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A METHOD OF CALCULATING MASSIVE FEYNMAN INTEGRALS

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A general method is proposed for calculating massive Feynman integrals on the basis of a representation of the massive denominators in the form of Mellin–Barnes integrals. This method is used to obtain expressions for some classes of single-loop massive Feynman integrals of propagator and vertex type (for arbitrary values of the powers of the denominators and dimension of space). The results are presented in the form of hypergeometric functions, making it possible to investigate different ranges of variation of the momenta.

Introduction

Because of the need to make numerous different calculations in gauge theories (QCD, electroweak model, etc.), it is very important to develop methods that permit exact calculation of various types of Feynman diagrams containing both massless and massive particles. This is true, in particular, for calculation of interaction cross sections and decay widths in various orders of perturbation theory, for the investigation of the coefficient functions in operator expansions, for the renormalization-group analysis of β functions, anomalous dimensions, and invariant charges, for study of the behavior of Green's functions, the problem of anomalies, etc. In many cases it is most convenient for the calculation of the corresponding Feynman integrals to use dimensional regularization [1,2] (see also the review [3]), the use of which makes it possible, in particular, to preserve gauge invariance at all stages.

At the present time, the greatest successes have been achieved in the development of methods of calculation of massless Feynman integrals of propagator type (i.e., dependent on a single external momentum): the method of Gegenbauer polynomials [4,5], integration by parts [6], the uniqueness method (see, for example, [7]), and also some other methods [8–10] (see also the review [11]). Massless integrals of vertex type (with two independent external momenta) have a much more complicated structure, and some cases were investigated in [12–14].

At the same time, for the calculation of many processes, especially those including heavy particles, one cannot avoid the use of Feynman integrals with massive denominators. However, at the present time not many exact expressions are known for different dimensionally regularized massive Feynman integrals (see, for example, [1,15–17] and the references given there).

In the present paper, we propose a general method for obtaining exact solutions for

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Feynman integrals containing massive denominators. The idea of the method is to represent the massive denominators in the form of Mellin–Barnes integrals with subsequent calculation of the corresponding massless integrals. This method makes it possible to obtain results for arbitrary values of the dimension of space n and indices of the lines (degrees of the corresponding denominators), making it possible to use the obtained expressions in both dimensional and analytic regularization. These results can be expressed in terms of functions of hypergeometric type, so that different ranges of variation of the momenta can be investigated (see below). We note that we can, without loss of generality, consider only scalar integrals, since all integrals with Lorentz tensor structure in the numerator can be reduced to scalar integrals in accordance with formulas of the type given, for example, in [18–19].

The paper is arranged as follows. Section 1 is devoted to exposition of the general idea of the proposed method, which can be used for arbitrary (including many-loop) massive integrals. In Sec. 2, the general technique is illustrated by the example of the calculation of massive Feynman integrals of propagator type. In Sec. 3, we calculate some classes of massive integrals of vertex type. In the Conclusions, we formulate and discuss the main results of the paper.

1. Representation for Massive Denominators

Suppose we have a Feynman integral that contains one or several massive denominators of the form

$$\frac{1}{(k^2-m^2+i0)^{\beta}}, \tag{1}$$

where k is the momentum of the corresponding line, m is a mass (for different lines, the masses may be different), and β is the index of the line (the degree of the denominator). In addition, the integral can also have massless denominators. The infinitesimally small addition (+i0) determines the usual "causal" method of passing around the singularities in the pseudo-Euclidean space. In what follows, we shall assume that all squares of the momenta in the denominators have such additions (including those in the employed expansions and integral representations), and we shall not write them out explicitly.

As is well known, the direct calculation of such massive integrals using standard methods (α representation or Feynman parameters) involves great difficulties in the calculation of the parametric integrals, and general expressions have been obtained only for the simplest cases. There is another procedure for obtaining asymptotic expansions of such integrals with respect to parameters of the type m^2/p^2 (p is an external momentum) associated with the use of the R^* operation (see [20–22]).

We expand the considered denominator (1) in a series in m^2/k^2:

$$\frac{1}{(k^2-m^2)^{\beta}}=\frac{1}{(k^2)^{\beta}(1-m^2/k^2)^{\beta}}=\frac{1}{(k^2)^{\beta}}\,{}_1F_0\left(\beta\,\middle|\,\frac{m^2}{k^2}\right)=\frac{1}{(k^2)^{\beta}}\frac{1}{\Gamma(\beta)}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{m^2}{k^2}\right)^{j}\Gamma(\beta+j). \tag{2}$$

If we now formally substitute this expansion in the integrand, we obtain an infinite sum of integrals in which the massive denominator (1) is replaced by massless denominators. It is obvious that if all the massive denominators in the considered Feynman integral are expanded in this manner, then it will be reduced to the sum of the corresponding massless integrals. In particular, if there is just one external momentum p, then we obtain in this manner an expansion in m^2/p^2. However, it is easy to see in simple examples that the obtained result will be incorrect. This is because the expansion (2) is valid only in the region $|k^2| > m^2$, while the integration is over all k, including the region in which $|k^2| < m^2$. In this region, the expansion (2) must, in general, be replaced by a different one:

$$\frac{1}{(k^2-m^2)^{\beta}}=\frac{1}{(-m^2)^{\beta}(1-k^2/m^2)^{\beta}}=\frac{1}{(-m^2)^{\beta}}\,{}_1F_0\left(\beta\,\middle|\,\frac{k^2}{m^2}\right)=$$
$$\frac{1}{(-m^2)^{\beta}}\frac{1}{\Gamma(\beta)}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{k^2}{m^2}\right)^{j}\Gamma(\beta+j). \tag{3}$$

Thus, the correct expansion of the denominator (1) is a combination of (2) and (3) with corresponding θ functions (for further work with such expressions, it is better to go over to Euclidean variables). However, the presence of the θ functions greatly complicates the corresponding massless integrals, and the entire gain from the reduction of the massive integrals to the massless ones is almost lost. One of the ways out of the resulting situation is to use the procedure proposed in [20–22]. It is as follows. Suppose we use for denominators of the form (1) the expansion (2) and compensate the "incorrectness" of this expansion in the region $|k^2| < m^2$ by appropriately chosen counterterms (a general prescription for the construction of such counterterms is given). This makes it possible to obtain correct asymptotic expansions of the corresponding integrals in powers of m^2/p^2.

We propose a different method for calculating massive Feynman integrals. The idea of the method is to use the Mellin–Barnes representation for the function ${}_1F_0$,

$$ {}_1F_0(\beta|z) \equiv \frac{1}{(1-z)^\beta} = \frac{1}{\Gamma(\beta)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds\, (-z)^s \Gamma(-s)\Gamma(\beta+s), \tag{4} $$

where the contour in the complex plane of s separates the "left" series of poles of the integrand Γ functions from the "right" poles (in what follows, all such integrals will be understood in precisely this sense). To calculate the integral (4), we can use the residue theorem, closing the contour at infinity in the right or left half-plane in order to make the integrand decrease (depending on the value of $|z|$). For example, for $|z| < 1$ we must close the contour in (4) on the right, and for $|z| > 1$ on the left (and the obtained expression is equal to the sum over the residues of $\Gamma(-s)$ or $\Gamma(\beta + s)$, respectively). In this manner, we obtain the well-known expressions for the analytic continuation of the hypergeometric functions (see, for example, [23]). Thus, the main formula of the method is

$$ \frac{1}{(k^2-m^2)^\beta} = \frac{1}{(k^2)^\beta} \frac{1}{\Gamma(\beta)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \left(-\frac{m^2}{k^2}\right)^s \Gamma(-s)\Gamma(\beta+s) \tag{5} $$

(we repeat that all squares of momenta contain infinitesimally small imaginary additions: $k^2 \leftrightarrow k^2 + i0$). The advantage of this method is that formula (5) contains both (2) and (3): for $|k^2| > m^2$, we obtain (summing over the residues of $\Gamma(-s)$) the expansion (2), and for $|k^2| < m^2$ (summing over the residues of $\Gamma(\beta + s)$) the expansion (3). At the same time, we can use the ordinary expressions for the massless integrals, replacing the corresponding index β by $(\beta + s)$. The use of this method also has a number of other helpful properties, which will be noted below in the calculation of definite classes of integrals.

It should be noted that the appropriateness of using the Mellin–Barnes representation for the hypergeometric functions (and also the Mellin transform) in the calculation of one-dimensional integrals has already been noted (see, for example, [24,25]). In particular, it was used in [26,19,13] to study parametric integrals that arise when the α representation is used to calculate certain Feynman integrals. In particular, the Mellin transform was used in [27] to analyze α-parametrized integrals in the investigation of singularities and the asymptotic behavior of massive Feynman amplitudes. Note also that in [28] a study was made of some aspects of the calculation of massive integrals of propagator type by using a single Mellin transformation on the square of the external momentum and considering the Mellin transforms of such integrals. Our proposed technique of the Mellin–Barnes representation directly for the massive denominators differs from these approaches, and from our point of view is more convenient for calculating definite classes of massive Feynman integrals.

2. Integrals of Propagator Type

We now consider examples of the use of the basic formula of the method (5) for integrals of propagator type (containing one external momentum). In this section, we shall operate with single-loop integrals of the form

$$ J(\alpha, \beta; m_1, m_2) \equiv \int \frac{d^n k}{(k^2-m_1{}^2)^\alpha ((p-k)^2-m_2{}^2)^\beta} \tag{6} $$

where $n = 4 - 2\varepsilon$ is the spacetime dimension (in the framework of dimensional regularization [1,2]). A special case of the integral (6) with $\alpha = \beta = 1$ was considered in particular in [15].

We give first a detailed treatment of a well-known simple example: $m_1 = 0$, $m_2 \equiv m$. Applying (5) to the integrand, we obtain

$$J(\alpha,\beta;0,m)=\int\frac{d^nk}{(k^2)^\alpha((p-k)^2-m^2)^\beta}=$$

$$\frac{1}{\Gamma(\beta)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}ds(-m^2)^s\Gamma(-s)\Gamma(\beta+s)J^{(0)}(\alpha,\beta+s), \tag{7}$$

where the symbol $J^{(0)}$ denotes the corresponding massless integral, the result for which is well known:

$$J^{(0)}(\alpha,\beta)\equiv J(\alpha,\beta;0,0)=\int\frac{d^nk}{(k^2)^\alpha((p-k)^2)^\beta}=$$

$$\pi^{n/2}i^{1-n}(p^2)^{n/2-\alpha-\beta}\frac{\Gamma(n/2-\alpha)\Gamma(n/2-\beta)\Gamma(\alpha+\beta-n/2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n-\alpha-\beta)}. \tag{8}$$

Substituting (8) in (7) and replacing the variable of integration s by $(n/2 - \alpha - \beta - s)$ (replacements of such type do not violate the condition for separating by a contour the right and left series of poles — all that happens is that "right" and "left" change places), we have

$$J(\alpha,\beta;0,m)=\pi^{n/2}i^{1-n}(-m^2)^{n/2-\alpha-\beta}\frac{\Gamma(n/2-\alpha)}{\Gamma(\alpha)\Gamma(\beta)}\times$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}ds\left(-\frac{p^2}{m^2}\right)^s\frac{\Gamma(-s)\Gamma(\alpha+s)\Gamma(\alpha+\beta-n/2+s)}{\Gamma(n/2+s)} \tag{9}$$

(here and in what follows, the phase is defined in such a way that $i^{1-n}(-m^2)^{n/2}=i(m^2)^{n/2}$). Closing the contour of integration on the right, we obtain

$$J(\alpha,\beta;0,m)=\pi^{n/2}i^{1-n}(-m^2)^{n/2-\alpha-\beta}\frac{\Gamma(n/2-\alpha)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{p^2}{m^2}\right)^j\frac{\Gamma(\alpha+j)\Gamma(\alpha+\beta-n/2+j)}{\Gamma(n/2+j)}=$$

$$\pi^{n/2}i^{1-n}(-m^2)^{n/2-\alpha-\beta}\frac{\Gamma(n/2-\alpha)\Gamma(\alpha+\beta-n/2)}{\Gamma(n/2)\Gamma(\beta)}\,{}_2F_1\left(\begin{matrix}\alpha,\alpha+\beta-n/2\\ n/2\end{matrix}\middle|\frac{p^2}{m^2}\right), \tag{10}$$

where ${}_2F_1$ is Gauss's hypergeometric function. If we close the contour of integration in (9) on the left, we obtain a result in the form of functions of m^2/p^2:

$$J(\alpha,\beta;0,m)=\pi^{n/2}i^{1-n}(p^2)^{n/2-\alpha-\beta}\left\{\frac{\Gamma(n/2-\alpha)\Gamma(n/2-\beta)\Gamma(\alpha+\beta-n/2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n-\alpha-\beta)}\times\right.$$

$$\left.{}_2F_1\left(\begin{matrix}\alpha+\beta-n/2,\alpha+\beta-n+1\\ \beta-n/2+1\end{matrix}\middle|\frac{m^2}{p^2}\right)+\left(-\frac{m^2}{p^2}\right)^{n/2-\beta}\frac{\Gamma(\beta-n/2)}{\Gamma(\beta)}\,{}_2F_1\left(\begin{matrix}\alpha,\alpha-n/2+1\\ n/2-\beta+1\end{matrix}\middle|\frac{m^2}{p^2}\right)\right\}. \tag{11}$$

Naturally, the same result can be obtained from (10) by applying the well-known formula of analytic continuation of the function ${}_2F_1$ (see, for example, [23]). Note also that other formulas of analytic continuation of ${}_2F_1$ [23] make it possible to express the obtained result in the variables $(m^2 - p^2)/m^2$ and $(p^2 - m^2)/p^2$ (and their inverses). Such expansions are of interest for investigation of the behavior of the Green's functions near the mass shell. In particular, going over to the variable $(p^2 - m^2)/p^2$, we reproduce the result obtained in [29] by means of the α representation.

Returning to the expression (11), we note that on formal substitution of the expansion (2) we would obtain only the first function ${}_2F_1$ in the curly brackets, and the result would be incorrect. Application of the procedure [20-22] here reduces to term-by-term recovery of the expansion coefficients of the second function ${}_2F_1$, which in our approach is obtained automatically.

Note also that the passage to the limit $\alpha \to 0$ in (10) gives

$$J(0,\beta;0,m)=\pi^{n/2}i^{1-n}(-m^2)^{n/2-\beta}\frac{\Gamma(\beta-n/2)}{\Gamma(\beta)}. \tag{12}$$

This result agrees with the well-known result of [1]. By means of the proposed method, the expression (12) can also be obtained directly by using the property [9]

$$J^{(0)}(0, n/2+i\xi)=i\frac{2\pi^{n/2}}{\Gamma(n/2)}\pi\delta(\xi).$$

We now consider another interesting special case of the integral (6), when $m_1 = m_2 \equiv m$:

$$J(\alpha,\beta;m,m)=\int\frac{d^nk}{(k^2-m^2)^\alpha((p-k)^2-m^2)^\beta}. \tag{13}$$

Applying the general formula (5) twice to the denominator of (13), and using (8), we obtain

$$J(\alpha,\beta;m,m)=\pi^{n/2}i^{1-n}(p^2)^{n/2-\alpha-\beta}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\frac{1}{(2\pi i)^2}\times$$

$$\iint\limits_{-i\infty}^{i\infty} ds\,dt\left(-\frac{m^2}{p^2}\right)^{s+t}\Gamma(-s)\Gamma(-t)\frac{\Gamma(n/2-\alpha-s)\Gamma(n/2-\beta-t)\Gamma(\alpha+\beta-n/2+s+t)}{\Gamma(n-\alpha-\beta-s-t)}. \tag{14}$$

Making the change of variable $t = n/2 - \alpha - \beta - s - u$, and also using Barnes's lemma to calculate the integral over s (see, for example, [24,23]),

$$\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} ds\,\Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s)=\frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)},$$

we obtain

$$J(\alpha,\beta;m,m)=\pi^{n/2}i^{1-n}(-m^2)^{n/2-\alpha-\beta}[\Gamma(\alpha)\Gamma(\beta)]^{-1}\times$$

$$\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} du\left(-\frac{p^2}{m^2}\right)^u\frac{\Gamma(-u)\Gamma(\alpha+u)\Gamma(\beta+u)\Gamma(\alpha+\beta-n/2+u)}{\Gamma(\alpha+\beta+2u)}. \tag{16}$$

Hence, closing the contour of integration to the right or to the left and using the well-known doubling formula for the Γ function ($\Gamma(2z)=2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+1/2)$), we can obtain the results

$$J(\alpha,\beta;m,m)=\pi^{n/2}i^{1-n}(-m^2)^{n/2-\alpha-\beta}\frac{\Gamma(\alpha+\beta-n/2)}{\Gamma(\alpha+\beta)}\,{}_3F_2\left(\begin{matrix}\alpha,\beta,\alpha+\beta-n/2\\(\alpha+\beta)/2,\,(\alpha+\beta+1)/2\end{matrix}\middle|\frac{p^2}{4m^2}\right), \tag{17}$$

$$J(\alpha,\beta;m,m)=\pi^{n/2}i^{1-n}(p^2)^{n/2-\alpha-\beta}\left\{\frac{\Gamma(n/2-\alpha)\Gamma(n/2-\beta)\Gamma(\alpha+\beta-n/2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n-\alpha-\beta)}\times\right.$$

$${}_3F_2\left(\begin{matrix}\alpha+\beta-n/2,\,(\alpha+\beta-n+1)/2,\,(\alpha+\beta-n+2)/2\\\alpha-n/2+1,\,\beta-n/2+1\end{matrix}\middle|\frac{4m^2}{p^2}\right)+$$

$$\left(-\frac{m^2}{p^2}\right)^{n/2-\alpha}\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)}\,{}_3F_2\left(\begin{matrix}\beta,\,(\beta-\alpha+1)/2,\,(\beta-\alpha+2)/2\\n/2-\alpha+1,\,\beta-\alpha+1\end{matrix}\middle|\frac{4m^2}{p^2}\right)+$$

$$\left.\left(-\frac{m^2}{p^2}\right)^{n/2-\beta}\frac{\Gamma(\beta-n/2)}{\Gamma(\beta)}\,{}_3F_2\left(\begin{matrix}\alpha,\,(\alpha-\beta+1)/2,\,(\alpha-\beta+2)/2\\n/2-\beta+1,\,\alpha-\beta+1\end{matrix}\middle|\frac{4m^2}{p^2}\right)\right\} \tag{18}$$

(with regard to the hypergeometric functions encountered in the paper, including ${}_3F_2$, see the Appendix). Note that, expanding (17) with respect to $\varepsilon = (4 - n)/2$ in the case $\alpha = \beta = 1$, we obtain the well-known result (see, for example, [30]) expressed in terms of elementary functions.

Finally, we consider the general case of the integral (6). Double application of formula (5) (with allowance for the expression (8)) gives

$$J(\alpha,\beta;m_1,m_2)=\pi^{n/2}i^{1-n}(p^2)^{n/2-\alpha-\beta}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\frac{1}{(2\pi i)^2}\times$$

$$\iint\limits_{-i\infty}^{i\infty} ds\,dt\left(-\frac{m_1^2}{p^2}\right)^s\left(-\frac{m_2^2}{p^2}\right)^t\Gamma(-s)\Gamma(-t)\frac{\Gamma(n/2-\alpha-s)\Gamma(n/2-\beta-t)\Gamma(\alpha+\beta-n/2+s+t)}{\Gamma(n-\alpha-\beta-s-t)}. \tag{19}$$

Calculating the contour integrals, we obtain

$$J(\alpha,\beta;m_1,m_2)=\pi^{n/2}i^{1-n}(-m_2^2)^{n/2-\alpha-\beta}\left\{\frac{\Gamma(n/2-\alpha)\Gamma(\alpha+\beta-n/2)}{\Gamma(n/2)\Gamma(\beta)}\times\right.$$

$$F_4(\alpha,\alpha+\beta-n/2;n/2,\alpha-n/2+1\,|\,p^2/m_2^2,m_1^2/m_2^2)+$$

$$\left(\frac{m_1^2}{m_2^2}\right)^{n/2-\alpha}\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)}F_4(\beta,n/2;n/2,n/2-\alpha+1\,|\,p^2/m_2^2,m_1^2/m_2^2)\Big\},\tag{20}$$

where F_4 is Appell's hypergeometric function of two variables [31,23] (see also the Appendix). If we consider the case $m_2 < m_1$, then in the expression (20) we must make the substitution $(m_1, \alpha) \leftrightarrow (m_2, \beta)$. From (19), we can also obtain the result in the variables m_1^2/p^2 and m_2^2/p^2:

$$\begin{aligned}J(\alpha,\beta;m_1,m_2)&=\pi^{n/2}i^{1-n}(p^2)^{n/2-\alpha-\beta}\Big\{\frac{\Gamma(n/2-\alpha)\Gamma(n/2-\beta)\Gamma(\alpha+\beta-n/2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n-\alpha-\beta)}\times\\&F_4(\alpha+\beta-n/2,\alpha+\beta-n+1;\alpha-n/2+1,\beta-n/2+1\,|\,m_1^2/p^2,m_2^2/p^2)+\\&\left(-\frac{m_1^2}{p^2}\right)^{n/2-\alpha}\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)}F_4(\beta,\beta-n/2+1;n/2-\alpha+1,\beta-n/2+\\&1\,|\,m_1^2/p^2,m_2^2/p^2)+\left(-\frac{m_2^2}{p^2}\right)^{n/2-\beta}\frac{\Gamma(\beta-n/2)}{\Gamma(\beta)}F_4(\alpha,\alpha-n/2+1;\\&\alpha-n/2+1,n/2-\beta+1\,|\,m_1^2/p^2,m_2^2/p^2)\Big\}.\end{aligned}\tag{21}$$

Thus, for the general integral (6) we have constructed the representation (19), from which we can obtain the result in the form of hypergeometric functions for different relations between the masses m_1 and m_2 and the momentum p (for example (20) and (21)).

Note that from the expression (20) for $\alpha = \beta = 1$ and $p^2 = 0$ we obtain the well-known result (see, for example, [21])

$$J(1,1;m_1,m_2)\,|_{p^2=0}=-i\pi^{n/2}\Gamma(1-n/2)\frac{(m_2^2)^{n/2-1}-(m_1^2)^{n/2-1}}{m_2^2-m_1^2}.$$

3. Integrals of Vertex Type

In this section, we consider examples of the application of the proposed technique to single-loop "triangle" integrals of vertex type (with two independent external momenta) containing massive denominators (see Fig. 1).

It is obvious that application of formula (5) to massive denominators requires information about the corresponding massless integrals:

$$J^{(0)}(\mu,\nu,\rho)\equiv\int\frac{d^nr}{(r^2)^\mu((p-r)^2)^\nu((q-r)^2)^\rho}\tag{22}$$

(as before, we understand the causal method of avoiding the singularities in the pseudo-Euclidean space). A general expression for the integrals (22) was obtained in [13] and can be represented in the form

$$\begin{aligned}J^{(0)}(\mu,\nu,\rho)&=\pi^{n/2}i^{1-n}[\Gamma(\mu)\Gamma(\nu)\Gamma(\rho)\Gamma(n-\mu-\nu-\rho)]^{-1}\times\\&\{(k^2)^{n/2-\mu-\nu-\rho}\Gamma(\mu)\Gamma(\mu+\nu+\rho-n/2)\Gamma(n/2-\mu-\nu)\Gamma(n/2-\mu-\rho)\times\\&F_4(\mu,\mu+\nu+\rho-n/2;\mu+\nu-n/2+1,\mu+\rho-n/2+1\,|\,p^2/k^2,q^2/k^2)+\\&(q^2)^{n/2-\mu-\rho}(k^2)^{-\nu}\Gamma(\nu)\Gamma(n/2-\rho)\Gamma(n/2-\mu-\nu)\Gamma(\mu+\rho-n/2)\times\\&F_4(\nu,n/2-\rho;\mu+\nu-n/2+1,n/2-\mu-\rho+1\,|\,p^2/k^2,q^2/k^2)+\\&(p^2)^{n/2-\mu-\nu}(k^2)^{-\rho}\Gamma(\rho)\Gamma(n/2-\nu)\Gamma(\mu+\nu-n/2)\Gamma(n/2-\mu-\rho)\times\\&F_4(\rho,n/2-\nu;n/2-\mu-\nu+1,\mu+\rho-n/2+1\,|\,p^2/k^2,q^2/k^2)+(p^2)^{n/2-\mu-\nu}(q^2)^{n/2-\mu-\rho}(k^2)^{\mu-n/2}\times\\&\Gamma(n-\mu-\nu-\rho)\Gamma(n/2-\mu)\Gamma(\mu+\nu-n/2)\Gamma(\mu+\rho-n/2)\times\\&F_4(n-\mu-\nu-\rho,n/2-\mu;n/2-\mu-\nu+1,n/2-\mu-\rho+1\,|\,p^2/k^2,q^2/k^2)\},\end{aligned}\tag{23}$$

where $k \equiv q - p$, and F_4 (as in (20) and (21)) is Appell's hypergeometric function of two variables (see (A.2)). In particular, if one of the parameters μ, ν, ρ is zero, then in (23) there remain, respectively, only the first, second, or third terms in the curly brackets and we obtain the well-known result (8), whereas for $\mu + \nu + \rho = n$ only the fourth term "survives," and we obtain the uniqueness relation (see, for example, [7]). For our

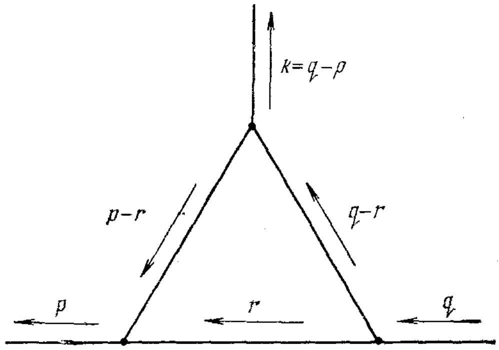

Fig. 1

purposes, it will be convenient to use the representation

$$J^{(0)}(\mu,\nu,\rho)=\pi^{n/2}i^{1-n}(k^2)^{n/2-\mu-\nu-\rho}[\Gamma(\mu)\Gamma(\nu)\Gamma(\rho)\Gamma(n-\mu-\nu-\rho)]^{-1}\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int ds\,dt\left(\frac{p^2}{k^2}\right)^s\times$$
$$\left(\frac{q^2}{k^2}\right)^t\Gamma(-s)\Gamma(-t)\Gamma(n/2-\mu-\nu-s)\Gamma(n/2-\mu-\rho-t)\Gamma(\mu+s+t)\Gamma(\mu+\nu+\rho-n/2+s+t), \qquad (24)$$

from which we can readily obtain both (23) and the corresponding expressions in terms of functions of other dimensionless momentum variables. We note that representations of such type were used in some special cases (for n = 4) in [26].

We now consider vertex integrals with one massive denominator:

$$J_1(\mu,\nu,\rho;m)\equiv\int\frac{d^nr}{(r^2-m^2)^\mu((p-r)^2)^\nu((q-r)^2)^\rho}. \qquad (25)$$

Use of the basic formula (5) of the method gives

$$J_1(\mu,\nu,\rho;m)=\frac{1}{\Gamma(\mu)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}du(-m^2)^u\Gamma(-u)\Gamma(\mu+u)J^{(0)}(\mu+u,\nu,\rho).$$

Substituting here the representation (24) and going over from the variable u to (n/2 − μ − ν − ρ − s − t − u), we obtain

$$J_1(\mu,\nu,\rho;m)=\pi^{n/2}i^{1-n}(-m^2)^{n/2-\mu-\nu-\rho}[\Gamma(\mu)\Gamma(\nu)\Gamma(\rho)]^{-1}\times$$
$$\frac{1}{(2\pi i)^3}\int_{-i\infty}^{i\infty}\int\int ds\,dt\,du\left(-\frac{p^2}{m^2}\right)^s\left(-\frac{q^2}{m^2}\right)^t\left(-\frac{k^2}{m^2}\right)^u\Gamma(-s)\Gamma(-t)\Gamma(-u)\Gamma(n/2-\nu-\rho-u)\times$$
$$\frac{\Gamma(\mu+\nu+\rho-n/2+s+t+u)\Gamma(\nu+s+u)\Gamma(\rho+t+u)}{\Gamma(n/2+s+t+u)}. \qquad (26)$$

Note that here we have two series of poles in the right half-plane of the variable u (due to Γ(−u) and Γ(n/2 − ν − ρ − u)). Calculating the integrals (26), we obtain the result

$$J_1(\mu,\nu,\rho;m)=\pi^{n/2}i^{1-n}(-m^2)^{n/2-\mu-\nu-\rho}\left\{\frac{\Gamma(\mu+\nu+\rho-n/2)\Gamma(n/2-\nu-\rho)}{\Gamma(\mu)\Gamma(n/2)}\times\right.$$
$$\Phi_1\left[\begin{matrix}\mu+\nu+\rho-n/2,\nu,\rho\\ n/2;\ \nu+\rho-n/2+1\end{matrix}\middle|\frac{p^2}{m^2},\frac{q^2}{m^2},-\frac{k^2}{m^2}\right]+\left(-\frac{k^2}{m^2}\right)^{n/2-\nu-\rho}\frac{\Gamma(n/2-\nu)\Gamma(n/2-\rho)\Gamma(\nu+\rho-n/2)}{\Gamma(\nu)\Gamma(\rho)\Gamma(n-\nu-\rho)}\times$$
$$\left.\Phi_1\left[\begin{matrix}\mu,\quad n/2-\rho,\quad n/2-\nu\\ n-\nu-\rho;\ n/2-\nu-\rho+1\end{matrix}\middle|\frac{p^2}{m^2},\frac{q^2}{m^2},-\frac{k^2}{m^2}\right]\right\}, \qquad (27)$$

where Φ_1 is a function of hypergeometric type that can be expressed in terms of Lauricella's generalized function of three variables (see (A.4)):

$$\Phi_1\left[\begin{matrix}a_1, a_2, a_3\\ c; \quad d\end{matrix}\middle| z_1, z_2, z_3\right] = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}\frac{z_1^{j_1}z_2^{j_2}z_3^{j_3}}{j_1!j_2!j_3!}\frac{(a_1)_{j_1+j_2+j_3}(a_2)_{j_1+j_3}(a_3)_{j_2+j_3}}{(c)_{j_1+j_2+j_3}(d)_{j_3}} =$$

$$F^{3:0;\,0;\,0}_{1:0;\,0;\,1}\left[\begin{matrix}(a_1:1,1,1), \ (a_2:1,0,1), \ (a_3:0,1,1)\\ (c:1,1,1): \qquad (d:1)\end{matrix}\middle| z_1, z_2, z_3\right], \tag{28}$$

where $(a)_j \equiv \Gamma(a+j)/\Gamma(a)$ is the Pochhammer symbol. Note that the general formula (26) makes it possible to go over to other dimensionless variables (for example, m^2/p^2, m^2/q^2, etc.). Note also that for $\mu = 0$ formula (27) corresponds to the well-known result of (8) (as it must).

It is sometimes of interest to consider symmetric deviation from the mass shell with respect to two ends of the corresponding Feynman diagram, $q^2 = p^2$ (see, for example, [19,13]). Then the function Φ_1 can be represented as a generalized hypergeometric function of two variables:

$$\Phi_1\left[\begin{matrix}a_1, a_2, a_3\\ c; \quad d\end{matrix}\middle| z, z, z_3\right] = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{z^j z^l}{j!\,l!}\frac{(a_1)_{j+l}(a_2+a_3)_{j+2l}(a_2)_l(a_3)_l}{(c)_{j+l}(a_2+a_3)_{2l}(d)_l} =$$

$$F^{2:0;\,2}_{1:0;\,2}\left[\begin{matrix}(a_1:1,1), \ (a_2+a_3:1,2):(a_2:1), \ (a_3:1)\\ (c:1,1):(a_2+a_3:2), \quad (d:1)\end{matrix}\middle| z, z_3\right]. \tag{29}$$

Because the sum over j in (29) represents the function ${}_2F_1$, we can here (as in the expression (10)) readily obtain, by means of the formulas of analytic continuation [23], expansions with respect to the variables $1/z$, $(1-z)$, $(z-1)/z$, etc., which are often used to investigate asymptotic behavior in different regions.

We now consider a vertex integral with two massive denominators and the same mass m:

$$J_2(\mu, \nu, \rho; m) = \int \frac{d^n r}{(r^2)^{\mu}((p-r)^2-m^2)^{\nu}((q-r)^2-m^2)^{\rho}}. \tag{30}$$

Application of (5) to the massive denominators gives

$$J_2(\mu,\nu,\rho;m) = \frac{1}{\Gamma(\nu)\Gamma(\rho)}\frac{1}{(2\pi i)^2}\iint_{-i\infty}^{i\infty} dv\, dw\, (-m^2)^{v+w}\times$$

$$\Gamma(-v)\Gamma(-w)\Gamma(\nu+v)\Gamma(\rho+w)J^{(0)}(\mu, \nu+v, \rho+w).$$

Substituting, further, the expression for $J^{(0)}$ (24), going over to the variable u by means of the substitution $w = n/2 - \mu - \nu - \rho - s - t - v - u$, and calculating the integral over v by means of Barnes's lemma (15), we obtain

$$J_2(\mu,\nu,\rho;m) = \pi^{n/2} i^{1-n}(-m^2)^{n/2-\mu-\nu-\rho}[\Gamma(\mu)\Gamma(\nu)\Gamma(\rho)]^{-1}\times$$

$$\frac{1}{(2\pi i)^3}\iiint_{-i\infty}^{i\infty} ds\, dt\, du\left(-\frac{p^2}{m^2}\right)^s\left(-\frac{q^2}{m^2}\right)^t\left(-\frac{k^2}{m^2}\right)^u\Gamma(-s)\Gamma(-t)\Gamma(-u)\times$$

$$\frac{\Gamma(\mu+\nu+\rho-n/2+s+t+u)\Gamma(\mu+s+t)}{\Gamma(n/2+s+t+u)}\frac{\Gamma(\nu+s+u)\Gamma(\rho+t+u)\Gamma(n/2-\mu+u)}{\Gamma(\nu+\rho+s+t+2u)}. \tag{31}$$

Hence, closing the contours with respect to the variables s, t, and u on the right, we can obtain the expression

$$J_2(\mu,\nu,\rho;m) = \pi^{n/2} i^{1-n}(-m^2)^{n/2-\mu-\nu-\rho}\frac{\Gamma(\mu+\nu+\rho-n/2)\Gamma(n/2-\mu)}{\Gamma(\nu+\rho)\Gamma(n/2)}\times$$

$$\Phi_2\left[\begin{matrix}\mu+\nu+\rho-n/2, \ \mu, \ \nu, \ \rho; \ n/2-\mu\\ n/2, \qquad \nu+\rho\end{matrix}\middle| \frac{p^2}{m^2}, \frac{q^2}{m^2}, \frac{k^2}{m^2}\right], \tag{32}$$

where the function Φ_2 can also be expressed in terms of Lauricella's generalized function of three variables (see (A.4)):

$$\Phi_2\left[\begin{matrix}a_1, a_2, a_3, a_4; \ b\\ c_1, \ c_2\end{matrix}\middle| z_1, z_2, z_3\right] =$$

$$\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}\frac{z_1^{j_1}z_2^{j_2}z_3^{j_3}}{j_1!\,j_2!\,j_3!}\frac{(a_1)_{j_1+j_2+j_3}(a_2)_{j_1+j_2}(a_3)_{j_1+j_3}(a_4)_{j_2+j_3}(b)_{j_3}}{(c_1)_{j_1+j_2+j_3}(c_2)_{j_1+j_2+2j_3}}=$$

$$F^{4:0;0;1}_{2:0;0;0}\left[\begin{matrix}(a_1:1,1,1),\ (a_2:1,1,0),\ (a_3:1,0,1),\ (a_4:0,1,1):(b:1)\\(c_1:1,1,1),\ (c_2:1,1,2)\end{matrix}\middle|\, z_1,z_2,z_3\right]. \tag{33}$$

In the case of symmetric deviation from the mass shell ($q^2 = p^2$), and bearing in mind that in our case $c_2 = a_3 + a_4 = \nu + \rho$, we can represent Φ_2 in terms of Kampé de Feriet's function (see (A.3))

$$\Phi_2\left[\begin{matrix}a_1,\ a_2,\ a_3,\ a_4;\ b\\ c_1,\ a_3+a_4\end{matrix}\middle|\, z,z,z_3\right]=\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{z^j z_3^{\,l}}{j!\,l!}\frac{(a_1)_{j+l}(a_2)_j(a_3)_l(a_4)_l(b)_l}{(c_1)_{j+l}(a_3+a_4)_{2l}}=$$

$$F^{1:1;3}_{1:0;2}\left[\begin{matrix}a_1:a_2;\ \ a_3,\ \ a_4,\ \ b\\ c_1:(a_3+a_4)/2,\ (a_3+a_4+1)/2\end{matrix}\middle|\, z,\ \frac{z_3}{4}\right], \tag{34}$$

where we have also used the doubling formula for the Γ function. As in the case (29), the sum over j represents ${}_2F_1$ and can be analytically extended to other variables.

Finally, we consider a vertex integral with three massive denominators when all the masses are the same:

$$J_3(\mu,\nu,\rho;m)\equiv\int\frac{d^n r}{(r^2-m^2)^{\mu}((p-r)^2-m^2)^{\nu}((q-r)^2-m^2)^{\rho}}. \tag{35}$$

Using formula (5), we can express it in terms of the already considered integral J_2:

$$J_3(\mu,\nu,\rho;m)=\frac{1}{\Gamma(\mu)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dv(-m^2)^v\Gamma(-v)\Gamma(\mu+v)\,J_2(\mu+v,\nu,\rho;m).$$

Substituting here the representation (31) and again using Barnes's lemma (15), we obtain the expression

$$J_3(\mu,\nu,\rho;m)=\pi^{n/2}i^{1-n}(-m^2)^{n/2-\mu-\nu-\rho}[\Gamma(\mu)\Gamma(\nu)\Gamma(\rho)]^{-1}\times$$

$$\frac{1}{(2\pi i)^3}\int\!\!\int\!\!\int_{-i\infty}^{i\infty}ds\,dt\,du\left(-\frac{p^2}{m^2}\right)^s\left(-\frac{q^2}{m^2}\right)^u\left(-\frac{k^2}{m^2}\right)^u\Gamma(-s)\Gamma(-t)\Gamma(-u)\times$$

$$\frac{\Gamma(\mu+\nu+\rho-n/2+s+t+u)\Gamma(\mu+s+t)\Gamma(\nu+s+u)\Gamma(\rho+t+u)}{\Gamma(\mu+\nu+\rho+2s+2t+2u)}. \tag{36}$$

Hence, proceeding as in the previous cases, we obtain the symmetric result

$$J_3(\mu,\nu,\rho;m)=\pi^{n/2}i^{1-n}(-m^2)^{n/2-\mu-\nu-\rho}\frac{\Gamma(\mu+\nu+\rho-n/2)}{\Gamma(\mu+\nu+\rho)}\times$$

$$\Phi_3\left[\begin{matrix}\mu+\nu+\rho-n/2,\ \mu,\ \nu,\ \rho\\ \mu+\nu+\rho\end{matrix}\middle|\,\frac{p^2}{m^2},\ \frac{q^2}{m^2},\ \frac{k^2}{m^2}\right], \tag{37}$$

where Φ_3 can also be represented in terms of a generalized hypergeometric function of three variables (see (A.4)):

$$\Phi_3\left[\begin{matrix}a_1,a_2,a_3,a_4\\ c\end{matrix}\middle|\,z_1,\ z_2,\ z_3\right]=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}\frac{z_1^{j_1}z_2^{j_2}z_3^{j_3}}{j_1!\,j_2!\,j_3!}\frac{(a_1)_{j_1+j_2+j_3}(a_2)_{j_1+j_2}(a_3)_{j_1+j_3}(a_4)_{j_2+j_3}}{(c)_{2j_1+2j_2+2j_3}}=$$

$$F^{4:0;0;0}_{1:0;0;0}\left[\begin{matrix}(a_1:1,1,1),\ (a_2:1,1,0),\ (a_3:1,0,1),\ (a_4:0,1,1)\\(c:2,2,2)\end{matrix}\middle|\,z_1,z_2,z_3\right]=$$

$$F^{4:0;0;0}_{2:0;0;0}\left[\begin{matrix}(a_1:1,1,1),\ (a_2:1,1,0),\ (a_3:1,0,1),\ (a_4:0,1,1)\\(c/2:1,1,1),\ ((c+1)/2:1,1,1)\end{matrix}\middle|\,z_1,z_2,z_3\right] \tag{38}$$

(here we have also used the doubling formula for the Γ function). In particular, when $p^2 = q^2 = k^2 = 0$, we obtain the well-known result (12) (in which $\beta = \mu + \nu + \rho$).

If we consider the case $q^2 = p^2$ ($z_1 = z_2 \equiv z$), then we readily obtain the result in the form of a generalized hypergeometric function of two variables (see (A.4)):

$$\Phi_3\left[\begin{matrix} a_1,\ a_2,\ a_3,\ a_4 \\ c \end{matrix}\middle| z, z, z_3\right] = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{z^j z_3{}^l}{j!\,l!} \frac{(a_1)_{j+l}(a_3+a_4)_{j+2l}(a_2)_j(a_3)_l(a_4)_l}{(c)_{2j+2l}(a_3+a_4)_{2l}} =$$

$$F^{2:1,\,2}_{2:0;\,1}\left[\begin{matrix} (a_1{:}1,1),\ (a_3+a_4{:}1,2){:}(a_2{:}1);\ (a_3{:}1),\ (a_4{:}1) \\ (c/2{:}1,1),\ ((c+1)/2{:}1,1);\ (a_3+a_4/2) \end{matrix}\middle| \frac{z}{4}\ \frac{z_3}{4}\right] \tag{39}$$

Note that reduction formulas of the type (29), (34), and (39) can be readily obtained from the corresponding Mellin-Barnes representations (26), (31), and (36) by means of Barnes's lemma (15).

We consider a simple special example of application of the general formula (37). If $p^2 = q^2 = 0$, then from (37) and (38) (or from (39)) we find that

$$J_3(\mu,\nu,\rho;m)\,|_{p^2=q^2=0} = \pi^{n/2} i^{1-n}(-m^2)^{n/2-\mu-\nu-\rho}\times$$

$$\frac{\Gamma(\mu+\nu+\rho-n/2)}{\Gamma(\mu+\nu+\rho)}\, {}_3F_2\left(\begin{matrix} \mu+\nu+\rho-n/2,\ \nu,\ \rho \\ (\mu+\nu+\rho)/2,\ (\mu+\nu+\rho+1)/2 \end{matrix}\middle| \frac{k^2}{4m^2}\right). \tag{40}$$

Such integrals are needed, in particular, in the calculation of the diagram corresponding to production of the Higgs boson in the process of the synthesis of gluons through a heavy-quark loop (see, for example, [32]).

For example, for the specific integral with $\mu = \nu = \rho = 1$ we can go to the limit $n \to 4$, and we obtain

$$J_3(1,1,1;m)\Big|_{\substack{p^2=q^2=0\\ n=4}} = -\frac{i\pi^2}{2m^2} f\left(\frac{k^2}{4m^2}\right),$$

where

$$f(z) \equiv {}_3F_2\left(\begin{matrix} 1,\ 1,\ 1 \\ {}^3/_2,\ 2 \end{matrix}\middle| z\right).$$

Using the formulas, given in [33], we obtain

$$f(z) = \begin{cases} z^{-1}\arcsin^2\sqrt{z}, & z\geqslant 0, \\ -z^{-1}\ln^2(\sqrt{1-z}+\sqrt{-z}), & z\leqslant 0. \end{cases}$$

Concluding this section, we note that we have represented the results for the integrals (25), (30), and (35) in the form of hypergeometric functions of the variables p^2/m^2, q^2/m^2, and k^2/m^2 (27), (32), (37), since in these variables the obtained expressions take their most compact form. Expansions with respect to other variables can be obtained from the general representations (26), (31), and (36).

Conclusions

In this paper, we have considered a general method of calculating Feynman integrals that contain massive denominators. The Mellin-Barnes representation (5) enables us to reduce the massive Feynman integrals to massless ones. At the same time, in contrast to the expansions in the series (2) and (3), the representation (5) is true for all relations between the momentum and the mass. If we know an expression for the massless integral with arbitrary index of the line corresponding to the massive denominator, then the massive integral can also be calculated. Note that we have calculated the integrals in pseudo-Euclidean space; however, it is clear that the transition to the Euclidean case does not present difficulties.

The method makes it possible to calculate the integrals for arbitrary line indices and dimension of space, and therefore it can be used in both dimensional and analytic regularization. In particular, this makes it possible to express by a single formula all results for the class of Feynman integrals in the most interesting case of integer indices (in practice, such integrals can be calculated recursively). In addition, expressions for integrals with arbitrary line indices can be used in an investigation of the compatibility of solutions of power-law form with dynamical integral equations for Green's functions (for example, in investigation of the infrared behavior of quantum chromodynamics). Note also that the method may be helpful in the case when massless singularities are regularized by the introduction of a small mass.

As a rule, the obtained expressions for the integrals can be represented in the form of hypergeometric functions of dimensionless combinations of squares of the momenta and masses. This is extremely helpful, since, using the formulas of analytic continuation, it is possible to go over from certain variables to others and investigate different ranges of variation of the momenta. In particular, to investigate processes with heavy particles it is convenient to use functions of arguments of the form p^2/m^2, and for light particles functions of arguments of the form m^2/p^2. It is also possible to investigate regions near the mass shells of the particles, and also the behavior near threshold values of the momenta.

In the present paper, we have illustrated the application of the proposed method for the example of classes of single-loop massive Feynman integrals of propagator and vertex type. So far as we know, some of the results have been obtained for the first time. For definite (integer) values of the powers of the denominators, and also after expansion with respect to $\varepsilon = (4 - n)/2$, the general formulas simplify appreciably; at the same time, it is convenient to use the formulas given in the reference book [33]. Great simplifications are also achieved by various subsidiary conditions (for example, vanishing of some line index or square of a momentum, treatment of certain momenta on the mass shell, etc.). For the known limiting cases of such kind, the obtained formulas give the correct results.

It is clear that the considered examples by no means exhaust the results that can be obtained by the proposed method. In particular, it can be used to calculate many-loop integrals with massive denominators, vertex integrals with larger number of external lines, integrals in the axial gauge, etc. We hope that continuation of investigations in this direction will make it possible to increase the number of exactly calculable diagrams in quantum field theory.

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Appendix

In this Appendix, we give definitions of the hypergeometric functions encountered in the present work (more detailed information about these functions can be found, for example, in [23,24,31,34,35]). Note that expansions of these functions in other ranges of variation of the variables can be obtained by means of analytic continuation (for this, it is convenient to represent the corresponding functions in the form of Mellin–Barnes integrals).

The generalized hypergeometric function of one variable is defined by

$$ {}_AF_B\left(\begin{matrix} a_1, \ldots, a_A \\ b_1, \ldots, b_B \end{matrix}\middle| z\right) = \sum_{j=0}^{\infty} \frac{(a_1)_j \ldots (a_A)_j}{(b_1)_j \ldots (b_B)_j} \frac{z^j}{j!}, \tag{A.1} $$

where $(a)_j \equiv \Gamma(a + j)/\Gamma(a)$ is the Pochhammer symbol.

Appell's hypergeometric function of two variables F_4 has the form

$$ F_4(a, b; c, d | z_1, z_2) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \frac{(a)_{j_1+j_2} (b)_{j_1+j_2}}{(c)_{j_1} (d)_{j_2}} \frac{z_1^{j_1} z_2^{j_2}}{j_1!\, j_2!}. \tag{A.2} $$

A more general hypergeometric function of two variables (Kampé de Feriet's function) is defined by

$$ F^{A:B;\,B'}_{C:D;\,D'}\left[\begin{matrix} a_1, \ldots, a_A : b_1, \ldots, b_B;\ b_1', \ldots, b_{B'}' \\ c_1, \ldots, c_C : d_1, \ldots, d_D\ ;\ d_1', \ldots, d_{D'}' \end{matrix}\middle| z_1, z_2\right] = $$

$$ \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \frac{(a_1)_{j_1+j_2} \cdots (a_A)_{j_1+j_2} (b_1)_{j_1} \cdots (b_B)_{j_1} (b_1')_{j_2} \cdots (b_{B'}')_{j_2}}{(c_1)_{j_1+j_2} \cdots (c_C)_{j_1+j_2} (d_1)_{j_1} \cdots (d_D)_{j_1} (d_1')_{j_2} \cdots (d_{D'}')_{j_2}} \frac{z_1^{j_1} z_2^{j_2}}{j_1! j_2!}. \tag{A.3} $$

For example, it is readily seen that $F_4 = F^{2:0;0}_{0:1;1}$.

Finally, the generalized Lauricella function of N variables [35] has the form

$$F^{A:B^{(1)};\ldots;B^{(N)}}_{C:D^{(1)};\ldots;D^{(N)}}\left[\begin{matrix}[a:\alpha^{(1)},\ldots,\alpha^{(N)}]:[b^{(1)}:\beta^{(1)}];\ldots;[b^{(N)}:\beta^{(N)}]\\ [c:\gamma^{(1)},\ldots,\gamma^{(N)}]:[d^{(1)}:\delta^{(1)}];\ldots;[d^{(N)}:\delta^{(N)}]\end{matrix}\middle| z_1,\ldots,z_N\right]=$$

$$\sum_{j_1=0}^{\infty}\cdots\sum_{j_N=0}^{\infty}\frac{\prod\limits_{i=1}^{A}(a_i)_{\alpha_i^{(1)}j_1+\ldots+\alpha_i^{(N)}j_N}}{\prod\limits_{i=1}^{C}(c_i)_{\gamma_i^{(1)}j_1+\ldots+\gamma_i^{(N)}j_N}}\,\frac{\prod\limits_{i=1}^{B^{(1)}}(b_i^{(1)})_{\beta_i^{(1)}j_1}\cdots\prod\limits_{i=1}^{B^{(N)}}(b_i^{(N)})_{\beta_i^{(N)}j_N}}{\prod\limits_{i=1}^{D^{(1)}}(d_i^{(1)})_{\delta_i^{(1)}j_1}\cdots\prod\limits_{i=1}^{D^{(N)}}(d_i^{(N)})_{\delta_i^{(N)}j_N}}\,\frac{z_1^{j_1}\ldots z_N^{j_N}}{j_1!\ldots j_N!}, \qquad (A.4)$$

where we have used the notation

$$[a:\alpha^{(1)},\ldots,\alpha^{(N)}]\equiv(a_1:\alpha_1^{(1)},\ldots,\alpha_1^{(N)}),\ldots,(a_A:\alpha_A^{(1)},\ldots,\alpha_A^{(N)});$$

$$[b^{(M)}:\beta^{(M)}]\equiv(b_1^{(M)}:\beta_1^{(M)}),\ldots,(b_{B(M)}^{(M)}:\beta_{B(M)}^{(M)});\quad M=1,\ldots,N;$$

$$[c:\gamma^{(1)},\ldots,\gamma^{(N)}]\equiv(c_1:\gamma_1^{(1)},\ldots,\gamma_1^{(N)}),\ldots,(c_C:\gamma_C^{(1)},\ldots,\gamma_C^{(N)});$$

$$[d^{(M)}:\delta^{(M)}]\equiv(d_1^{(M)}:\delta_1^{(M)}),\ldots,(d_{D(M)}^{(M)}:\delta_{D(M)}^{(M)});\quad M=1,\ldots,N.$$

In (A.4), it is understood that all α, β, γ, δ are non-negative integers, although this formula can be generalized to all non-negative values of these parameters if all the Pochhammer symbols are represented in terms of the corresponding Γ functions (see, for example, [34–35]). Note that for N = 2 the function (A.4) is sometimes called the generalized Kampé de Feriet function.

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FRACTALS IN QUANTUM THEORY: ANALYTICAL AND NUMERICAL APPROACHES

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Quantum systems whose evolution has a fractal nature are considered for the example of the evolution of a wave packet in quantum mechanics. Quantum states in which the evolution of the expectation values of certain operators are described by fractal curves are constructed. The fractal dimensions of these curves are calculated. The presence of an exact analytical result makes it possible to compare the different methods of calculating the fractal dimensions.

1. Introduction

In recent years, many papers have been devoted to the random behavior of quantum systems, the dimensions of quantum-mechanical trajectories, and other related questions. Some papers have been devoted exclusively to quantum-mechanical systems [1,2]. It has been shown that in the general case the dimension of a trajectory in quantum mechanics varies from $d = 1$ to $d = 2$. The maximal value $d = 2$ is achieved in the essentially quantum case, while the value $d = 1$ corresponds to the classical limit. Other papers have considered the quantization of classical systems in which there is already chaotic behavior [3]. Unfortunately, it was found that the quantum dynamics of the considered systems is quasiperiodic.

It has also been suggested that in the case of quantization of systems with degenerate energy levels (such as a potential with two minima) chaotic behavior of the expectation

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