$$t \to \tilde{t} = e^{\omega}(t+1) - 1 \simeq t + \omega(t+1), \quad q \to \tilde{q} = e^{\omega/2} q \approx q + (\omega/2) q.$$

Then instead of (18) we have different C_0 and C:

$$C_0 = -(t+1)\frac{p^2}{2} + \frac{pq}{2}, \quad C = -(t+1)\left(\frac{p^2}{2} + \frac{g}{q^2}\right) + \frac{pq}{2},$$

this being manifested on the right-hand side of (13) in the appearance of the domain of integration $[0, e^{\omega} - 1]$.

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HAMILTONIAN FORMALISM OF WEAKLY NONLINEAR HYDRODYNAMIC SYSTEMS

M. V. Pavlov

A study is made of systems of quasilinear equations that are diagonalizable and Hamiltonian and have the condition $\partial_i v_i \equiv 0$, where $u_t^i = v^i(u)u_x^i$, $i = 1, \ldots, N$. The conservation laws of such systems are found, together with the metric and Poisson bracket. For definite examples it is shown how solutions are found. The conditions for the existence of solutions and continuity of commuting flows are found.

The fundamentals of the Hamiltonian formalism for systems of quasilinear equations were first laid in [1] for the description of averaged analogs of completely integrable problems (Korteweg-de Vries equation, sinh-Gordon equation, etc.). This therefore led to the identification of a class of equations possessing a Hamiltonian and Poisson bracket. The further development of this mathematical formalism occurred in [2], where it was shown that if the original system is also diagonalizable then it possesses an infinite set of conservation laws. Moreover, it was also shown that for this conditions of semi-Hamiltonicity are sufficient (see [2]). However, examples of application of the developed theory did not exist.

All the necessary basic definitions are given in [1-4]. The condition given in [2],

$$\partial_j\left(\frac{\partial_k v_i}{v_k - v_i}\right) = \partial_k\left(\frac{\partial_j v_i}{v_j - v_i}\right), \quad i \neq j \neq k,$$

is called semi-Hamiltonicity of the system.

For the existence of a Hamiltonian formalism, it is also necessary for the metric to have zero curvature (see [1]). As was communicated to the author by S. P. Tsarev, the only identically nonzero components of the Riemann tensor are $R^i_{kki} = 0$ in the notation of [5] or [6].

The notation in the present paper corresponds to that of [1,2].

For the system $u_t^i = v^i(u)u_x^i$ we seek conservation laws in the form

$$\frac{\partial f^i}{\partial t} = \frac{\partial}{\partial x}(f^i v^i), \tag{1}$$

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where $f^i = f^i(u)$. After differentiation with respect to x and t, we obtain

$$\sum_{k=1}^{N}\left(f_i \frac{\partial v^i}{\partial u^k} + v^i \frac{\partial f^i}{\partial u^k} - v^k \frac{\partial f^i}{\partial u^k}\right) u_x^k = 0. \tag{2}$$

Each term in this sum is equal to zero. Since we consider Hamiltonian systems, it follows from (2) that

$$\partial_i v^i = 0, \quad k = i, \tag{3}$$

$$f^i = \sqrt{g_{ii}}\mu_i(u^i), \quad k \neq i, \tag{4}$$

where g_{ii} is the metric that occurs in the definition of the Poisson bracket (see [1] or [2]), and $\mu_i(u^i)$ is an arbitrary function. We recall

<u>DEFINITION.</u> A weakly linear system is defined as a diagonal system of quasilinear equations with the condition $\partial_i v^i \equiv 0$ (see, for example, [7]).

It can be seen from this that conservation laws of the form (1) exist only for weakly nonlinear systems. Thus,

$$H = \sum_{k=1}^{N} \sqrt{g_{kk}}\,\mu_k(u^k), \quad G = \sum_{k=1}^{N} \sqrt{g_{kk}}\,\mu_k(u^k) v_k(u), \quad \text{where} \quad \frac{\partial H}{\partial t} = \frac{\partial G}{\partial x} \tag{5}$$

is a conservation law for the system $u_t^i = v^i(u)u_x^i$, $i = 1, \ldots, N$, where $\partial_i v^i \equiv 0$.

As can be seen from (5), the condition of Theorem 3 of [2] on the completeness of the hydrodynamic integrals is satisfied. The integrals we have found generate a complete set of commuting flows in the given system.

A characteristic feature of weakly nonlinear systems is the possibility of performing the calculations in many cases explicitly and completely.

We consider the system of two equations (see [2])

$$\Gamma_{ik}^{\ i} = \frac{\partial_k v_i}{v - v} = \partial_k \ln\sqrt{g_{ii}}. \tag{6}$$

In our case, this relation is readily integrated, whence

$$g_{11} = \frac{\mu(u)}{(v_2 - v_1)^2}, \quad g_{22} = \frac{\eta(v)}{(v_2 - v_1)^2}, \tag{7}$$

μ and η are found from the condition $R^i_{kki} = 0$,

$$H = \frac{A(u) + B(v)}{u - v}, \tag{8}$$

where A and B are arbitrary functions.

The simplest example of a system of N equations is $u_t^i = v^i(u)u_x^i$, where $v^i(u) = \sum_{k=1}^{N} u_k - u_i$.

For it,

$$g_{ii} = \frac{\mu_i(u^i)}{\prod_{m\neq i}(u_m - u_i)^2}, \quad H = \sum_{k=1}^{N} \frac{A_k(u^k)}{\prod_{m\neq k}(u_m - u_k)} \tag{9}$$

Hydrodynamic systems with rational v^i, as, for example, (9), are more readily obtained as the semiclassical limit in the sense of [8] of the original equation rather than as the averaged equations from [1].

It was shown in [2] that semi-Hamiltonian (and, a fortiori, Hamiltonian) diagonalizable hydrodynamic systems are described by the implicit formula

$$w^i(u) = x + v^i(u)t, \tag{10}$$

where w^i is a commuting flow. However, it was not noted in [2] that in the case of weakly nonlinear systems (this is the only exception to Theorem 6) flows with $\partial_i w^i \equiv 0$ do not give the solutions (10) since u_x^i and u_t^i become infinite.

PROPOSITION. A weakly nonlinear system has solutions generated by commuting flows with the condition $\partial_i w^i \not\equiv 0$.

The proof is by parallel differentiation of (10) with respect to x and t.

We consider in more detail example (9). The Born–Infeld equation can be written in the form

$$\begin{cases} u_t^1 = u^2 u_x^1, \\ u_t^2 = u^1 u_x^2. \end{cases} \tag{11}$$

Systems of two equations identically satisfy the condition of semi-Hamiltonicity, and therefore the metric has the form (7). From the requirement $R^i_{kki} = 0$ and the definition of the metric we obtain up to a constant for the Hamiltonian system (11)

$$g_{11} = -g_{22} = 1/(u_1-u_2)^2. \tag{12}$$

Since the operation of covariant differentiation is defined, $w^i = \nabla^i\nabla_i H$, where H in (8) and (9) (see [2]), we can find the commuting flows

$$w^1 = (u_1-u_2)A''(u_1) - A'(u_1) - B'(u_2), \quad w^2 = -(u_1-u_2)B''(u_2) - A'(u_1) - B'(u_2). \tag{13}$$

Taking $w^i = v^i$, we find $A^0(u_1)$ and $B^0(u_2)$, i.e., the Hamiltonian of the system (11):

$$H = \frac{A^0(u_1)+B^0(u_2)}{u_1-u_2}, \quad A^0(u_1) = \frac{A_0}{2}u_1^2, \quad B^0(u_2) = \frac{B_0}{2}u_2^2, \quad A_0+B_0 = -1.$$

From the condition $\partial_i w^i \not\equiv 0$, we find a restriction on the commuting flows for the system (11):

$$A'''(u_1) \neq 0, \quad B'''(u_2) \neq 0. \tag{14}$$

It can be seen that the system (11) is a special case of the system (9) for N = 2. It is natural to consider the multicomponent case

$$u_t^i = \left(\sum_{k=1}^{N} u_k - u_i\right) u_x^i, \quad \Gamma_{ik}^i = \frac{1}{u_i-u_k}, \quad g_{ii} = \frac{\mu_i(u^i)}{\prod\limits_{m\neq i}(u_m-u_i)^2}.$$

From the condition $R^i_{kki} = 0$ for N = 3 we find that μ_i, μ_j, μ_k are constants related by

$$\frac{1}{\mu_i} + \frac{1}{\mu_j} + \frac{1}{\mu_k} = 0, \tag{15}$$

i.e., the metric is a one-parameter family. For N > 3, we can readily show that metrics of zero curvature do not exist, i.e., the systems (9) are semi-Hamiltonian but do not have a Hamiltonian formalism.

We consider the two systems

$$\begin{cases} u_t^i = (u^j+u^k)u_x^i, \\ u_t^j = (u^i+u^k)u_x^j, \\ u_t^k = (u^i+u^j)u_x^k, \end{cases} \tag{16a}$$

$$\begin{cases} u_t^i = u^j u^k u_x^i, \\ u_t^j = u^i u^k u_x^j, \\ u_t^k = u^i u^j u_x^k. \end{cases} \tag{16b}$$

Both systems are a generalization of (11) to the three-component case. The system (16a) goes over into (11) for $u^k = 0$, and (16b) does so for $u^k = 1$. For them,

$$\Gamma_{ik}^i = \frac{1}{u_i-u_k}, \quad g_{ii} = \frac{\mu_i}{(u_i-u_j)^2(u_i-u_k)^2}, \quad H = \sum_{i=1}^{3} \frac{A_i(u^i)}{\prod\limits_{m\neq i}(u_m-u_i)}.$$

The commuting flows have the form

$$w^i = \frac{1}{\mu_i}A_i''(u^i)(u_i-u_j)(u_i-u_k) + \frac{1}{\mu_k}A_k'(u^k)(u_j-u_k) -$$

$$\frac{1}{\mu_j}A_j'(u^j)(u_j-u_k)-\frac{1}{\mu_i}A_i'(u^i)(u_j+u_k-2u_i)+\frac{2}{\mu_i}A_i(u^i)+\frac{2}{\mu_j}A_j(u^j)+\frac{2}{\mu_k}A_k(u^k). \tag{17}$$

As in the previous example, the restriction on the commuting flows is $A_i'''(u^i) \not\equiv 0$, etc. Both systems (16a) and (16b) are weakly nonlinear, and therefore from $A_i'''(u^i) \equiv 0$ we obtain for the Hamiltonian of the systems (16a) and (16b)

$$A_i(u^i)=N_iu_i, \tag{18}$$

where $N_i/\mu_i+N_j/\mu_j+N_k/\mu_k=1$ (in case (16a)), and

$$A_i(u^i)=\frac{M_i}{2}u_i^2, \tag{19}$$

where $M_i/\mu_i+M_j/\mu_j+M_k/\mu_k=1$ (in case (16b)).

It can be seen from (13) and (17) that if the arbitrary functions A_i are twice differentiable then the flows are also twice differentiable functions.

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