$$\bar{V}^2 = \frac{1}{\sqrt{r_1(r_3-r_2)}} K\left(\frac{r_2(r_3-r_1)}{r_1(r_3-r_2)}\right). \tag{B.5}$$

$$\bar{V}^3 = \frac{1}{\sqrt{r_2(r_3-r_1)}} K\left(\frac{r_3(r_2-r_1)}{r_2(r_3-r_1)}\right). \tag{B.6}$$

Applying the operator L to (B.1)–(B.3) and the operator T to (B.4)–(B.6), we obtain series of solutions of the form

$$V_n = L^n(V^1, V^2, V^3), \tag{B.7}$$

$$\bar{V}_n = T^n(\bar{V}^1, \bar{V}^2, \bar{V}^3). \tag{B.8}$$

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MINIMAL TORI IN THE FIVE-DIMENSIONAL SPHERE IN $\mathbb{C}^3$

R. A. Sharipov

The class of surfaces that have a certain property (called complex-normal) in the five-dimensional sphere in $\mathbb{C}^3$ is considered. It is shown that the minimal tori in this class are described by the equation $u_{z\bar{z}} = e^{-2u} - e^u$, which can be integrated by the inverse scattering method. The construction of finite-gap minimal tori that are complex-normal in the five-dimensional sphere in $\mathbb{C}^3$ is described.

1. Introduction

Minimal surfaces in multidimensional spaces arise naturally as classical trajectories of relativistic strings with the Nambu [1] and Polyakov [2] Lagrangians. Geometrically, these are surfaces of zero mean curvature, whose embedding in an enveloping space is specified in many known cases by equations that can be integrated by the inverse scattering method [3–5]. The minimal surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ are described by the Liouville equation $u_{z\bar{z}}=e^u$, which is nonlinear but can be linearized by a Bäcklund transformation [6,7]. In spaces of higher dimension, and also in curved spaces, the equations of embedding of minimal surfaces can no longer be linearized, but they do possess matrix Lax pairs (see [5]), and this makes a fairly efficient investigation of their solutions possible.

In this paper, we consider minimal tori in the five-dimensional sphere $S^5 \subset \mathbb{C}^3$, the embedding of which is described by the Bullough–Dodd–Zhiber–Shabat equation

$$u_{z\bar{z}} = e^{-2u} - e^u. \tag{1.1}$$

Using the construction of finite-gap solutions of this equation from [8], we construct

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finite-gap minimal tori that are complex-normal in S^5. The situation here is analogous to the one considered in [9,10], in which significant progress was achieved in the description of tori of constant mean curvature in $\mathbb{R}^3$, S^3, and H^3 on the basis of finite-gap solutions of the sine–Gordon equation $u_{z\bar{z}}=\sin u$. Equation (1.1) was considered in the context of affine geometry in [11].

2. Complex-Normal Surfaces in the Hermitian Sphere in $\mathbb{C}^3$ and Their Scalar Invariants

We consider the space $\mathbb{C}^3$ with the standard Hermitian scalar product

$$\langle A|B\rangle=\sum_{i=1}^{3}\bar{A}_iB_i \tag{2.1}$$

and the associated Euclidean scalar product

$$(A|B)=\mathrm{Re}\langle A|B\rangle. \tag{2.2}$$

Let $r(x^1, x^2)$ be a vector function with values in $\mathbb{C}^3$ that determines the embedding of the real two-dimensional surface T in the sphere S_R of radius R in $\mathbb{C}^3$. We denote by E_1 and E_2 the tangent vectors

$$E_1=\partial_1 r=\frac{\partial r}{\partial x^1},\quad E_2=\partial_2 r=\frac{\partial r}{\partial x^2}.$$

The scalar product (2.1) induces a Hermitian metric on T

$$h_{ij}=\langle E_i|E_j\rangle=g_{ij}+i\omega_{ij}, \tag{2.3}$$

whose real part g_{ij} is the Riemannian metric induced by the scalar product (2.2), while the imaginary part

$$\omega_{ij}=(iE_i|E_j) \tag{2.4}$$

is skew-symmetric and determines a closed 2-form ω on T. The tensor field $\Omega^i_j = g^{ik}\omega_{kj}$ possesses zero trace, $\Omega^i_i = 0$, and defines the invariant $\det\Omega$ of the metric (2.3) on T.

DEFINITION. We shall say that an embedding $r\colon T\to M\subset\mathbb{C}^3$ of a real two-dimensional surface T in a real submanifold M of codimension one in $\mathbb{C}^3$ is a "complex-normal" embedding if the unit Euclidean normal N to M at every point of the surface T is Hermitian orthogonal to the tangent plane to T at this point, i.e., $\langle E_i|N\rangle = 0$.

For the complex-normal surface T we determine the vectors F_1 and F_2 that are Hermitian orthogonal to N and Euclidean orthogonal to the vectors E_1 and E_2, using the relation

$$F_i=iE_i+\Omega_i^sE_s. \tag{2.5}$$

The vectors F_1 and F_2 determine a further tensor field $f_{ij} = (F_i|F_j)$, which is related to g_{ij} and ω_{ij} by

$$f_{ij}=g_{ij}+\omega_{ik}g^{ks}\omega_{sj}.$$

For the associated tensor field $F^i_j = g^{ik}f_{kj}$ we obtain

$$F_j^i=\delta_j^i+g^{ik}\omega_{kr}g^{rs}\omega_{sj}=\delta_j^i+\Omega_r^i\Omega_j^r.$$

The scalar invariants of this field can be expressed in terms of the invariant $\det\Omega$ of the field Ω^i_j.

The vectors E_1 and E_2 form a frame in the tangent space to T, while the vectors F_1, F_2, N, and iN form a frame in the normal space. The dynamics of the first frame is determined by the equations

$$\partial_iE_j=\Gamma_{ij}^kE^k+T_{ij}^kF_k+b_{ij}N+id_{ij}N, \tag{2.6}$$

where Γ^k_{ij} are the coefficients of the metric connection on the surface T, determined by the well-known formula

$$\Gamma_{ij}^k={}^1/_2g^{ks}[\partial_ig_{sj}+\partial_jg_{is}-\partial_sg_{ij}]. \tag{2.7}$$

The dynamics of the vectors F_1 and F_2 is entirely determined by the dynamics of E_1 and E_2

from the relation (2.5). The dynamics of the vector of the unit normal to M is described by the equations

$$\partial_i N = L_i^k E_k + M_i^k F_k + iS_i N. \tag{2.8}$$

The tensor fields b_{ij}, d_{ij}, L_i^k, and M_i^k that occur in these equations as coefficients are connected by numerous relations, which follow from the choice of the frames. From the orthogonality of the vectors N and E_j, we obtain

$$b_{ij} = -L_i^k g_{kj}, \tag{2.9}$$

and from the orthogonality of iN and E_j we have

$$d_{ij} = M_i^k f_{kj} - L_i^k \omega_{kj}. \tag{2.10}$$

Differentiating (2.4) and using at the same time the relation (2.6), we obtain the equation

$$\nabla_s \omega_{ij} = T_{sj}^{\ k} f_{ki} - T_{si}^{\ k} f_{kj}, \tag{2.11}$$

by which the part of the tensor $t_{isj} = T_{is}^k f_{kj}$ skew-symmetric with respect to the indices i and j is completely determined.

In the case when the manifold M is a sphere S_R of radius R, the relations (2.8)–(2.11) described above simplify appreciably. In this case, the radius vector $r(x^1, x^2)$ is proportional to the normal r = RN, by virtue of which $E_i = \partial_i r = R\partial_i N$. Comparing this now with (2.8), we obtain

$$L_i^k = \frac{1}{R}\delta_j^i, \quad M_i^k = 0, \quad S_i = 0.$$

Further, from (2.9) and (2.10) we have for the matrices of the second quadratic forms

$$b_{ij} = -\frac{1}{R} g_{ij}, \quad d_{ij} = -\frac{1}{R}\omega_{ij}.$$

But the matrix d_{ij} is symmetric and the matrix ω_{ij} skew-symmetric, by virtue of which both these matrices are equal to zero: $d_{ij} = \omega_{ij} = 0$. Thus, in the case $M = S_R$ the vectors F_1 and F_2 for a complex-normal embedding of the surface T are identical to iE_1 and iE_2, and the relations (2.6) and (2.8) can be written in the form

$$\nabla_i E_j = T_{ij}^{\ k} F_k - \frac{1}{R} g_{ij} N, \quad \partial_i N = \frac{1}{R} E_i. \tag{2.12}$$

By virtue of (2.11), the tensor $T_{kij} = g_{ks} T_{ij}^s$ is symmetric with respect to all its indices. The Gauss, Peterson–Codazzi, and Ricci equations are obtained as consistency conditions of the equations (2.12). The Gauss equation has the form

$$R_{kij}^{s} = T_{jk}^{\ r} T_{ir}^{\ s} - T_{ik}^{\ r} T_{jr}^{\ s} + \frac{g_{jk}\delta_i^s - g_{ik}\delta_j^s}{R^2}, \tag{2.13}$$

where R_{kij}^s is the Riemann curvature tensor, determined by the metric connection (2.7) in accordance with the formula

$$R_{kij}^{s} = \partial_i \Gamma_{kj}^s - \partial_j \Gamma_{ki}^s - \Gamma_{ki}^r \Gamma_{rj}^s + \Gamma_{kj}^r \Gamma_{ri}^s. \tag{2.14}$$

In the given case, the Peterson–Codazzi and Ricci equations can be combined into a single equation, which has the form

$$\nabla_i T_j^{\ sk} - \nabla_j T_i^{\ sk} = 0. \tag{2.15}$$

The symmetric tensor T_{ijk} of the second quadratic forms has two second-order scalar invariants:

$$H^2 = T_i^{\ is} T_{js}^{\ \ j}, \quad k = T^{ijs} T_{ijs} \tag{2.16}$$

and a fourth-order invariant q, which is determined by the relation

$$q = T_{jk}^{\ \ i} T_s^{\ jk} T_{rp}^{\ \ s} T_i^{\ rp}. \tag{2.17}$$

By virtue of the specifics of the two-dimensional case (dim T = 2), the invariants (2.16) and (2.17) form a maximal set of functionally independent invariants of the symmetric tensor T_{ijk}. In addition, in the two-dimensional case (dim T = 2) it follows from the symmetry of the Riemann curvature tensor that Eq. (2.13) is equivalent to just one scalar equation, which relates the Gaussian curvature K of the surface T to the curvatures H and k of the tensor T_{ijk}:

$$2K=g^{kj}R^{s}_{ksj}=H^2-k+2R^{-2}. \quad (2.18)$$

The invariant H in (2.16), which is the length of the vector of the mean normal to the surface T,

$$Hn=T_i^{ik}F_k, \quad (2.19)$$

is the mean curvature of the surface T embedded in S, and the vector n tangent to the sphere in (2.19) is the unit vector of the mean normal to T.

3. Complex-Normal Tori of Zero Mean Curvature

The condition of vanishing of the mean curvature is a strong restriction on the class of considered surfaces, since the scalar equation H = 0 entails by virtue of (2.19) the vector equation $T_i^{ik} = 0$. When the symmetry of the tensor T_{ijk} is taken into account, this last equation means that there are just two independent components in this tensor. To exploit this circumstance, we use isothermal coordinates on the surface $x = x^1 = \mathrm{Re}\, z$ and $y = x^2 = \mathrm{Im}\, z$, in which the metric g_{ij} has the conformal form $g = 2R^2e^u dz d\bar{z}$. Then the components of the tensor T^k_{ij} can be expressed in terms of the two quantities A and B:

$$T_{11}{}^1=A,\quad T_{12}{}^2=T_{21}{}^2=T_{22}{}^1=-A,\quad T_{22}{}^2=B,\quad T_{12}{}^1=T_{21}{}^1=T_{11}{}^2=-B. \quad (3.1)$$

We calculate the coefficients of the metric connection Γ^k_{ij} in accordance with (2.7) for metric of the conformal form

$$\Gamma_{11}{}^1={}^1/_2u_x,\quad \Gamma_{11}{}^2=-{}^1/_2u_y,\quad \Gamma_{12}{}^1=\Gamma_{21}{}^1={}^1/_2u_y,\quad \Gamma_{22}{}^2={}^1/_2u_y,\quad \Gamma_{22}{}^1=-{}^1/_2u_x,\quad \Gamma_{12}{}^2=\Gamma_{21}{}^2={}^1/_2u_x, \quad (3.2)$$

after which we substitute (3.1) in the Peterson–Codazzi–Ricci equation (2.15). After appropriate calculations that take into account (3.2), we have

$$\partial_x(e^uA)=\partial_y(e^uB),\quad \partial_y(e^uA)=-\partial_x(e^uB). \quad (3.3)$$

It is readily seen that the relations (3.3) are identical to the Cauchy–Riemann conditions of holomorphicity of the function $G(z) = e^uA + ie^uB$.

The case $G(z) \equiv 0$ is trivial, since in this case the subspace generated by the vectors E, F, and N also contains their derivatives, by virtue of (2.12). This means that in the dynamics too these vectors belong to a certain real three-dimensional hyperplane in $\mathbb{C}^3$, and the surface T is a two-dimensional sphere, the central section of S_R with this hyperplane.

In the case $G(z) \not\equiv 0$, we consider a compact surface T of toric topology. Regarding z as a uniformizing parameter inherited from the universal covering $\mathbb{C}\to T$, we obtain $G(z) = \mathrm{const} \neq 0$, since in this case G(z) is holomorphic on the compact complex torus T. By a simultaneous change of scale along the x and y axes, which is equivalent to adding a constant to the function u(x, y), we can satisfy the condition $|G(z)| = 1$ and write $G(z) = \cos\vartheta + i \sin\vartheta$. We can now rewrite (3.1) in the form

$$\begin{gathered} T_{11}{}^1=e^{-u}\cos\vartheta,\quad T_{22}{}^2=e^{-u}\sin\vartheta,\quad T_{11}{}^2=-e^{-u}\sin\vartheta,\quad T_{22}{}^1=-e^{-u}\cos\vartheta,\\ T_{12}{}^1=T_{21}{}^1=-e^{-u}\sin\vartheta,\quad T_{12}{}^2=T_{21}{}^2=-e^{-u}\cos\vartheta. \end{gathered} \quad (3.4)$$

From a tensor T^k_{ij} of the form (3.4) we calculate the invariants k and q, which are defined by the relations (2.16) and (2.17),

$$k=2R^{-2}e^{-3u},\quad q=2R^{-4}e^{-6u}. \quad (3.5)$$

We calculate the curvature tensor R^s_{kij} from the connection (3.2) in accordance with (2.14) and, substituting it in the Gaussian equation in the form (2.18), we obtain an equation for the function u(x, y):

$$u_{xx}+u_{yy}=4e^{-2u}-4e^u,$$

which is identical to Eq. (1.1). The solution of this equation that corresponds to embedding of the two-dimensional torus in $S_R\subset\mathbb{C}^3$ is doubly periodic with a certain lattice of periods in the plane of the variables x and y. Below, we consider the class of finite-gap minimal surfaces in the sphere S_R, this class including compact two-dimensional tori that are complex-normal in this sphere.

4. Finite-Gap Solutions of the Equation $u_{z\bar{z}} = e^{-2u} - e^{u}$ and the Associated Orthonormal Frame in $\mathbb{C}^3$

We consider a Riemann surface Γ of even genus g with two distinguished points P_0 and P_∞, on which there is a meromorphic function $\lambda(P)$ with divisor of zero and poles $3P_0 - 3P_\infty$ and on which two involutions are defined: the holomorphic involution σ, which acts in accordance with $\lambda(\sigma P) = -\lambda(P)$, and an antiholomorphic involution τ of separating type such that

$$\lambda(\tau P)\overline{\lambda(P)}=1. \tag{4.1}$$

The Riemann surface Γ is divided by the ovals of the involution τ into two regions: the region Γ_0, which contains the point P_0, and the region Γ_∞, which contains the point P_∞. By virtue of the relation (4.1), all fixed ovals of the involution τ are projected into the unit circle on the complex λ plane. The number of these ovals does not exceed three. It is determined by the number of real tori on the Jacobian $\mathrm{Jac}(\Gamma)$, each of which consists of classes of divisors D of degree g satisfying the condition

$$D+\tau D-P_0-P_\infty=C, \tag{4.2}$$

where C is a divisor of the canonical class on Γ. By virtue of (4.2), every real divisor D determines a certain Abelian differential $\omega(P)$ of the third kind with zeros at the points of the divisor $D + \tau D$ and residues

$$\operatorname*{Res}_{P=P_0} \omega(P)=+i, \quad \operatorname*{Res}_{P=P_\infty} \omega(P)=-i$$

at the points P_0 and P_∞, where it has simple poles. Under the action of the anti-involution τ, the differential $\omega(P)$ transforms in accordance with the rule

$$\omega(\tau P)=\overline{\omega(P)}, \tag{4.3}$$

by virtue of which it is real on the τ ovals. The real torus T_0 is distinguished among the other real tori by the fact that for the divisors D from this torus the differential $\omega(P)$ is positive on all ovals of the anti-involution τ with respect to the natural orientation on $\partial\Gamma_\infty$.

Having fixed the torus T_0, we consider its subset consisting of the divisors that are invariant with respect to the composition $\tau\sigma$,

$$\tau D=\sigma D. \tag{4.4}$$

This subset is nonempty; it is a real torus T_0 in the Prymian $\mathrm{Prym}(\Gamma)$ of the Riemann surface Γ. For the divisors of such a torus, the relation (4.3) can be augmented by the relation

$$\omega(\sigma P)=\omega(P),$$

which follows from (4.4) and from the invariance of the points P_0 and P_∞ with respect to the involution σ.

We fix the local parameters $k^{-1}(P)$ and $q^{-1}(P)$ in the neighborhood of the distinguished points P_0 and P_∞ by means of the conditions

$$k^3(P)=\lambda(P), \quad \overline{k(\tau P)}=q(P). \tag{4.5}$$

Now, having fixed a positive divisor $D\in T_0\subset\mathrm{Prym}(\Gamma)$ of degree g, we construct a Baker–Akhiezer vector function $\psi(z, P)$ with values in $\mathbb{C}^3$ such that

$$\psi_1(P)=e^{ik(P)z}[k^{-1}(P)+\dots], \quad \psi_2(P)=e^{ik(P)z}[k^{-2}(P)+\dots], \quad \psi_3(P)=e^{ik(P)z}[k^{-3}(P)+\dots] \tag{4.6}$$

in the neighborhood of the distinguished point P_∞, and such that

$$\psi_1(P)=e^{iq(P)\bar{z}}[q^1(P)+\dots]e^{-u}, \quad \psi_2(P)=e^{iq(P)\bar{z}}[q^2(P)+\dots]e^{u}, \quad \psi_3(P)=e^{iq(P)\bar{z}}[q^3(P)+\dots] \tag{4.7}$$

in the neighborhood of the point P_0. The functions ψ_1, ψ_2, and ψ_3 are uniquely determined by the divisor D and by the conditions (4.6) and (4.7), and at the same time (see [8]) they satisfy the differential equations

$$\partial_z\psi_1=-u_z\psi_1+i\lambda\psi_3, \quad \partial_{\bar{z}}\psi_1=ie^{-2u}\psi_2, \quad \partial_z\psi_2=u_z\psi_2+i\psi_1, \quad \partial_{\bar{z}}\psi_2=ie^{u}\psi_3, \quad \partial_z\psi_3=i\psi_2, \quad \partial_{\bar{z}}\psi_3=i\lambda^{-1}e^{u}\psi_1, \tag{4.8}$$

the condition of compatibility of which is equivalent to Eq. (1.1). The condition $D \in T_0 \subset \mathrm{Prym}(\Gamma)$ ensures reality and nonsingularity of the finite-gap solution $u(z, \bar{z})$ of this equation, for which there is an explicit expression in Prym theta functions (see [8]). In relation to the functions ψ_1, ψ_2, and ψ_3, this same condition, expressed in the form (4.4), gives in conjunction with (4.5)

$$\psi_1(\sigma P)=-\lambda^{-1}(P)e^{-u}\overline{\psi_2(\tau P)}, \quad \psi_2(\sigma P)=\lambda^{-1}(P)e^{u}\overline{\psi_1(\tau P)}, \quad \psi_3(\sigma P)=-\lambda^{-2}(P)\overline{\psi_3(\tau P)}. \tag{4.9}$$

A remarkable feature of spectral problems associated with integrable nonlinear equations is the existence of bilinear forms — pairings or generalized Wronskians consistent with the Lax operators and, in the finite-gap case, possessing certain "resonance" properties. This last circumstance can be exploited in the construction of soliton-like solutions on a finite-gap background for these equations and in the construction of Cauchy kernels on Riemann surfaces. For the spectral problem (4.8), the pairing has the form

$$\Omega(P, Q)=\{\psi(P)\,|\,\psi(\sigma Q)\}=\psi_1(P)\psi_2(\sigma Q)\lambda(P)-\psi_2(P)\psi_1(\sigma Q)\lambda(P)-\psi_3(P)\psi_3(\sigma Q)\lambda^2(P). \tag{4.10}$$

Differentiaing (4.10) with respect to z and $\bar{z}$, and taking into account Eqs. (4.10), we obtain the relations

$$\partial_z\Omega(P, Q)=i[\lambda(Q)-\lambda(P)]\lambda(P)\psi_2(P)\psi_3(\sigma Q),$$

$$\partial_{\bar{z}}\Omega(P, Q)=ie^{u}[\lambda(P)\lambda^{-1}(Q)-1]\lambda(P)\psi_3(P)\psi_1(\sigma Q),$$

from which we see that when the arguments coincide the function (4.10) does not depend on z and $\bar{z}$. Moreover, the function $W(P) = \Omega(P, P)$ is meromorphic on Γ and can be calculated explicitly:

$$W(P)=\frac{i\,d\lambda(P)}{\lambda(P)\,\omega(P)}. \tag{4.11}$$

Because the covering $\lambda\colon \Gamma\to\mathbb{C}$ is a three-sheeted one, each value of the function $\lambda(P)$ is attained with multiplicity three, i.e., at three different points P_1, P_2, and P_3. The resonance property of $\Omega(P, Q)$ is

$$\Omega(P_i, P_j)=\begin{cases} W(P_i) & \text{for } P_i=P_j,\\ 0 & \text{for } P_i\neq P_j.\end{cases} \tag{4.12}$$

For every value of λ equal in modulus to unity, $|\lambda| = 1$, the points P_1, P_2, and P_3 lie on the ovals of the anti-involution τ and are unchanged by the action of τ. From them, we form a matrix $U = U(\lambda, z, \bar{z})$ of the form

$$U=\left\|\begin{matrix} \dfrac{e^{u/2}\psi_1(P_1)}{\sqrt{W(P_1)}} & \dfrac{e^{-u/2}\psi_2(P_1)}{\sqrt{W(P_1)}} & \dfrac{\psi_3(P_1)}{\sqrt{W(P_1)}} \\ \dfrac{e^{u/2}\psi_1(P_2)}{\sqrt{W(P_2)}} & \dfrac{e^{-u/2}\psi_2(P_2)}{\sqrt{W(P_2)}} & \dfrac{\psi_3(P_2)}{\sqrt{W(P_2)}} \\ \dfrac{e^{u/2}\psi_1(P_3)}{\sqrt{W(P_3)}} & \dfrac{e^{-u/2}\psi_2(P_3)}{\sqrt{W(P_3)}} & \dfrac{\psi_3(P_3)}{\sqrt{W(P_3)}} \end{matrix}\right\|. \tag{4.13}$$

With allowance for the invariance of the points P_1, P_2, and P_3 with respect to τ, and with allowance for (4.9), the resonance property (4.12) leads to the relation

$$e^{u}\psi_1(P_i)\overline{\psi_1(P_j)}+e^{-u}\psi_2(P_i)\overline{\psi_2(P_j)}+\psi_3(P_i)\overline{\psi_3(P_j)}=W(P_i)\delta_{ij},$$

which is equivalent to unitarity of a matrix U of the form (4.13). From this relation there also follow the reality and non-negativity of the values of the function (4.11) on the ovals of the anti-involution τ, indicating that the radicals in the matrix (4.13) are real. The columns of this unitary matrix U form a frame in $\mathbb{C}^3$:

$$L=U_1, \quad M=U_2, \quad N=U_3, \tag{4.14}$$

which consists of unit vectors that are Hermitian orthogonal to each other.

5. Finite-Gap Embeddings of Two-Dimensional Surfaces in $\mathbb{C}^3$

We consider the dynamics of the orthonormal frame (4.14). In the complex variables z and $\bar{z}$, it is determined by Eqs. (4.8) and has the form

$$\partial_z L = -\tfrac{1}{2}u_z L + i\lambda e^{u/2}N,\quad \partial_{\bar z} L = \tfrac{1}{2}u_{\bar z} L + ie^{-u}M,\quad \partial_z M = \tfrac{1}{2}u_z M + ie^{u}L,\quad \partial_{\bar z} M = -\tfrac{1}{2}u_{\bar z} M + ie^{u/2}N,$$
$$q_z N = ie^{u/2}M,\quad \partial_{\bar z} N = i\lambda^{-1}e^{u/2}L. \tag{5.1}$$

Going over to the real variables $x = x^1 = \mathrm{Re}\, z$ and $y = x^2 = \mathrm{Im}\, z$, we obtain from (5.1) the equations

$$\partial_x L = \tfrac{1}{2}u_y L + i\lambda e^{u/2}N + ie^{-u}M,\quad \partial_x M = -i\tfrac{1}{2}u_y M + ie^{-u}L + ie^{u/2}N,\quad \partial_x N = ie^{u/2}M + i\lambda^{-1}e^{u/2}L, \tag{5.2}$$

for the dynamics of the frame L, M, N with respect to x and analogous equations for the dynamics of this frame with respect to y:

$$\partial_y L = -i\tfrac{1}{2}u_x L - \lambda e^{u/2}N + e^{-u}M,\quad \partial_y M = i\tfrac{1}{2}u_x M - e^{u}L + e^{u/2}N,\quad \partial_y N = -e^{u/2}M + \lambda^{-1}e^{u/2}L. \tag{5.3}$$

We specify the embedding of the surface T in the sphere $S_R \subset \mathbb{C}^3$ parametrically by means of the function

$$r(x^1, x^2) = RN(x^1, x^2) = RN(x, y). \tag{5.4}$$

For the tangent vectors E_1 and E_2 in such an embedding we have

$$E_1 = iRe^{u/2}M + iR\lambda^{-1}e^{u/2}L,\quad E_2 = -Re^{u/2}M + R\lambda^{-1}e^{u/2}L. \tag{5.5}$$

Using the Hermitian orthogonality of the frame (4.14), we can readily establish that the vectors E_1 and E_2 are Hermitian orthogonal to the vector N of the unit normal to the sphere. Hence, the embedding (5.4) is complex-normal. Moreover, the metric g_{ij} determined in accordance with (2.3) is diagonal and has the conformal form $g = 2R^2e^u(dx^2 + dy^2)$.

Now, using Eqs. (5.2) and (5.3), recalling that λ is chosen with unit modulus, and writing $\lambda = \cos\vartheta + i\sin\vartheta$, we deduce from (5.5) the dynamics of the vectors E_1 and E_2:

$$\nabla_1 E_1 = e^{-u}\cos\vartheta F_1 - e^{-u}\sin\vartheta F_2 - 2Re^u N,\quad \nabla_2 E_1 = -e^{-u}\sin\vartheta F_1 - e^{-u}\cos\vartheta F_2,$$
$$\nabla_2 E_2 = -e^{-u}\cos\vartheta F_1 + e^{-u}\sin\vartheta F_2 - 2Re^u N, \tag{5.6}$$

with coefficients of the metric connection determined from (3.2). Comparing (5.6) with (2.12), we find the components of the tensor of second quadratic forms T^k_{ij} for the embedding (5.4). They have exactly the same form (3.4) as we obtained in Sec. 3 in the general consideration. The scalar invariants are given by the expressions (3.5). Compact finite-gap tori arise in the case when the function ψ_3 is doubly periodic. The question of the periodicity of such functions is standard in the theory of finite-gap integration and is usually solved by imposing some rather ineffective conditions on the choice of the Riemann surface Γ, namely, rationality of a certain number of ratios of the Abelian integrals on it.

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