

GAUGE THEORIES FOR GRAVITY ON A LINE

R. Jackiw

In Memoriam M. C. Polivanov

1. INTRODUCTION

Professor M. C. Polivanov and I met only a few times, during my infrequent visits to the Soviet Union in the 1970's and 1980's. His hospitality at the Moscow Steklov Institute made the trips a pleasure, while the scientific environment that he provided made them professionally valuable. But it is the human contact that I remember most vividly and shall now miss after his death. At a time when issues of conscience were both pressing for attention and difficult/dangerous to confront, Professor Polivanov made a deep impression with his quiet but adamant commitment to justice. I can only guess at the satisfaction he must have felt when his goal of gaining freedom for Yuri Orlov was attained, and even more so these days when human rights became defensible in his country; it is regrettable that he cannot now enjoy the future that he strived to attain.

One of our joint interests was the Liouville theory [1, 2], which in turn can be viewed as a model for gravity in two-dimensional space-time. Some recent developments in this field are here summarized and dedicated to Polivanov's memory, with the hope that he would have enjoyed knowing about them.

We study lower-dimensional gravity both for pedagogical reasons – one expects that the dimensional reduction affords sufficient simplification to permit thorough analysis, while still retaining enough useful content to inform the physical (3 + 1)-dimensional problem – and also, if one is lucky, for practical applications, e.g., idealized cosmic strings are described by (2 + 1)- dimensional gravity, while the still lower-dimensional models are used in statistical mechanics.

The drastic dimensional reduction to (1 + 1) dimensions – gravity on a line, i.e., *lineal* gravity – is not devoid of interest, provided dynamical equations are not based on the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, which vanishes identically in two dimensions.

In a proposal of several years ago [3], it was suggested that gravity equations be based on the Riemann scalar R , the simplest entity that encodes in two dimensions all local geometric information about space-time. Moreover, in an action formulation it is necessary to introduce an additional scalar field, which acts as a Lagrange multiplier that enforces the equation of motion for R . Thus we are dealing with scalar-tensor theories, or – to use the contemporary string nomenclature – “dilaton” gravities.

Since the initial proposal, various models have been studied. Here I shall describe two that are selected by their group theoretical properties: they can be formulated as gauge theories based on groups relevant to space-time: de Sitter or anti-de Sitter (in (1 + 1) dimensions both groups are $SO(2, 1)$, although the geometries are different) and Poincaré. The first of these is the one proposed originally [3]; it is governed by the action

$$I_1 = \int d^2x \sqrt{-g} \eta (R - \Lambda) . \quad (1)$$

The second is “string-inspired” and has been recently studied for purposes of modeling (on a line!) black hole physics [4]; its action is

$$\bar{I}_2 = \int d^2x \sqrt{-\bar{g}} e^{-2\varphi} (\bar{R} - 4\bar{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \Lambda) . \quad (\bar{2})$$

Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology. Published in *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 92, No. 3, pp. 404–414, September, 1992. Original article submitted June 17, 1992.

(Notation: time and space carry the metric tensor $g_{\mu\nu}$ with signature $(1, -1)$. The two-vector $x^\mu = (t, x)$ will be frequently presented in light-cone components $x^\pm \equiv \frac{1}{\sqrt{2}}(t \pm x)$. Tangent space components are labeled by Latin letters a, b, \dots , and the Minkowski metric tensor $h_{ab} = \text{diag}(1, -1)$ raises/lowers these indices. Also we use the anti-symmetric tensor ϵ^{ab} , $\epsilon^{01} = 1$.)

In (1), R is the scalar curvature built from $g_{\mu\nu}$, η is a world scalar Lagrange multiplier related to the dilaton, while Λ is a cosmological constant. In $(\bar{2})$ we temporarily use an over-bar to denote a differently scaled metric tensor $\bar{g}_{\mu\nu}$ from which \bar{R} is constructed, while φ is the dilaton. Formula $(\bar{2})$ arises naturally from string theory, restricted to a two-dimensional target space, with the anti-symmetric tensor field identically vanishing. In the string context, matter is taken to couple to $\bar{g}_{\mu\nu}$; for our purposes in the absence of matter it is convenient to redefine variables by $\bar{g}_{\mu\nu} = e^{2\varphi} g_{\mu\nu}$, $\eta = e^{-2\varphi}$. Then $(\bar{2})$ becomes

$$I_2 = \int d^2x \sqrt{-g} (\eta R - \Lambda) \quad (2)$$

but it is to be remembered that because of the redefinition, the “physical” metric tensor is $g_{\mu\nu}/(-2\eta)$. Note that (2) is invariant against shifting η by a constant, because $\sqrt{-g} R$ is a total derivative.

It is seen that the two models (1) and (2) differ in the placement of the Lagrange multiplier with the cosmological term: in (1) η multiplies Λ ; in (2) the η factor is absent from Λ . Of course in the limit $\Lambda = 0$, the difference disappears.

We now describe the interesting gauge group structure of (1) and (2) which we name *(anti) de Sitter gravity* and *extended Poincaré gravity*, respectively.

2. (ANTI) DE SITTER GRAVITY

The equations of motion that follow from varying η and $g_{\mu\nu}$ in (1) are

$$R = \Lambda \quad (3)$$

$$(\mathcal{D}_\mu \mathcal{D}_\nu - g_{\mu\nu} \mathcal{D}^2) \eta + \frac{\Lambda}{2} g_{\mu\nu} \eta = 0 \quad (4a)$$

The second equation, with \mathcal{D}_μ the space-time covariant derivative, can be decomposed into traceless and trace parts

$$(\mathcal{D}_\mu \mathcal{D}_\nu - \frac{1}{2} g_{\mu\nu} \mathcal{D}^2) \eta = 0 \quad (4b)$$

$$(\mathcal{D}^2 - \Lambda) \eta = 0 \quad (4c)$$

The above geometric dynamics may be presented in a gauge theoretical fashion [5]. To this end one uses the (anti) de Sitter group with Lorentz generator J and translation generators P_a satisfying the $SO(2,1)$ algebra (for $\Lambda \neq 0$)

$$[P_a, J] = e_a{}^b P_b, \quad [P_a, P_b] = -\frac{\Lambda}{2} \epsilon_{ab} J \quad (5)$$

The gauge connection one-form is introduced $A = A_\mu dx^\mu$ and expanded in terms of the generators,

$$A = e^a P_a + \omega J \quad (6)$$

where e^a_μ is the *Zweibein* and ω_μ is the spin-connection. The curvature two-form

$$F = dA + A^2 \quad (7)$$

becomes

$$F = f^a P_a + fJ = (De)^a P_a + \left(d\omega - \frac{\Lambda}{4} e^a \epsilon_{ab} e^b\right) J \quad (8)$$

$$(De)^a \equiv de^a + \epsilon^a{}_b \omega e^b \quad (9)$$

It is seen that $d\omega$ is the scalar curvature density and f^a is the torsion density, each expressed in terms of e^a and ω , which at this stage are independent variables:

The Lagrange density

$$\mathcal{L}'_1 = \sum_{A=0}^2 \eta_A F^A = \eta_a (De)^a + \eta_2 \left(d\omega - \frac{\Lambda}{4} e^a \epsilon_{ab} e^b\right) \quad (10)$$

$$F^A = (f^a, f), \quad \eta_A = (\eta_a, \eta_2)$$

is gauge invariant: the three field strengths F^A transform covariantly according to the three-dimensional adjoint representation, while the Lagrangian multiplet triplet η_A transforms by the coadjoint representation.

The equation obtained from (10) by varying η_a gives the condition of vanishing torsion and allows evaluating the spin connection in terms of the *Zweibein*

$$\omega = e^a (h_{ab} \epsilon^{\mu\nu} \partial_\mu e_\nu^b) / \det e \quad (11)$$

The equation which follows upon variation of η_2 regains (3) once (11) is used. Variation of e^a and ω produces equations for the Lagrange multipliers η_a and η_2 , respectively, the latter of course coinciding with η in the geometric formulations (1), (3) and (4)

$$d\eta_a + \epsilon_a{}^b \omega \eta_b - \frac{\Lambda}{2} \epsilon_{ab} \eta_2 e^b = 0 \quad (12a)$$

$$d\eta_2 + \eta_a \epsilon^a{}_b \omega e^b = 0 \quad (12b)$$

Upon taking a space-time covariant derivative of (12b) and using (12a) to eliminate η_a , we recover (4). Finally we see that when ω is eliminated from \mathcal{L}'_1 with the help of (11), so that the torsion (9) vanishes, what remains is the Lagrange density of (1), expressed in terms of *Zweibeine*.

Thus the geometric formulation of this gravity theory is contained within the (anti) de Sitter group theoretical framework for solutions with $\det e \neq 0$, but see below.

Explicit classical solutions to the equations are easy to find. Working within the geometric framework, we use coordinate invariance to choose a conformally flat metric tensor

$$g_{\mu\nu} = h_{\mu\nu} \exp 2\sigma \quad (13)$$

Then (3) becomes the Liouville equation,

$$\square\sigma = \frac{\Lambda}{2} \exp 2\sigma \quad (14)$$

studied in [1]. Its general solution depends on two arbitrary functions of the two light-cone variables, $F(x^+)$, $G(x^-)$,

$$\exp 2\sigma = \frac{F'(x^+)G'(x^-)}{\left(1 - \frac{\Lambda}{4} FG\right)^2} \quad (15)$$

whose derivatives fulfill the consistency condition $F'G' > 0$. But the residual coordinate invariance within the conformal gauge allows choosing $F(x^+) = x^+$, $G(x^-) = x^-$, hence

$$\exp 2\sigma = \frac{1}{\left(1 - \frac{\Lambda}{8} x^2\right)^2} \quad (16)$$

In conformal gauge, (4b) reduces to

$$\partial_\mu V_\nu + \partial_\nu V_\mu - h_{\mu\nu} h^{\alpha\beta} \partial_\alpha V_\beta = 0 \quad (17)$$

where V_μ is defined by

$$V_\mu \exp 2\sigma = \partial_\mu \eta \quad (18)$$

Equation (17) is just the (flat-space) conformal Killing equation with solutions in terms of arbitrary functions of a single light-cone variable:

$$V_- = V_-(x^+), \quad V_+ = V_+(x^-) \quad (19)$$

Finally, the remaining equation (4c) together with (18) restricts these functions, so that the solution for η takes the form

$$\eta = \frac{\alpha_a x^a + \alpha_2 \left(1 + \frac{\Lambda}{8} x^2\right)}{1 - \frac{\Lambda}{8} x^2} \quad (20)$$

where α_a is a constant two-vector and α_2 is a constant scalar.

The *Zweibein* and spin connection of the gauge theoretical formulation are given by related formulas. The former, the “square root” of the metric tensor, becomes (apart from an arbitrary Lorentz transformation on the tangent-space indices)

$$e_\mu^a = \delta_\mu^a \exp \sigma = \frac{1}{1 - \frac{\Lambda}{8} x^2} \delta_\mu^a \quad (21)$$

while the latter is

$$\omega_\mu = -h_{\mu\alpha} \epsilon^{\alpha\beta} \partial_\beta \sigma \quad (22)$$

The Lagrange multiplier η_2 coincides with η , while Eq. (12) for η_a is solved by

$$\eta_a \exp \sigma = \epsilon_a^\mu \partial_\mu \eta \quad (23)$$

Of course the general solution is an arbitrary coordinate transformation of the above.

Finally we observe that the gauge theoretical formulation allows an alternative group theoretical presentation of solutions. The field equations following from (10), upon respective variation of η_A and A , are

$$F = 0 \quad (24)$$

$$dH + [A, H] = 0 \quad (25)$$

A , F and $H = \eta_a h^{ab} P_b + \frac{2}{\Lambda} \eta_2 J$ belong to the $SO(2, 1)$ algebra (the factor $2/\Lambda$ is a consequence of the group metric). Equation (24) implies that A is a pure gauge given by an arbitrary element U of the $SO(2, 1)$ group,

$$A = U^{-1} dU \quad (26)$$

while the Lagrange multiplier is then determined by (25) to be

$$H = U^{-1} \Phi U \quad (27)$$

where Φ is a constant element in the algebra. The explicit group and algebra elements that correspond to the above solution, Eqs. (20)–(23), are

$$U = \exp \left(\frac{i\pi}{\sqrt{1 - \frac{\Lambda}{8} x^2}} \left(-\frac{1}{2} x^a \epsilon_a^b P_b + J \right) \right) \quad (28)$$

and

$$\Phi = \frac{2}{\Lambda} \alpha_a \epsilon^{ab} P_b - \alpha_2 J \quad (29)$$

U is unique up to a constant gauge transformation.

Within the gauge theoretical framework, an even simpler solution to (24) and (25) is available: $A = 0$, $H = \Phi$, which makes no sense geometrically: not only $\det e$, but both the connections e^a and ω vanish! But, in fact, use can be made of such solutions: when presented with a geometrically singular configuration, perform any gauge transformation producing non-singular connections, for example, with the group element U above. So we see that the group theoretical framework, even in its $\det e = 0$ sector, contains adequate information for encoding the gravity theory.

3. EXTENDED POINCARÉ GRAVITY

Equations of motion of the string-inspired gravitational theory (2) are, from varying η

$$R = 0 \tag{30}$$

and from varying $g_{\mu\nu}$

$$(\mathcal{D}_\mu \mathcal{D}_\nu - g_{\mu\nu} \mathcal{D}^2)\eta + \frac{\Lambda}{2} g_{\mu\nu} = 0 \tag{31a}$$

which is equivalent to

$$\mathcal{D}_\mu \mathcal{D}_\nu \eta = \frac{\Lambda}{2} g_{\mu\nu} . \tag{31b}$$

Note that (31a) differs from (4a) by the absence of η in the last term.

To give a gauge theoretical formulation [6], we make use of the *centrally extended* Poincaré group, whose algebra is

$$[P_a, J] = \epsilon_a{}^b P_b, \quad [P_a, P_b] = \epsilon_{ab} I \tag{32}$$

where the central element I commutes with P_a and J . Consequently the connection A and curvature F now become

$$A = e^a P_a + \omega J + aI \tag{33}$$

$$F = dA + A^2 = f^a P_a + fJ + gI \tag{34}$$

$$= (De)^a P_a + d\omega J + \left(da + \frac{1}{2} e^a \epsilon_{ab} e^b\right) I .$$

Here a and g are the additional connection and curvature associated with the central element in the algebra.

This magnetic-like extension of the Poincaré group may be viewed as an unconventional contraction of the de Sitter group: The ordinary Poincaré algebra (Eq. (32) without the central element) is the $\Lambda \rightarrow 0$ contraction of the $SO(2, 1)$ algebra (5). However, owing to the well-known ambiguity of two-dimensional angular momentum, in (5) one may replace J by $J - 2I/\Lambda$ before taking the $\Lambda \rightarrow 0$ limit, which then leaves (32).

The extension reflects a 2-cocycle in the composition law for representatives of the Poincaré group. If the group acts on coordinates x^a by

$$x^a \longrightarrow \bar{x}^a = \mathcal{M}^a{}_b x^b + q^a, \tag{35a}$$

where \mathcal{M} is a finite Lorentz transformation

$$\mathcal{M}^a{}_b = \delta^a{}_b \cosh \alpha + \epsilon^a{}_b \sinh \alpha \tag{35b}$$

and q^a is a finite translation, the composition law for these is

$$\mathcal{M}_{(12)} = \mathcal{M}_1 \mathcal{M}_2 \tag{36a}$$

$$q_{(12)} = q_1 + \mathcal{M}_1 q_2 \tag{36b}$$

However, the composition law for a representation $G(\mathcal{M}, q)$ containing the extension (32) in its algebra acquires a 2-cocycle

$$G(\mathcal{M}_1, q_1) G(\mathcal{M}_2, q_2) = \exp\left\{\frac{i}{2} q_1^a \epsilon_{ab} (\mathcal{M}_1 q_2)^b\right\} G(\mathcal{M}_1 \mathcal{M}_2, q_1 + \mathcal{M}_1 q_2) \tag{37}$$

(I is represented by $i = \sqrt{-1}$).

A finite gauge transformation, generated by the gauge function Θ ,

$$\Theta = \theta^a P_a + \alpha J + \beta I , \quad (38)$$

produces the following transformations on the connections:

$$\begin{aligned} e^a &\rightarrow \bar{e}^a = (\mathcal{M}^{-1})^a_b (e^b + \epsilon^b_c \theta^c \omega + d\theta^b) \\ \omega &\rightarrow \bar{\omega} = \omega + d\alpha \\ a &\rightarrow \bar{a} = a - \theta^a \epsilon_{ab} e^b - \frac{1}{2} \theta^2 \omega + d\beta + \frac{1}{2} d\theta^A \epsilon_{ab} \theta^b . \end{aligned} \quad (39)$$

The multiplet of curvatures $F^A = (f^a, f, g)$ transforms by the adjoint 4×4 representation of the extended group,

$$\begin{aligned} f^a &\rightarrow \bar{f}^a = (\mathcal{M}^{-1})^a_b (f^b + \epsilon^b_c \theta^c f) \\ f &\rightarrow \bar{f} = f \\ g &\rightarrow \bar{g} = g - \theta^a \epsilon_{ab} f^b - \frac{1}{2} \theta^2 f \end{aligned} \quad (40)$$

or

$$F^A \rightarrow \bar{F}^A = \sum_{B=0}^3 (U^{-1})^A_B F^B, \quad U = \begin{pmatrix} \mathcal{M}^a_b & -\epsilon^a_c \theta^c & 0 \\ 0 & 1 & 0 \\ \theta^c \epsilon_{cd} \mathcal{M}^d_b & -\theta^2/2 & 1 \end{pmatrix} . \quad (41)$$

The upper left 3×3 block in U comprises the adjoint representation of the conventional Poincaré group with q^a of (35) identified with $-\epsilon^a_c \theta^c$, while the fourth row and column arise from the extension. Note that in the above realization of the gauge action on F , the extension is not visible: I is represented by $\mathbf{0}$. On the other hand, an additional connection and curvature (a, g) are present.

In this representation, the extended algebra possesses a non-singular Killing metric, which is unavailable without the extension

$$h_{AB} = \begin{pmatrix} h_{ab} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} . \quad (42)$$

It is true that ${}^T U h U = h$; this allows raising and lowering the indices (A, B) .

An invariant Lagrange density is now constructed with an extended multiplet of Lagrange multipliers η_A ,

$$\begin{aligned} \mathcal{L}'_2 &= \sum_{A=0}^3 \eta_A F^A = \eta_a (De)^a + \eta_2 d\omega + \eta_3 \left(da + \frac{1}{2} e^a \epsilon_{ab} e^c \right) \\ F^A &= (f^a, f, g), \quad \eta_a = (\eta_a, \eta_2, \eta_3) , \end{aligned} \quad (43)$$

which obey the coadjoint transformation law,

$$\eta_A \rightarrow \bar{\eta}_A = \sum_{B=0}^3 \eta_B U^B_A \quad (44)$$

or in components

$$\begin{aligned} \eta_a &\rightarrow \bar{\eta}_a = (\eta_b - \eta_3 \epsilon_{bc} \theta^c) \mathcal{M}^b_a \\ \eta_2 &\rightarrow \bar{\eta}_2 = \eta_2 - \eta_a \epsilon^a_b \theta^b - \frac{1}{2} \eta_3 \theta^2 \\ \eta_3 &\rightarrow \bar{\eta}_3 = \eta_3 . \end{aligned} \quad (45)$$

Using the invariant metric (42), other group invariants may be constructed

$$\mathcal{F}^2 = \sum_{A,B=0}^3 *F^A h_{AB} F^B \quad (46)$$

$$M = -\frac{2}{\Lambda} \sum_{A,B=0}^3 \eta_A h^{AB} \eta_B \quad (47)$$

where $*F^A$ is the 0-form $\frac{1}{2}\epsilon^{\mu\nu} F_{\mu\nu}^A$, dual to the 2-form F^A .

We recognize in (43) the torsion $(De)^a$ and curvature $d\omega$ densities, which vanish as a consequence of varying η_a and η_2 , respectively. Thus Eq. (30) is regained. The Lagrange multiplier η in (2) corresponds to η_2 in the present formulas and the equation for it, obtained by varying ω , is as in the (anti) de Sitter model, (12b),

$$d\eta_2 + \eta_a \epsilon^a{}_b e^b = 0 \quad , \quad (48a)$$

while the equation for η_a , obtained by varying e^a , differs from (12a),

$$d\eta_a + \epsilon_a{}^b \omega \eta_b + \eta_3 \epsilon_{ab} e^b = 0 \quad . \quad (48b)$$

We need a value for η_3 to close the system (48). The equation for that multiplier is obtained by varying a ,

$$d\eta_3 = 0 \quad , \quad (48c)$$

and a constant, cosmological solution

$$\eta_3 = -\frac{\Lambda}{2} \quad (48d)$$

renders (48b) similar to (12a),

$$d\eta_a + \epsilon_a{}^b \omega \eta_b - \frac{\Lambda}{2} \epsilon_{ab} e^b = 0 \quad (48e)$$

except that there is no factor of η_2 in the last, cosmological term of (48e). This, of course, has the consequence that when (48a) and (48e) are combined as before, the second-order equation that emerges for $\eta = \eta_2$ reproduces (31).

The remaining equation of the gauge theoretical formulation, obtained by varying η_3

$$da = -\frac{1}{2} e^a \epsilon_{ab} e^b \quad (49)$$

and allowing evaluation of a , has no counterpart in the geometric formulation. Equation (49) can always be locally integrated because the right side is a two-form, hence closed in two dimensions. However, in general, there will be singularities in a , since upon integrating (49) over a two-space, the right side gives the total “volume,” which could be a well-defined non-vanishing quantity, while the left side always integrates to zero if the manifold is closed and bounded, and a is non-singular.

Note that upon eliminating ω in \mathcal{L}'_2 with the zero-torsion equation $(De)^a = 0$ and evaluating η_3 at $-\Lambda/2$, \mathcal{L}'_2 coincides with the Lagrange density in (2), now expressed in terms of *Zweibeine*, apart from the total derivative $-\Lambda/2 da$, which does not contribute to equations of motion.

Thus, here again, the group theoretical formulation reproduces the geometric one for solutions with $\det e \neq 0$, but again see below. However, the former is more flexible: Eq. (48c) is satisfied with vanishing η_3 ; this corresponds to a vanishing cosmological constant. Thus the gauge theory built on the *extended* Poincaré group possesses as a solution a *non-extended* system. It is interesting therefore that here the cosmological term is an integration constant, and not inserted *a priori* into the theory.

Finding explicit solutions is straightforward. In the geometric formulation, (3) is solved by a flat metric tensor

$$g_{\mu\nu} = h_{\mu\nu} \quad . \quad (50)$$

Then (31) immediately gives

$$-2\eta = M - \frac{\Lambda}{2}(x - x_0)^2 \quad (51)$$

with M and x_0 being integration constants, the former reflecting the η -translation invariance mentioned earlier.

Interest in the model [4] derives precisely from the above “black-hole” solution with mass M [in terms of the “physical” metric $g_{\mu\nu}/(-2\eta)$], located at x_0 . An arbitrary coordinate transformation of this configuration produces the general solution.

The gauge theoretical counterparts of the above are a flat *Zweibein* (apart from a constant tangent-space Lorentz transformation)

$$e_\mu^a = \delta_\mu^a \quad (52)$$

and a vanishing spin connection

$$\omega = 0 \quad (53)$$

Taking in (48c) the cosmological solution for η_3 allows solving (48e) for η_a

$$\eta_a = \frac{\Lambda}{2}\epsilon_{a\mu}(x^\mu - x_0^\mu) \quad (54)$$

and from (48a) $\eta_2 = \eta$ is recovered to be as in (51). Finally (49) is solved for a

$$a_\mu = \frac{1}{2}\epsilon_{\mu\nu}x^\nu \quad (55)$$

with a pure gauge contribution $\partial_\mu\chi$ left arbitrary. The potential in (55) corresponds to a constant “magnetic field,” as is appropriate with our “magnetic-like” extension of translations.

Note the two invariants defined in (46) and (47): \mathcal{F}^2 vanishes since F^A does, while M is recognized as the “black hole” mass.

The gauge theoretical solution may, of course, also be presented in a group theoretical fashion, since the equations are of the same form as in (24) and (25), with all quantities belonging to the *extended* algebra and group. The explicit formulas, corresponding to the “black hole” solution, Eqs. (50) – (55), are as follows. The group element U that leads to the pure gauge connection $A = U^{-1}dU$ is

$$U = \exp x^a P_a \quad (56)$$

up to a constant gauge transformation. The constant algebra element Φ that gives $H = \eta_a h^{ab} P_b - \eta_3 J - \eta_2 I = U^{-1}\Phi U$ is (placement of η_2 and η_3 dictated by the group metric (42), *viz.* $\eta^A = h^{AB}\eta_B$)

$$\Phi = \frac{\Lambda}{2}x_0^a \epsilon_a{}^b P_b + \frac{\Lambda}{2}J + \left(\frac{M}{2} - \frac{\Lambda}{4}x_0^2\right)I \quad (57)$$

As in the (anti) de Sitter model, we see that after a further gauge transformation we pass to the geometrically singular configuration $A = 0$, $H = \Phi$. This gives an especially succinct account of the relevant geometric information: Φ encodes the integration constants, which characterize the intrinsic geometry – the cosmological constant Λ , the “black hole” mass M and location x_0 . A geometry is built with these characteristics once a gauge transformation is performed, say with the above U , to obtain non-singular connections.

4. CONCLUSION

The two models considered here are special: their geometric dynamics possess a gauge theoretical formulation. The extended Poincaré model exhibits the intriguing possibility of a cosmological term that

is an integration constant, as are the “black hole” mass M and location x_0 ; all three are encoded in the Lagrange multipliers of the theory.

Both models can also be obtained by dimensional reduction from $(2 + 1)$ dimensions: To obtain (anti) de Sitter gravity in its geometric formulation one begins [3] with the Einstein theory/Hilbert action (with cosmological term), suppresses dependence on the third dimension, sets $g_{\mu 2}$ to zero for $\mu = 0, 1$ and g_{22} to η^2 ; for the gauge theoretical formulation one starts with the *Dreibein*-spin connection form of the theory, which also is equivalent to a Chern–Simons, $\mathcal{O}(2, 2)$ or $\mathcal{O}(3, 1)$ model [7]. Extended Poincaré gravity can be similarly constructed, but the higher-dimensional theory has to be suitably extended by an Abelian ideal. Indeed it is found that *both* the (anti) de Sitter and extended Poincaré $(1 + 1)$ -dimensional theories arise as *different* dimensional reductions of a *single*, extended $(2 + 1)$ -dimensional gravity [8]. This and another interesting topic – the coupling of matter consistently with the gauge principle [9] – are beyond the scope of our review. In yet a further investigation one could study non-topological theories in which invariants (46) and/or (47) are added to the Lagrange density (43).

In conclusion, we note that dynamics determined by a group has been familiar in physics since the invention of the Yang–Mills theory. However, the examples described here offer a new possibility: in the Lie algebra that determines a gauge theory one can allow an extension. This gives rise to richer dynamics within the same group theoretical structure, and in the gravity model studied above produces the cosmological constant.

ACKNOWLEDGEMENTS

The review was prepared with the assistance of D. Cangemi, particularly in finding the explicit solutions; this I gratefully acknowledge.

This work is supported in part by funds provided by the U. S. Department of Energy (D.O.E.) under contract #DE-AC02-76ER03069.

REFERENCES

- [1] Dzhordzhadze G., Pogrebkov A., Polivanov M., *General Solutions of the Cauchy Problem for the Liouville Equation* $\varphi_{tt}(t, x) - \varphi_{xx}(t, x) = (1/2)m \exp \varphi(t, x)$., Dokl. Akad. Nauk. SSSR. **243** (1978), 318; Sov. Phys. Dokl. **23** (1978), 828; *Singular Solutions of the Equation* $\square \varphi + (m^2/2) \exp \varphi = 0$ and *Dynamics of Singularities.*, Theor. Mat. Fiz. **40** (1979), 221; Theor. Math. Phys. **40** (1979), 706.
- [2] D’Hoker E., Jackiw R., *Classical and Quantal Liouville Field Theory.*, Phys. Rev. D. **26** (1982), 3517; *Space-Translation Breaking and Compactification in the Liouville Theory.*, Phys. Rev.Lett. **50** (1983), 1719; D’Hoker E., Freedman D., Jackiw R., *SO(2 + 1)-Invariant Quantization of the Liouville Theory.*, Phys. Rev.D. **28** (1983), 2583.
- [3] Teitelboim C., *Gravitation and Hamiltonian Structure in Two Space-Time Dimensions.*, Phys. Lett. **126B** (1983), 41; *The Hamiltonian Structure of Two-Dimensional Space-Time and Its Relation with the Conformal Anomaly.*, Quantum Theory of Gravity, Ed. S. Christensen. Bristol: Adam Hilger, 1984; Jackiw R., *Liouville Field Theory: A Two-Dimensional Model for Gravity?*, Quantum Theory of Gravity, Ed. S.Christensen. Bristol: Adam Hilger, 1984; *Lower-Dimensional Gravity.*, Nucl. Phys. vol. B252, 1985, p. 343.
- [4] Verlinde H., *Black Holes and Strings in Two Dimensions.*, The Sixth Marcel Grossman Meeting on General Relativity, Ed. H. Sato. Singapore: World Scientific, 1992; Callan C., Giddings S., Harvey A., Strominger A., *Evanescent Black Holes*, vol. 45, 1992, p. 1005 (Quantization of the model is discussed in these papers).
- [5] Fukuyama T., Kamimura K., *Gauge Theory of Two-Dimensional Gravity.*, Phys. Lett. **160B** (1985), 259; Isler K., Truengerberger C., *Gauge Theory of Two-Dimensional Quantum Gravity.*, Phys. Rev. Lett. **63** (1989), 834; Chamseddine A., Wyler D., *Gauge Theory of Topological Gravity in 1 + 1 Dimensions.*, Phys. Lett. B **228** (1989), 75 (Quantization of the model is discussed in these papers).
- [6] Cangemi D., Jackiw R., *Gauge Invariant Formulations of Lineal Gravity.*, Phys. Rev. Lett. (to appear).
- [7] Achúcarro A., Townsend P., *A Chern–Simons Action for Three-Dimensional anti-de Sitter Supergravity Theories.*, Phys. Lett. B **180** (1986), 89; Witten E., *2 + 1-Dimensional Gravity as an Exactly Solvable System.*, Nucl. Phys. **B311** (1988/89), 46.
- [8] Cangemi D. (to appear).
- [9] Grignani G., Nardelli G. (to appear).