SHOCK WAVES IN ONE-DIMENSIONAL MODELS WITH CUBIC NONLINEARITY

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Shock waves are described qualitatively for a class of one-dimensional models with cubic nonlinearity (of the type of the modified Korteweg—de Vries equation): $u_t - 6u^2u_x + \gamma u_{xxx} = \nu u_{xx}$. Both the integrable and the nonintegrable case are considered. The behavior of a shock wave in the limit $t \to \infty$ is considered.

INTRODUCTION

The analytic description of shock waves is a classical problem of mathematical physics that goes back to Riemann. Traditional examples apply to gas dynamics. In these examples, the mathematical representation of a shock wave is a discontinuous (generalized) solution $u(x, t)$ of a partial differential equation.

From the physical point of view, such solution behavior is characteristic for models without dispersion and without viscosity, for example, for the ordinary wave equation $u_{tt} - u_{xx} = 0$. In the wave equation, discontinuities present in the initial condition propagate with constant velocity, but "new" discontinuities are not formed. The reason for this is the linearity of the wave equation.

In nonlinear problems of gas dynamics and hydrodynamics, the formation of new discontinuities is associated with the phenomenon of "breaking."

As an illustration, we consider the modified Hopf equation

$$u_t - 6u^2 u_x = 0 \tag{1}$$

with a smooth initial condition monotonic in x having the form

$$u(x, 0) \to u_\pm, \qquad x \to \pm\infty, \tag{2}$$

$$0 \le u_- < u_+, \quad u_\pm = \text{const}. \tag{3}$$

It is readily seen that after a finite time t the solution $u(x, t)$ of the Cauchy problem (1)—(2) "breaks," and a shock wave of finite amplitude $\hat{A} = u_+ - u_-$ is formed and propagates "to the left" with the constant velocity

$$k = -2 \cdot \frac{u_+^3 - u_-^3}{u_+ - u_-}. \tag{4}$$

If the inequality in (3) has the opposite sign, $0 \le u_- > u_+$, the solution goes over with the passage of time to the self-similar regime $u = u(x/t)$, i.e., it becomes a "rarefaction wave" or a "simple Riemann wave" (the terminology adopted in gas dynamics [1]).

For the numerical and analytic study of shock waves in gas dynamics and hydrodynamics the method of "vanishing viscosity," is used, i.e., to the original (as a rule, hyperbolic) system of equations a viscosity term of the type νu_{xx}, $\nu > 0$, is added, and the limit $\nu \to 0$ is investigated. The idea behind this method is that viscosity "smooths" a discontinuity, and one can hope for approximation of a shock wave by smooth solutions of a dissipative system (see [1]).

For Eq. (1), the viscosity regularization has the form

$$u_t - 6u^2 u_x = \nu u_{xx}, \qquad \nu > 0. \tag{5}$$

This is the modified (cubic nonlinearity) form of the well-known Burgers equation (MB) (see [2]).

It is easy to show that the MB equation (5) in the case (2), (3) has smooth solutions of the traveling-wave type $u(x - kt)$ with the same velocity $k < 0$ as in (4). In accordance with the results of [3], the solutions $u_\nu(x, t)$ of the MB model approach

St. Petersburg Branch of the V. A. Steklov Mathematics Institute, Russian Academy of Sciences. Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 97, No. 2, pp. 191-212, November, 1993. Original article submitted July 7, 1992.

certain solutions of (1) in the limit $\nu \to 0$.

Equations (1) and (5) are models without allowance for dispersion. Allowance for dispersion introduces qualitative changes into the shock-wave structure. This was first noted in the fifties by Sagdeev. Gurevich and Pitaevskii (see [4]) studied in more detail the effect of "dispersion" oscillations (collisionless shock wave) for the simple example of the Korteweg—de Vries (KdV) equation.

For our original equation (1), the corresponding dispersion regularization is

$$u_t - 6u^2 u_x + \gamma u_{xxx} = 0. \tag{6}$$

For $\gamma = 1$, this is the canonical expression of the integrable modified KdV equation (MKdV) [4]. The shock waves in the model (6), (2) inherit many qualitative features of KdV theory. However, there are also differences, and these will be described below (see also [5,6]).

Finally, one can make a "synthetic" regularization of the model (1) by taking into account both dispersion and viscosity effects (see [7]):

$$u_t - 6u^2 u_x + u_{xxx} = \nu u_{xx}, \qquad \nu > 0. \tag{7}$$

This is the MKdV—Burgers model. We must expect that, depending on the choice of the parameter ν, the solutions of the model (7), (2) will preserve certain features of the models (5) and (6). This expectation will be made more precise in the text.

This paper is devoted to study of the qualitative structure of shock waves* in the models (5), (6), and (7) and clarification of the specific features that are introduced (compared with the KdV case) by the cubic nonlinearity in the behavior of shock waves. We shall pay particular attention to the asymptotic behavior as $t \to \infty$. The simple self-similar regime established at long times can be interpreted as an attractor of our system; such an interpretation is valid not only in the dissipative but also in the integrable case [6,8,3,5].

We note that the case of the opposite sign of the nonlinearity, which leads to a different picture of shock waves, was investigated earlier in [5].

Because all the considered models are invariant with respect to the substitution $u \to -u$, we shall assume throughout that

$$u_- \geq 0. \tag{8}$$

This does not affect the generality of our analysis.

1. MODIFIED BURGERS EQUATION

In this section, we investigate traveling shock waves

$$u = u_0(x - kt), \qquad \tau := x - kt, \tag{9}$$

with asymptotic behavior (2) for the model (5). Substitution of the ansatz (9) in (5) leads after one integration to the equation

$$\nu u_\tau = F(u), \qquad F(u) = -2u^3 - ku - A, \qquad A = \text{const}. \tag{10}$$

Taking into account the boundary conditions (2), we find that the velocity k is given by the expression (4). It follows from (10) that u_+ and u_- are two neighboring zeros of the function $F(u)$ and that the sum of all three zeros is $u_+ + u_- + u_0 = 0$. In addition, on the solution $u_0(\tau)$ we necessarily have

$$\operatorname{sign} F(u) = \operatorname{sign}(u_+ - u_-). \tag{11}$$

It follows from this and the form of the function $F(u)$ that the position of the zeros u_+ and u_- in the graph of $F(u)$ is determined by one of two possibilities (Figs. 1 and 2).

With allowance for Figs. 1 and 2, it is more convenient to formulate the condition (11) in the form of the inequality

$$F'(u)\big|_{u=u_-} \geq 0. \tag{12}$$

We then obtain

Proposition 1. *Equation (5) has a solution of the type of a traveling "shock" wave* (2), (9) *in one of two cases*:

$$u_+ > u_-, \tag{13.1}$$

*In all that follows, we understand by "shock wave" a solution $u(x, t)$ with discontinuous asymptotic behaviors (2) as $x \to \pm\infty$.

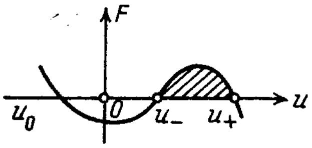

Fig 1

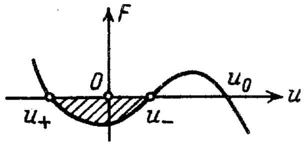

Fig 2

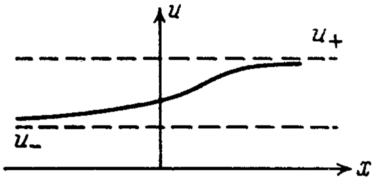

Fig 3

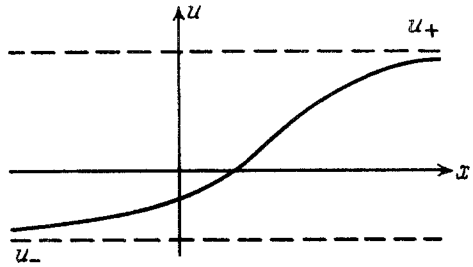

Fig 4

$$u_+ \leq -2u_-. \tag{13.2}$$

The function $u(\tau)$ is monotonic and smooth with respect to τ (Figs. 3 and 4).

We shall not describe here other types of shock waves for the MB model (5).

The traveling shock wave considered above possesses the property of "orbital" stability. More precisely we have

Proposition 2. *Consider the following perturbation of the wave* (9), (13):

$$u(x,0) = u_0(x) + v(x), \qquad v(x) \to 0, \qquad |x| \to \infty. \tag{14}$$

The function $v(x)$ is assumed to decrease rapidly. Then the solution of the Cauchy problem (5), (14) *has asymptotic behavior*

$$u(x,t) \to u_0(\tau + C), \qquad C = \text{const}, \tag{15}$$

where $C \in \mathbf{R}$ is determined by the condition

$$\int_{-\infty}^{\infty} (u(x,0) - u_0(x+C))\, dx = 0. \tag{16}$$

The proof of this theorem follows from the general results of [3]. We shall not give it here. We merely mention that the very fact of nontrivial stability "up to a shift" can be readily deduced from the following heuristic argument.

We consider a small perturbation of the solution u_0:

$$u(x,t) = u_0(\tau) + \varepsilon v(\tau), \qquad \varepsilon \ll 1, \qquad v(\tau) \to 0, \qquad |\tau| \to \infty.$$

The linearized MB equation is

$$v_t - kv_\tau + 6(u_0^2 v)_\tau = \nu v_{\tau\tau}. \tag{17}$$

We seek a solution of it by the Fourier method:

$$v = e^{-\lambda t} w(\tau). \tag{18}$$

Then for the function $w(\tau)$ we obtain

$$\left(-\lambda + 6(u_0^2)_\tau\right) w + (6u_0^2 - k) w_\tau = \nu w_{\tau\tau}.$$

By means of a Liouville transformation

$$w = \psi \cdot \exp\left(\frac{1}{2\nu} \int_{\tau_0}^{\tau} (6u_0^2 - k)\, d\tau\right) \tag{19}$$

this equation can be reduced to the ordinary Schrödinger equation

$$-\nu\psi_{\tau\tau} + p(\tau)\psi = \lambda\psi, \tag{20}$$

with potential

$$p(\tau) = \left((6u_0^2)_\tau + \tfrac{1}{4\nu}(6u_0^2 - k)^2\right) \to p_\pm > 0, \tau \to \pm\infty.$$

It is known from quantum mechanics that the wave function $\psi_0(\tau)$ corresponding to the minimal eigenvalue λ_0 does not have zeros (see, for example, [9]). On the other hand, such a function ψ_0 is well known — it corresponds to $\lambda_0=0$ and can be calculated in accordance with the expression (19) from the solution $v_0(\tau)=w(\tau)$ of the linearization (17):

$$v_0(\tau) = \partial_\tau u_0(\tau). \tag{21}$$

[For complete rigor, we should also verify that the constructed function $\psi_0(\tau)$ belongs to $L_2(\mathbf{R})$, but this is an elementary calculation.] Since the function $v_0(\tau)$ and, therefore, $\psi_0(\tau)$ does not have zeros on the axis τ, it follows that $\lambda_0=0$ is truly the minimum eigenvalue, and therefore $v_0(\tau)$ is the unique Fourier harmonic (18) that does not decrease as $t \to +\infty$. However, by the very definition the vector $v_0=\partial u_0/\partial\tau$, which belongs to the tangent space to $u_0(\tau)$, is responsible for only a phase shift $\tau \to \tau + C$ (see [10])!.

Proposition 2 shows that a solution of the type of a traveling shock wave is an attractor of the dissipative MB system.

2. THE MKdV—BURGERS MODEL

We now discuss the structure of the shock waves in the model (7). As before, we restrict ourselves to the class of traveling waves $u=u(x-kt)$. Substituting the ansatz (9) in (7) and integrating once, we obtain

$$-\nu u_\tau + u_{\tau\tau} = -F(u), \tag{22}$$

where the function $F(u)$ has the form (10). It follows from this, first, that the velocity k is, as before, given by (4). Second, making in (22) the substitution $\tau \to -\tau$, we obtain the equation

$$u_{\tau\tau} = -\frac{\partial \hat{U}(u)}{\partial u} - \nu u_\tau, \qquad \hat{U} = -\left(\frac{u^4}{2} + \frac{ku^2}{2} + Au\right), \qquad A = \text{const}, \tag{23}$$

which describes motion (with friction) in the field of the potential $\hat{U}(u)$. The boundary conditions are

$$u \to \underset{(+)}{u_-}, \qquad \tau \to \underset{(-)}{+}\infty. \tag{24}$$

The graph of the potential $\hat{U}(u)$ is shown in Fig. 5.

It is obvious that for fulfillment of (24) u_+ and u_- must coincide with points of extrumum of the function $\hat{U}(u)$, and the point u_- must be a minimum, i.e.,

$$\hat{U}''(u)\Big|_{u=u_-} \geq 0. \tag{25}$$

The last condition is obviously identical to (12). It is interesting that the criterion (12) does not depend on the presence of dispersion. Thus, we have

Proposition 3. *The conditions for the existence of traveling shock waves* (2), (9) *in the model* (7) *are the same as for* (13), *i.e., there are two different types of shock wave depending on the choice of* u_+ *in Fig.* 5.

We discuss the nature of these solutions $u_0(\tau)$. It is to be expected that for $\nu \gg 1$ the function $u_0(\tau)$ resembles the monotonic shock waves shown in Figs. 3 and 4. On the other hand, for $\nu \ll 1$ we must expect the appearance of the oscillations typical for dispersion problems. This does occur. The graphs $u_0(\tau)$ in the cases $\nu \geq \nu_{cr}$ and $\nu < \nu_{cr}$ are shown in Figs. 6 and 7 in the case $u_- < u_+$. Of course, such behavior of the function $u_0(\tau)$ is readily predicted on the basis of the mechanical interpretation of (23).

We calculate the value ν_{cr} of the critical viscosity. Linearizing (7) on the background of the constant $u=c$, we obtain

$$u = c + \varepsilon v(\tau), \qquad \varepsilon \ll 1, \qquad c = \{u_+, u_-\}.$$

The characteristic exponents of this equation are

$$\lambda_{1,2}(c) = \frac{\nu \pm \sqrt{\nu^2 + 4(k + 6c^2)}}{2}. \tag{26}$$

It follows from this that $\lambda_{1,2}(u_-) \in \mathbf{R}$, while $\lambda_{1,2}(u_+) \in \mathbf{R}$ for $\nu \geq \nu_{cr}$ and have nonvanishing imaginary part when $\nu < \nu_{cr}$, where

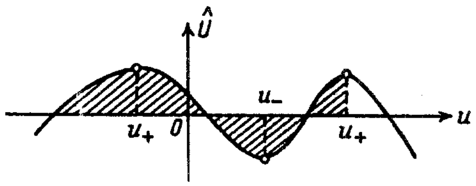

Fig 5

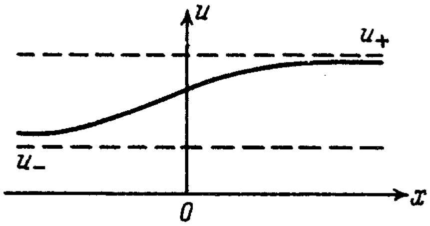

Fig 6

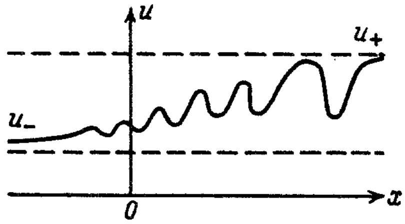

Fig 7

$$\nu_{\rm cr} = 2\sqrt{|6u_-^2 + k|}. \tag{27}$$

If in (7) we change the sign of the dispersion term u_{xxx}, then the condition for the existence of a shock wave is changed (see [5]). However, the value of the critical viscosity $\nu_{\rm cr}^-$ in this case is equal to (27). (In [5] it was incorrectly stated that $\nu_{\rm cr}^- > \nu_{\rm cr}$.)

It would be interesting to prove a theorem on the attractor nature of the solution $u_0(x-kt)$ as we did above for the MB model. That the result is true, at least for $\nu \geq \nu_{\rm cr}$, is intuitively clear (and, in the case of the KdV—B model, it has been rigorously proved [8]). The most interesting case is that of low viscosity, $\nu < \nu_{\rm cr}$, and the limit $\nu \to 0$, since for $\nu = 0$ the shock waves acquire a qualitatively new nature (collisionless or dissipationless shock waves). We now turn to the description of them.

3. THE MKdV MODEL. SCATTERING THEORY

The integrable model (6) with $\gamma = 1$ admits the Lax representation

$$\psi_x = U(\lambda)\psi, \qquad \psi_t = V(\lambda)\psi \tag{28}$$

with U—V pair of operators of the form [4]

$$U(\lambda) = -i\lambda\sigma_3 + \begin{pmatrix} 0 & iu \\ -iu & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V(\lambda) = 4\lambda^2 U(\lambda) + \lambda V_1 + V_0. \tag{29}$$

We do not need the precise expressions for V_1 and V_0. As before, we attempt to find a solution of the type of the traveling shock wave (2), (9). Repeating the arguments of Sec. 2, we arrive at the equation of motion of a material point in the field $\hat{U}(u)$:

$$u_{\tau\tau} = -\frac{\partial \widehat{U}(u)}{\partial u}. \tag{30}$$

Since there is no "friction" in this case [$\nu = 0$ in (23)], the only type of solution $u(\tau)$ with boundary conditions (2) for $u_+ \neq u_-$ corresponds to a "transition" from one peak of the potential $\hat{U}(u)$ to the other peak (see Fig. 5); moreover, in accordance with the conservation of energy

$$\widehat{U}(u_+) = \widehat{U}(u_-),$$

where u_+ and u_- are the coordinates of the maxima of $\hat{U}(u)$. A simple analysis taking into account (23) shows that this is possible only if $A = 0$ and

$$u_+ = -u_-. \tag{31}$$

Thus, the only type of traveling shock wave $u_0(\tau)$ in the MKdV model

$$u_t - 6u^2 u_x + u_{xxx} = 0 \tag{32}$$

has the kink form (31). It is also easy to find an explicit expression for this solution (see, for example, [11]). In the spectral interpretation, this solution $u_0(\tau)$ corresponds to the point $\lambda=0$ of the discrete spectrum for the U operator (29) with potential of the form (2), (31).

It can be shown that this solution is an attractor as $t\to\infty$ for the model (32) with the boundary conditions (2), (31) provided there are no other points of the discrete spectrum of the U operator. The proof of this fact is standard for soliton theory [4] and is based on the observation that the solitonless part of the solution is damped at large times.

The question arises of how we describe other [different from (31)] types of shock wave. For example, how do we describe the simplest shock wave (2) of the form

$$u_+ > u_- \geq 0? \tag{33}$$

The numerical experiment of [11] showed that the behavior of the solution $u(x, t)$ of the Cauchy problem for the model (32) with initial condition $u(x, 0)$ of the type (2), (33) qualitatively resembles the KdV case, namely, there arises a region of oscillations that expands in a self-similar manner with increasing t.

What is the asymptotic behavior of the solution $u(x, t)$ as $t\to+\infty$? Some results on the behavior of the shock wave $u(x, t)$ in the neighborhood of the wave front (in the region of large-scale, solitonlike oscillations) were obtained in [12]. The problem of obtaining a uniform asymptotic behavior (for all x) remained open.

In this section, we describe the leading term of the uniform (in x) asymptotic behavior of $u(x, t)$ as $t\to\infty$ in the absence of a discrete spectrum. Our exposition is based on the methods developed in the theory of the KdV equation [13] and the nonlinear Schrödinger equation (NS equation) [14].

In agreement with the general philosophy of the inverse scattering method, we first construct a form of scattering theory for the U operator (29) with potential (2), (33). For this we determine the Jost solutions ψ_+ and ψ_-, which are vector solutions of the system (28) determined by the asymptotic behaviors

$$\psi_\pm(x,t,\lambda) \to e_\pm, \qquad x \to \pm\infty, \qquad e_{\underset{(-)}{+}} = \begin{pmatrix} 1 \\ \dfrac{(-\lambda + z_{\underset{(-)}{+}})}{u_{\underset{(-)}{+}}} \end{pmatrix} \exp\left(-iz_{\underset{(-)}{+}}\left(x + 4\left(\lambda^2 + \frac{u^2_{\underset{(-)}{+}}}{2}\right)t\right)\right). \tag{34}$$

Here,

$$z_{\underset{(-)}{+}}(\lambda) = \sqrt{\lambda^2 - u^2_{\underset{(-)}{+}}}$$

are equations of spectral curves Γ_+ and Γ_- of genus zero. We stipulate that

$$\operatorname{Im} z_{\underset{(-)}{+}} \geq 0$$

on the upper sheets Γ^+ of these Riemann surfaces. We make the natural identification of the upper Γ^+ and also lower Γ^- sheets on the curves Γ_+, Γ_-.

It can be shown in the standard manner [4] that the function $\psi_+(\lambda)$ is analytic on the lower sheet $\Gamma^-\backslash\infty^-$, and the function $\psi_-(\lambda)$ analytic on $\Gamma^+\backslash\infty^+$. On the continuous spectrum E_-

$$E_{\underset{(+)}{-}} = \left]-\infty, -u_{\underset{(+)}{-}}\right] \cup \left[u_{\underset{(+)}{-}}, +\infty\right[$$

we have the scattering relation

$$\psi_+(P) = \psi_-(P)a(P) + \psi_-(\sigma P)b(P), \qquad \lambda \in E_-. \tag{35}$$

Here, $P=(\lambda, z_-)$ is a point of the curve Γ_-, σ is the involution that interchanges the sheets; and $a(P)$ and $b(P)$ are certain scalar functions. It is readily seen that

$$a(P)a(\sigma P) - b(P)b(\sigma P) = \frac{z_+}{z_-}(P), \qquad \lambda \in E_-,$$

and that

$$a(P) = \{\psi_+(P), \psi_-(\sigma P)\}/z_-(P), \quad b(P) = -\{\psi_+(P), \psi_-(P)\}/z_+(P), \tag{36}$$

where $\{\varphi, \psi\} := \det(\varphi, \psi)$.

It follows from (36) that the function $a(P)$ can be analytically continued to the sheet Γ^-, and that

$$a(P) \to 1, \quad b(P) \to 0 \quad \text{as} \quad P \to \infty^{\pm}. \tag{37}$$

The function $a(P)$ may have on Γ^- a finite number of simple zeros that belong to the interval $]-u_-, u_-[$ and form the discrete spectrum of the U operator (29). In this paper, we assume that $a(P) \neq 0$ on Γ^-, which corresponds to consideration of the solitonless section.

We shall assume that in the neighborhood of the ends of the spectrum E_- the function $a(P)$ has a simple pole

$$(\text{in the local variable } a(P) \sim 1/k, \quad \lambda(P) \sim \underset{(-)}{+} u_- \; k = \sqrt{\lambda \overset{-}{(+)} u_-}). \tag{38}$$

In accordance with (36), this corresponds to the general situation. The condition (38) is usually called the condition of "absence of virtual roots." In terms of the reflection coefficient $r(P)$,

$$r(P) := b(P)/a(P),$$

the condition (38) has the form

$$r(\pm u_-) = -1. \tag{39}$$

Let τ be the antiinvolution of complex conjugation, which does not interchange the sheets Γ_-, i.e., τ: $\lambda \to \bar{\lambda}$. It is easy to show that

$$\begin{cases} a(P) = b(\tau P), \\ r(P) r(\tau P) = 1, \quad \lambda(P) \in E_*, \end{cases} \tag{40}$$

on the simple branch of the spectrum $E_* = E_- \setminus E_+$. In addition, we have the inequalities

$$0 \le r(P) r(\sigma P) < 1, \quad \lambda(P) \in E_+. \tag{41}$$

Note that for $\lambda(P) \in E_+$ the operations τ and σ act in the same way: $\tau P = \sigma P$.

Besides the fairly familiar (see [12,14]) symmetries τ and σ, our model has a further symmetry — the reflection

$$\pi : \lambda \to -\lambda.$$

We stipulate that π does not change the sheets Γ. The obvious π symmetry of the U—V pair (29) leads to the following reduction of the Jost functions ψ:

$$\sigma_2 \psi_{\underset{(-)}{+}}(\pi P) = \psi_{\underset{(+)}{-}}(P) m_{\underset{(+)}{-}}(P), \tag{42}$$

with some nondegenerate function

$$m_{\underset{(+)}{-}}(P),$$

which does not depend on x and t. Here and in what follows, σ_i, $i = 1, 2, 3$, are Pauli matrices. The reduction (42) entails the π symmetry of the scattering coefficients $a(P)$, $b(P)$, but we do not need these expressions.

4. THE MKdV MODEL. MATRIX RIEMANN PROBLEM

The general scheme of [13,6] for investigating shock waves for integrable systems with real spectrum is based on solution of the inverse scattering problem in the form of a matrix Riemann problem. We formulate the Riemann problem that we need for our purposes.

1. Let $\Psi(P)$ be a 2×2 matrix that is a piecewise analytic function on $\mathbb{C}$ and as $P \to \infty^{\overset{+}{(-)}}$ has asymptotic behavior of the form

$$\Psi(P) \to A_{\substack{+\\(-)}} \left(I + \frac{V^{\substack{+\\(-)}}}{\lambda} + O(\lambda^{-2}) \right) \exp\left(\underset{(-)}{+} i\lambda\sigma_3(x + 4\lambda^2 t) \right),$$

$$A_+ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad A_- = \operatorname{diag}\left((-2\lambda)^{-1}, (-2\lambda)\right). \tag{43}$$

2. Suppose that outside the spectrum E_- the function $\Psi(P)$ is analytic and on the cut E_- has discontinuity of the form

$$\Psi(P - i0) = \Psi(P + i0)G(P), \qquad P \in E_-, \quad G(P) = -\begin{pmatrix} r^{(+)}(\tau P) & z_-^{(+)}(P) \\ \frac{r(P)r(\tau P)-1}{z_-^{(+)}(P)} & r^{(+)}(P) \end{pmatrix}. \tag{44}$$

Here, the superscript $(+)$ means the value on the lower edge of the cut E_-. It follows from (40) that the matching matrix $G(P)$ becomes an upper triangular matrix on the simple branch of the spectrum E_*.

3. det $\Psi(P) = 1$ and $\Psi(P)$ is a regular (without singularities) function on $\mathbf{C}$.

Proposition 4. *The solution $u(x, t)$ of the Cauchy problem* (2), (33) *for the MKdV equation* (32) *can be found from the solution $\Psi(P)$ of the above Riemann problem by means of the formula*

$$u(x,t) = V_{12}^-, \tag{45}$$

where V^- is the matrix coefficient of the expansion of the function Ψ (43).

To prove Proposition 4, we note that the exact solution of the Riemann problem is given by a matrix formed from the Jost functions:

$$\Psi(P) := (\psi_+(\sigma P), \psi_-(P)) \begin{pmatrix} \Delta^{-1}(P) & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Delta = \{\psi_+(\sigma P), \psi_-(P)\}; \qquad P \in \Gamma^+ \tag{46}$$

The properties (43)—(45) can be verified by direct calculation.

Proposition 5. *The solution $\Psi(P)$ of the above matrix Riemann problem is unique.*

The proof follows immediately from Liouville's theorem and the asymptotic behaviors (43).

Thus, we have reduced the investigation of the original Cauchy problem for $u(x, t)$ to an auxiliary linear matching problem. We show that the Riemann problem admits an explicit asymptotic solution as $t \to +\infty$.

In this solution, an important role is played by a Whitham deformation $\tilde{\Gamma}$ of the curve Γ_- into the curve Γ_+ with a dependence of the deformation on the deformation parameter

$$\xi = x/t, \qquad \xi \in \mathbb{R},$$

and the explicit expressions of finite-gap theory for the MKdV equation are also important. Bearing in mind that these expressions have not apparently been published hitherto, we give them in the following section.

5. THE MKdV MODEL. REAL FINITE-GAP SOLUTIONS

The model (32) is the summit of "third order" in the integrable NS hiererachy (see [4]). Therefore, there are many similarities between the finite-gap expressions for the NS equation and MKdV equation. However, there is also an important difference. The fact is that in order to identify real solutions of the model (32) it is necessary to take into account the additional symmetry π of the spectral Riemann surface Γ. This additional reduction can be made in various ways. One can make the substitution $\lambda \to \sqrt{\lambda}$ in the U—V pair (28) and construct a spectral "finite-gap" curve Γ, as two-sheeted covering of the curve $z = \sqrt{\lambda}$ (see [15]).

Here, we prefer to take a different route, namely, modifying in the necessary manner the well-known finite-gap expressions for the NS equation (the lowest representative of the integrable hierarchy), we use the technique of [16] to satisfy the additional symmetry π.

We now turn to the detailed exposition. Let Γ be a nonsingular hyperelliptic curve of even genus $g = 2k$ that admits the symmetries π and τ and is given by the equation

$$\mu^2(\lambda) = \prod_{j=1}^{2k+1} (\lambda^2 - \lambda_j^2), \qquad \lambda_j \in \mathbb{R}, \qquad k \in \mathbb{N}.$$

We choose the canonical basis (a_i, b_i), $i = 1, \ldots, g$, of one-dimensional homologies on Γ in such a way that

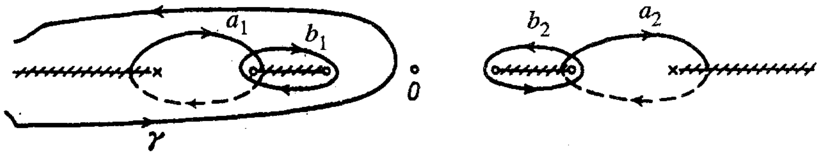

Fig 8

$$\begin{cases} \tau a = a, & \tau b = -b, \\ \pi a = \mathbb{P}a, & \pi b = \mathbb{P}b. \end{cases} \tag{47}$$

Here, $a=(a_1, \ldots, a_g)^T$, $b=(b_1, \ldots, b_g)^T$; $\mathbb{P}$ is the permutation matrix $\mathbb{P}_{ij}=\delta_{i,g+1-i}$ (see Fig. 8, $g=2$).

We normalize the holomorphic differentials $\omega_1, \ldots, \omega_g$ by the condition $\oint_{a_j}\omega_i = \delta_{ij}$; the corresponding B matrix

$$B_{ij} = \oint_{b_j} \omega_j, \qquad i,j = 1,\ldots,g,$$

determines the Riemann theta function

$$\Theta(z) = \sum_{m\in\mathbb{Z}^g} \exp\left(\pi i(2\langle z,m\rangle + \langle Bm,m\rangle)\right).$$

We define the Abelian integrals $\Omega_i(P)$, $i=1, 2, 3$, which have vanishing a periods and singularities of the form

$$\Omega_1(P) \to \mp(\lambda + w_1), \qquad \Omega_2(P) \to \mp(4\lambda^3 + w_2), \qquad \Omega_3(P) \to \mp(\ln\lambda + w_3), \qquad P \to \infty^{\pm}. \tag{48}$$

The expression for the complex finite-gap solution $v(x, t)$ of the model (32) has the form (cf. [14])

$$v(x,t) = \frac{\Theta(\Omega + d - n)}{\Theta(\Omega + d)} e^{-w_3}. \tag{49}$$

Here $\Omega=(Vx+Wt)/2\pi$, where V and W are the vectors of the b periods of the integrals $\Omega_1(P)$ and $\Omega_2(P)$;

$$n = \int_{\infty^-}^{\infty^+} \omega = \int_\gamma \omega,$$

where the path γ does not intersect the basis cycles.

Note that (49) can be obtained from the expression (6) of [14] if it is noted that by virtue of the π symmetry $w_1=w_2=0$.

We now formulate restrictions on the phase vector d that enables us to separate real solutions: $v=\bar{v}$. As in [16], we can show that

$$\Theta(z) = \Theta(\mathbb{P}z) = \overline{\Theta}(\bar{z}) \tag{50}$$

and, in addition,

$$\mathbb{P}\Omega = \Omega, \qquad \bar{n} = -n = \mathbb{P}n. \tag{51}$$

From these identities, after obvious calculations, we obtain

Proposition 6. *We choose the phase vector $d\in\mathbb{R}^g$ in such a way that*

$$\mathbb{P}d = d + M, \qquad M \in \mathbb{Z}^g. \tag{52}$$

Then (49) *determines a real finite-gap solution of the MKdV problem* (32).

It should be noted that by virtue of the symmetry π the g-dimensional theta function Θ decomposes into a sum of products of theta functions of dimension $k=g/2$ (see [17]) and the dynamics with respect to x and t is restricted in accordance with (51) to the "even" part of the torus Jac(Γ), which is the k-dimensional torus $\mathbb{T}^k$.

It was precisely this torus that arose in our case as Jacobian Jac(Γ_0) of the curve Γ_0 of genus k in the construction of finite-gap solutions by a method analogous to that of [15].

To conclude this section, we note that expressions for the corresponding potential $v(x, t)$ of the Baker—Akhiezer ψ function $e(x, t)$ has the same form as in (14) (at the same time $\beta=1$), and therefore we do not give them here.

6. WHITHAM DEFORMATIONS

We consider the following ansatz of the solutions $u(x, t)$ of the MKdV model:

$$u(x,t) = v\left(x,t \mid \Gamma(\xi), d\right) + O(\varepsilon), \qquad \varepsilon \ll 1, \quad \xi = x/t, \tag{53}$$

i.e., we assume that in the leading order in ε the solution is a weakly modulated finite-gap solution. A necessary condition for uniform boundedness of the correction in (53) is the (Whitham) system of equations on the bifurcation points λ_i of the curve $\Gamma(\xi)$:

$$\left(\xi + s(\lambda_i \mid \vec{\lambda})\right) \partial_\xi \lambda_i = 0, \qquad i = 1, \ldots, g. \tag{54}$$

Here, $s(\lambda|\vec{\lambda})$ is a rational function of $\lambda \in \mathbb{C}$. The function $s(\lambda) \to 4\lambda^2$, $\lambda \to \infty$, and therefore the number of its zeros in $\mathbb{C}$ exceeds by two the number of poles. At the first glance, this circumstance hinders the transfer of the technique of [14] to the case of the model (32). The function $s(\lambda)$ is not monotonic on the spectrum E (the union of the hatched regions in Fig. 8). We note, however, that the function $s(\lambda)$ is even:

$$s(\lambda) = s(-\lambda). \tag{55}$$

Therefore, it is sufficient to investigate its properties for $\lambda \geq 0$. Repeating the arguments of [14], we prove

Proposition 7. *Suppose that for some $\xi = \xi_0$ the bifurcation point λ_j in (54) moves: $\partial_\xi \lambda_j|_{\xi=\xi_0} \neq 0$; then:*

1) *the bifurcation point $\lambda = \lambda_j$ moves to the left: $\partial_\xi \lambda_j|_{\xi=\xi_0} < 0$;*

2) *the bifurcation point $\lambda = -\lambda_j$ moves to the right;*

3) *all other bifurcation points are fixed.*

We define the stationary point $P_0 \in E$ as solution of the equation

$$\left(d\Omega_1(P)\xi + d\Omega_2(P)\right)|_{P=P_0} = 0. \tag{56}$$

It follows from (55) and Proposition 7 that the points σP_0, πP_0, $\sigma\pi P_0$, which together with P_0 are stationary points, form a complete set (there are no other stationary points). The following analog of Lemma 1 of [14] is helpful for what follows.

Proposition 8. *We consider for $P \in \Gamma^+$, $\lambda \geq 0$ the function*

$$f(P) = \operatorname{Im}\left(\Omega_1(P)\xi + \Omega_2(P)\right).$$

We assume that there exists $P_0 \in E$. Then the function $f(P)$ does not change sign within each of the gaps of the spectrum:

$$\operatorname{sign} f(P) = \begin{cases} 1, & \lambda(P) > \lambda(P_0), \\ -1, & \lambda(P) < \lambda(P_0), \end{cases} \qquad \text{for} \quad \lambda(P) \in \mathbb{R}_+ \setminus E. \tag{57}$$

We now discuss at the qualitative level of rigor the application of these results to our problem of the shock wave in Sec. 4. In accordance with the numerical results of [11] there is at long times a ξ-dependent region of transition from u_- to u_+ that is filled with oscillations. We assume that these oscillations can be described by a solution of the form (53), where $\Gamma(\xi)$ is the curve of Sec. 5 of genus $g=2$. Then the bifurcation points λ_i of these curves satisfy the system (54). It is readily seen that by virtue of Proposition 7 there exists a unique curve $\Gamma(\xi)$ possessing the natural properties

$$\Gamma(\xi) \to \Gamma_\pm, \qquad \xi \to \pm\infty, \tag{58}$$

where Γ_+ and Γ_- are the curves of genus 0 of Sec. 4. This deformation $\Gamma_- \to \Gamma_+$ has the form shown in Fig. 9 (Whitham rearrangement):

$$\Gamma(\xi) = \begin{cases} \Gamma_-, & \xi < \xi_i = -6(2u_+^2 - u_-^2), \\ \tilde{\Gamma}(\xi), & \xi \in [\xi_i, \xi_f], \\ \Gamma_+, & \xi > \xi_f = -4(u_-^2 + u_+^2/2). \end{cases} \tag{59}$$

The bifurcation values ξ_i and ξ_f can be readily calculated using (54) and (59). (In [5] there is a misprint in the expression for ξ_f and ξ_i.) The qualitative picture of the solution $u(x, t)$ is shown in Fig. 10 (MKdV shock wave).

As follows from this figure, the nature of the shock wave (2), (33) is analogous to the KdV case [13] for $u_- < u_+$. In the neighborhood of $\xi = \xi_i$ low-amplitude, almost harmonic oscillations develop, and they grow with increasing ξ up to the point $\xi = \xi_f$, where the solution can be approximated [12] by a train of solitons. It is important that $u_- \cdot u_+ \geq 0$. If the inequality has the opposite sign, the solution contains the traveling wave of kink type discussed at the beginning of Sec. 4.

The above analysis of the Whitham rearrangements can be readily applied to other choices of u_+, u_-. For example, in the case $u_+ > u_- \geq 0$ it is easily shown that the deformation $\Gamma_- \to \Gamma_+$ can be made in a "regular" manner, without increasing

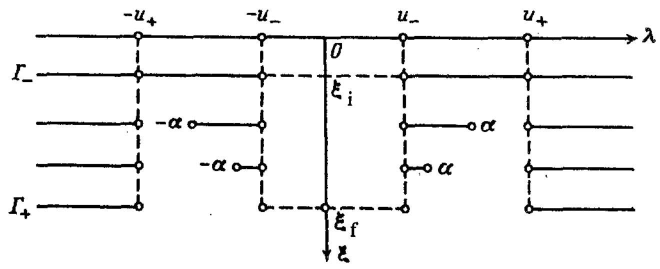

Fig 9

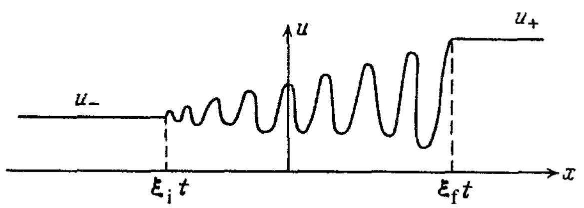

Fig 10

the genus of the curve $\tilde{\Gamma}(\xi)$ in the transition region [18]. The corresponding asymptotic solution $u(x, t)$ has the form of a rarefaction wave for the dissipationless equation (1), and this is in agreement with the numerical experiment of [11] (see also [5]).

7. CALCULATION OF THE PHASE OF THE SHOCK WAVE

The qualitative analysis of Sec. 6 is neither rigorous nor does it permit calculation of the shock-wave phase $d(x, t)$. Indeed, Whitham's theory describes modulation of the "action" type variables λ_i and does not provide a method for calculating the phase variables. To solve the problem of the phase, and also to establish the obtained asymptotic behavior, it is necessary to analyze the Riemann problem of Sec. 4.

Technically, this asymptotic analysis is not simple, but conceptually it follows the course of [13], and therefore in this section we adopt a more laconic style of exposition.

We present an explicit construction of the asymptotic solution $\Psi_A(P)$ to the Riemann problem formulated in Sec. 4. We shall assume in what follows that

$$t \to +\infty, \qquad \xi_i < \xi < \xi_f,$$

this corresponding to consideration of the most nontrivial Whitham oscillation zone. The function $\Psi_A(P)$ has the form

$$\Psi_A(P) = \left(\frac{\Phi(\sigma P)}{z_-(\sigma P)}, \Phi(P) \right), \qquad P \in \tilde{\Gamma}^+(\xi). \tag{60}$$

Here, $\tilde{\Gamma}^+$ is the upper sheet of the Whitham curve $\tilde{\Gamma}$ of genus $g=2$,

$$z^2(\lambda) = (\lambda^2 - u_-^2)\left(\lambda^2 - \alpha^2(\xi)\right),$$

where $\alpha(\xi)$ is a moving bifurcation point, uniquely determined by the condition (54):

$$(\xi + s(\alpha)) = 0, \qquad u_- < \alpha < u_+. \tag{61}$$

The function $\Phi(P)$ is given by an explicit expression in terms of a Cauchy-type integral on the curve $\tilde{\Gamma}$:

$$\Phi(P) = \tilde{e}_-(P) + \frac{1}{2\pi i} \int_{L \cup \partial \tilde{\Gamma}^+} M(P, Q) \tilde{f}(Q). \tag{62}$$

The contour of integration L is symmetric with respect to the involution π: $\lambda \to -\lambda$ and the involution σ: $z \to -z$:

$$\pi L = -L, \qquad \sigma L = L. \tag{63}$$

To determine L, it is sufficient to say that L is generated by a group with generators π and z from the contour γ_0:

$$\gamma_0 = [\alpha^+, u_+^+], \qquad \gamma_0 \in \tilde{\Gamma}^+. \tag{64}$$

Here, the superscript denotes the upper sheet of $\tilde{\Gamma}^+$; $\partial\tilde{\Gamma}^+$ is the boundary of the upper sheet, i.e., a contour that passes around the cuts on $\tilde{\Gamma}$ in such a way that the sheet $\tilde{\Gamma}^+$ remains on the left.

The analog of the Cauchy kernel, $M(P, Q)$, has the form

$$M(P,Q) = \frac{d\lambda(Q)}{\lambda(Q) - \lambda(P)} \cdot \frac{\{\tilde{e}_-(P), \tilde{e}_-(\sigma Q)\}}{\{\tilde{e}_-(Q), \tilde{e}_-(\sigma Q)\}}.$$

The Baker—Akhiezer function $\tilde{e}_-(P)$ is constructed on $\tilde{\Gamma}$ and has simple poles at the branch points $\lambda = \pm u$, i.e., has a phase shift d_0 [see (70)].

The function $\tilde{f}(P)$ is given by the expressions

$$\tilde{f}(P) = \begin{cases} -r(P)\left(A(\sigma P)\hat{e}(\sigma P) + B(\sigma P)\hat{e}(P)\right), & P \in \partial\tilde{\Gamma}^+, \\ \left(1 + r^{(+)}(P)\right) A(P)\hat{e}(P), & P \in \gamma_0 \cup \sigma\pi\gamma_0, \\ \left(1 - r^{(+)}(\tau P)\right) A(P)\hat{e}(P), & P \in \sigma\gamma_0 \cup \pi\gamma_0. \end{cases}$$

$$B(P) = -r(P)A(\sigma P)H\left(\lambda^2(P) - \lambda^2(P_0)\right), \quad P \in \partial\tilde{\Gamma}^+, \tag{65}$$

$$A(P) := \begin{cases} \alpha(P + i0) = \alpha^{(+)}(P), & P \in L, \\ \lim_{P' \to P} \alpha(P'), & P \in \partial\tilde{\Gamma}^+, \quad P' \in \tilde{\Gamma}^+. \end{cases} \tag{66}$$

Here, $H(x)$ is the Heaviside function, $\hat{e}(P)$ is the Baker—Akhiezer function with phase shift d (71), and P_0 is the stationary point determined in Sec. 6; the superscript $(+)$ denotes transition to the upper edge of the contour;

$$\alpha(P) = \hat{\theta}(P)\exp\left(-\frac{1}{2\pi i}\int_{L\cup\partial\tilde{\Gamma}^+} \widehat{M}(P,Q)\ln g(Q)\right). \tag{67}$$

Here, $\hat{M}(P, Q)$ is a multivalent analog of a Cauchy kernel:

$$\widehat{M}(P,Q) = \int_{+\infty}^{P} m(P', Q),$$

where $m(P, Q)$ is a meromorphic bidifferential normalized by the conditions (q is a local parameter)

$$\oint_{a_i(P)} m(P,Q) = 0, \qquad m(P,Q) \sim \frac{dq(P)dq(Q)}{(q(P) - q(Q))^2}, \qquad P \sim Q.$$

It follows from this that

$$\oint_{b_j(P)} m(P,Q) = 2\pi i\omega_j(Q).$$

The factor $\hat{\theta}(P)$ in (67) can be found from the condition that the function $\alpha(P)$ be single valued on $\tilde{\Gamma}$ and from the asymptotic behavior

$$\alpha(P) \to 1, \qquad P \to \infty^{\pm}, \tag{68}$$

$$\widehat{\Theta}(P) = \text{const} \cdot \frac{\theta\left(\int_{\infty^+}^{P} \omega + d\right)}{\theta\left(\int_{\infty^+}^{P} \omega + d_0\right)}. \tag{69}$$

Here, d_0 is a constant vector of the form

$$d_0 = -\left(\int_{\infty^+}^{u_+} + \int_{\infty^+}^{-u_+}\right)\omega - K, \tag{70}$$

K is a vector of Riemann constants on $\tilde{\Gamma}$, and the integration part does not intersect the basis cycles. The expressions (69) and (70) mean that the function $\hat{\Theta}(P)$ has simple poles at the branch points $\lambda = \pm u_+$.

The expression for the phase vector $d \in \mathbf{R}^2$ has the form [see (67) and (69)]

$$d = d_0 - \frac{1}{2\pi i}\int_{\partial\tilde{\Gamma}^+\cup L} (\ln g(Q))\,\omega(Q). \tag{71}$$

The function $g(P)$ is given by the expressions associated with (65),

$$g(P) = \begin{cases} 1 - r(P)r(\tau P)H\left(\lambda^2(P) - \lambda^2(P_0)\right), & P \in \partial\tilde{\Gamma}^+, \\ -r^{(+)}(P), & P \in \gamma_0 \cup \sigma\pi\gamma_0, \\ r^{(+)}(\tau P), & P \in \sigma\gamma_0 \cup \pi\gamma_0. \end{cases} \tag{72}$$

It follows from these expressions and from (67) that the function $A(P)$ does not have singularities at the branch points $\lambda = \pm u_+$. The zeros of the function $A(P)$ suppress the poles of $\tilde{e}_-(P)$, so that the function $\Phi(P)$ does not have singularities at these points. In addition, we note that $\Phi(P)$ satisfies on the contour $\partial\tilde{\Gamma}^+ \cup L$ the matching condition

$$\alpha_*^-(P) = \alpha_*^+(P)g(P), \qquad P \in \partial\tilde{\Gamma}^+ \cup L.$$

Here, as in (66),

$$\alpha_*^{\overset{+}{(-)}}(P) = \begin{cases} \alpha(P \underset{(-)}{+} i0), & P \in L, \\ \lim_{P'\to P} \alpha(P'), & P \in \partial\tilde{\Gamma}^+, \quad P' \in \tilde{\Gamma}^{\overset{+}{(-)}}. \end{cases} \tag{73}$$

From the requirement of fulfillment of the matching condition (44) for the function $\Psi_A(P)$ and the property of absence of a virtual level, (39), it follows that the function $\Phi(P)$ has zeros at the branch points $\lambda = \pm u_-$ on the transition from the lower sheet of $\tilde{\Gamma}^-$. Therefore, the first column of $\Psi_A(P)$ does not have zeros at these points [see (60)].

It turns out that the matrix matching conditions (44) are indeed satisfied in the limit $t \to +\infty$ for our ansatz $\Psi_A(P)$. More precisely, we have the following proposition.

Proposition 9. *The function $\Psi_A(P)$ solves asymptotically (as $t \to +\infty$) the Riemann problem of Sec. 4. In particular,*

$$\Psi_A(P + i0)G(P)\Psi_A^{-1}(P - i0) = I + \delta(P, \xi, t), \tag{74}$$

where the "small" perturbation δ has the structure

$$\delta(P, \xi, t) = \begin{cases} O\left((P - P_0)t^{-\varepsilon}\right), & |P - P_0| \geq t^{-\varepsilon}, \\ O(1), & |P - P_0| \leq t^{-\varepsilon}, \quad \varepsilon > 0, \end{cases} \qquad \int_{-\infty}^{\infty} d\eta\,\delta(\eta, t) \leq C \cdot t^{-\varepsilon}.$$

These expressions enable us to obtain (see [13,14]) the following estimate for the true ψ function, i.e., the solution of the Riemann problem:

$$\Psi(P) = \left(I + O(t^{-\varepsilon})\right)\Psi_A(P), \qquad |\operatorname{Im} P| > \varepsilon_1 > 0. \tag{75}$$

Returning now to the expressions of Sec. 4 [see (45)], and summarizing the above results, we obtain our final proposition.

Proposition 10. *The leading term in the $t \to +\infty$ asymptotic behavior of the solution $u(x, t)$ of the Cauchy problem for the MKdV model with boundary conditions of the shock-wave type* (2), (32) *under the condition of absence of a discrete spectrum is given by the Whitham-modulated two-gap solution $v(x, t \mid \tilde{\Gamma}(\xi), d(\xi))$, where the curve $\tilde{\Gamma}(\xi)$ is described in Sec.* 5, *and for the phase shift d the expression* (71) *holds.*

In general, the correction to the asymptotic behavior $v(x, t)$ decreases as a power with respect to t:

$$u(x,t) = v\left(x, t \mid \Gamma(\xi), d(\xi)\right) + \hat{\delta}(\xi, t). \tag{76}$$

In particular, $\hat{\delta} = O(t^{-1/2})$ for $\xi < \xi_i$ and $\hat{\delta} = O(t^{-N})$, $\forall N > 0$ for $\xi > \xi_f$ (see [13,14]).

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