

$$\langle \mu_i \rangle_{\nu=n} \leqslant \frac{\text{const}}{(i-n-1)(i-n-2)}.$$

Substituting in (23), we obtain

$$\limsup_{k \to \infty} \sup_{N > k} \left\langle \sum_{i \geqslant \nu+k} \mu_i \right\rangle \leqslant \lim_{k \to \infty} \frac{\text{const}}{k-2} = 0.$$

The "left tail" of the sum in (22) can be treated similarly.

Remark. The proof of Proposition 1.1 can be readily extended to a more general case, for example, when the density p(x) is continuous and positive on (a, b), where $a = \inf\{x : F(x) > 0\}$, $b = \sup\{x : F(x) < 1\}$. For this, it is merely necessary to prove the estimate (25). Note that for any $\varepsilon > 0$, setting $N_0 = [\varepsilon N]$, we can assume that $\nu \geqq N_0$, and the index i in (23) varies in the range $N_0 \leqq i \leqq N - N_0$ (at the same time, by virtue of (21) and Lemma 1.1, the error does not exceed 2ε). But by virtue of the law of large numbers $\xi_{(N_0)} \sim F^{-1}(\varepsilon) > a$, $\xi_{(N-N_0)} \sim F^{-1}(1-\varepsilon) < b$, and therefore for $N_0 \leqq i \leqq N - N_0 + 1$ all $\xi_{(i)}$ lie with high probability in a certain segment, and there, by virtue of the continuity and positivity of p(x), the inequalities $C_1 \leqq p(x) \leqq C_2$ do hold.

Thus, Proposition 1.1 establishes the fact of volume-uniform localization on the average for our model. The more subtle problem of almost certain localization still remains open. However, the following model example gives grounds for expecting a negative answer.

Example (for details, see [18]). We consider a random measure $d\tilde{\mu}^{(N)}$ with atoms $\tilde{\mu}_i = C_N u_i^{-2}$, $i = 1, \ldots, N$, normalized by the condition $\tilde{\mu}_1 + \ldots + \tilde{\mu}_N = 1$. Here, $u_1, u_2, \ldots$ are independent identically distributed random variables with uniform distribution on [0, 1]. It is easy to show that for this measure Lemma 1.1 and Proposition 1.1 hold. In addition, one can show that for the maximal atom $\mu_{1,N} = \max\{\tilde{\mu}_1, \ldots, \tilde{\mu}_N\}$ the following limit relations hold with probability 1:

$$\limsup_{N \to \infty} \tilde{\mu}_{1,N} = 1, \quad \liminf_{N \to \infty} \tilde{\mu}_{1,N} = 0.$$

This means that, on the one hand, almost certain localization holds with respect to some (random) subsequence $N' \to \infty$ and is expressed very strongly, since the entire mass is concentrated in one atom. On the other hand, there exists another (also random) subsequence $N'' \to \infty$ with respect to which the measure $d\tilde{\mu}^{(N)}$ dissipates, since all the atoms $\tilde{\mu}_i$ are uniformly small.

LITERATURE CITED

1. P. W. Anderson, Phys. Rev., 109, 1492 (1958).
2. B. Simon and B. Souillard, J. Stat. Phys., 36, 273 (1984).
3. R. Carmona, Acta Appl. Math., 4, 65 (1985).
4. L. A. Pastur, "Spectral theory of random self-adjoint operators," in: Reviews of Science and Technology, Ser. Probability Theory, Mathematical Statistics, Theory of Cybernetics, Vol. 25 [in Russian], VINITI, Moscow (1987), p. 3.
5. Ya. B. Zel'dovich, S. A. Molchanov, A. A. Ruzmaikin, and D. D. Sokolov, Zh. Eksp. Teor. Fiz., 89, 2061 (1985).
6. Ya. B. Zeldovich, S. A. Molchanov, A. A. Ruzmaikin, and D. D. Sokoloff, "Generating, diffusion, intermittency of random fields," in: Sov. Sci. Rev. Math. Phys., Vol. 7, Gordon and Breach, London (1987), p. 1.
7. Ya. B. Zel'dovich, S. A. Molchanov, A. A. Ruzmaikin, and D. D. Sokolov, Usp. Fiz. Nauk, 152, 3 (1987).
8. S. A. Molchanov, A. A. Ruzmaikin, and D. D. Sokolov, Usp. Fiz. Nauk, 145, 593 (1985).
9. S. A. Molchanov, "Ideas in the theory of random media," Paper No. 914-V88 deposited at VINITI, 04.01.88 [in Russian], Moscow (1988).
10. R. L. Stratonovich, Conditional Markov Processes [in Russian], Moscow State University, Moscow (1966).
11. V. K. Matskyavichus, Litov. Mat. Sb., 22, 128 (1982).
12. I. M. Lifshits, Zh. Eksp. Teor. Fiz., 55, 2408 (1968).
13. I. M. Lifshits, A. Yu. Grosberg, and A. R. Khokhlov, Usp. Fiz. Nauk, 127, 353 (1979).
14. M. Kac, "Mathematical mechanisms of phase transitions," in: Stability and Phase Transitions [Russian translations], Mir, Moscow (1973), p. 164.
15. I. M. Lifshits, S. A. Gredeskul, and L. A. Pastur, Introduction to the Theory of Disordered Systems [in Russian], Nauka, Moscow (1982).
16. Ya. G. Sinai, Theory of Phase Transitions. Rigorous Results [in Russian], Nauka, Moscow (1980).
17. W. Feller, An Introduction to Probability Theory and its Applications, Vol. 2, Wiley, New York (1966).
18. L. V. Bogachev, "The localization phenomenon for sequences of random discrete measures," Paper No. 4652-V88 deposited at VINITI, 13.06.88 [in Russian], Moscow (1988).