4.3. Spin System Defined on the Space $\mathscr{H}^{1/2}\oplus\mathscr{H}^{1/2}$. SO(3)-Invariant Environment. The generator L[·] takes the form

$$L[\rho]=-i\sum_{\alpha,\beta=\alpha_1}^{\alpha_2} h(\alpha\beta)[T_{00}(\alpha\beta),\rho]+\frac{1}{2}\sum_{\substack{\alpha,\beta,\alpha',\beta'\\ Q=0,\pm1}} c_1(\alpha\beta,\alpha'\beta')\times$$

$$\{[T_{1Q}(\alpha\beta),\rho T_{1Q}^+(\alpha'\beta')]+[T_{1Q}(\alpha\beta)\rho,T_{1Q}^+(\alpha'\beta')]\}+\frac{1}{2}\sum_{i,j=13}^{15} c_{ij}\{[F_i,\rho F_j^+]+[F_i\rho,F_j^+]\},$$

where $\sum_{\alpha\beta}h(\alpha\beta)=0$, $\mathbb{C}_1\geqslant 0$, $\{c_{ij}\}_{i,j=13}^{15}\geqslant 0$, $\mathbb{C}_1$ is a 4 × 4 matrix, and $\{c_{ij}\}_{i,j=13}^{15}$ is a 3 × 3 matrix.

Therefore, in this case too it is possible to write down explicitly relations between the dissipative parameters in the form of inequalities.

Remark 5. If we require O(3) invariance of the environment and the spaces $\mathscr{H}^{1/2}(\alpha_1)$ and $\mathscr{H}^{1/2}(\alpha_2)$ have different parities, then in accordance with Theorem 3 the relations we have given admit further simplification.

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BOUNDARY-VALUE PROBLEM FOR NONLINEAR SCHRÖDINGER EQUATION

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A mixed boundary-value problem on the half-line for the nonlinear Schrödinger equation and a generalization of it are studied by the inverse scattering method. A connection is established between the conservation laws and boundary conditions in integrable boundary-value problems for the higher nonlinear Schrödinger equations. It is shown that the generalized boundary-value problem requires simultaneous consideration of regular and singular solutions for the nonlinear Schrödinger equation with repulsion.

The 20-year development of the inverse scattering transform method has made it one of the most powerful tools for studying nonlinear evolution equations (see, for example, the monograph [1] and the bibliography given there). However, apart from rare exceptions the application of the inverse scattering method is restricted to problems on a finite interval with quasiperiodic boundary conditions or to problems on the complete axis. It is well known that for linear equations problems with local boundary conditions are also of great interest. There was recently proposed in [2,3] a generalization of the inverse scattering method for nonlinear equations with local boundary conditions, and this opens up the possibility of wide application of the method to nonlinear boundary-value problems.

In this paper we consider a mixed boundary-value problem on the half-line for the nonlinear Schrödinger equation (NSE) and a certain generalization of it. The NSE has been well studied in both the classical [1] and the quantum cases [4,5]. By means of the coordinate Bethe ansatz the quantum NSE has also been investigated with local boundary

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conditions [6,7]. These results can also be reproduced by means of the algebraic Bethe ansatz in the framework of the quantum inverse scattering method. An alternative approach to the treatment of local boundary conditions for nonlinear integrable equations is outlined in [8].

1. Boundary-Value Problem for Nonlinear Schrödinger Equation

We consider the following boundary-value problem for the NSE on the positive half-axis with mixed boundary condition at the left-hand end:

$$i\psi_t=-\psi''+2\varkappa|\psi|^2\psi, \tag{1}$$

$$\psi'(x)\mp\theta\psi(x)|_{x=0}=0, \quad \theta>0. \tag{2}$$

We shall assume that the function $\psi(x)$ decreases rapidly as $x \to \infty$. In limiting cases the boundary condition (2) goes over into a Neumann condition ($\theta \to 0$) or Dirichlet condition ($\theta \to \infty$).

This boundary-value problem possesses an infinite set of conservation laws and can be included in the inverse scattering scheme [3]. For this we continue $\psi(x)$ as an even function to the negative half-axis; at $x = 0$ the potential $\psi(x)$ and its derivatives can have discontinuities. Equation (1) holds for $x \neq 0$, and it is convenient to represent the boundary condition (2) in the form of the constraints

$$\Delta\psi'=\pm 2\theta\psi_0. \tag{3}$$

(For every function φ we denote $\varphi_\pm=\varphi(\pm 0)$, $\varphi_0={}^1/_2(\varphi_++\varphi_-)$, $\Delta\varphi=\varphi_+-\varphi_-$.) As is well known, the NSE is the condition of commutativity of two differential operators (see, for example, [1])

$$[\partial_x-Q(\lambda),\ \partial_t-V(\lambda)]=0, \tag{4}$$

where

$$Q(\lambda)=-\frac{i\lambda}{2}\sigma_3+\sqrt{\varkappa}(\bar\psi\sigma_++\psi\sigma_-), \quad V(\lambda)=i\left(\frac{\lambda^2}{2}+\varkappa|\psi|^2\right)+i\sqrt{\varkappa}[(\psi'+i\lambda\psi)\sigma_--(\bar\psi'-i\lambda\bar\psi)\sigma_+],$$

where σ_a, a = 1, 2, 3 are Pauli matrices. It can be represented in the integrated form

$$\partial_t T_0(x,y,\lambda)=V(x,\lambda)T_0(x,y,\lambda)-T_0(x,y,\lambda)V(y,\lambda). \tag{5}$$

Here, $T_0(x, y, \lambda)$ is the monodromy matrix of the auxiliary linear problem, satisfying the equation $\partial_x T_0(x, y, \lambda) = Q(x, \lambda)T_0(x, y, \lambda)$ and the initial condition $T_0(y, y, \lambda) = I$, where I is the 2×2 unit matrix. It is shown in [3] that the constraint (3) is equivalent to the equation

$$V_+(\lambda)K(\lambda)=K(\lambda)V_-(\lambda), \quad K(\lambda)=\lambda\pm i\theta\sigma_3. \tag{6}$$

We consider the function

$$T(x,y,\lambda)=\begin{cases} T_0(x,y,\lambda), & \text{if} \quad xy>0,\\ T_0(x,0,\lambda)K(\lambda)T_0(0,y,\lambda), & \text{if} \quad x>0>y,\\ T_0(x,0,\lambda)K^{-1}(\lambda)T_0(0,y,\lambda), & \text{if} \quad x<0<y.\end{cases} \tag{7}$$

Then the relations (4) for $x \neq 0$ and (6) are together equivalent to the equation

$$\partial_t T(x,y,\lambda)=V(x,\lambda)T(x,y,\lambda)-T(x,y,\lambda)V(y,\lambda), \tag{8}$$

which is completely analogous to (5) and makes it possible to include the boundary-value problem (1)-(2) in the scheme of the inverse scattering method.

2. Generalized Boundary-Value Problem

Now let $\psi(x)$ be an arbitrary function on the real axis. In the definition (7) we replace the matrix $K(\lambda)$ by $L(\lambda) = \lambda + iS^a\sigma_a$, the L operator of the lattice isotropic Heisenberg magnet. We shall assume that for $\varkappa<0$ the vector $\mathbf{S}=(S^1, S^2, S^3)$ is real, and for $\varkappa>0$ that S^3 is real while S^1 and S^2 are purely imaginary; in both cases we assume $S^aS^a = \theta^2 > 0$. Then the reality properties of $L(\lambda)$ and $T_0(\lambda)$ are the same and lead to the relation

$$\overline{T(x,y,\bar\lambda)}=\sigma T(x,y,\lambda)\sigma^{-1}, \quad \sigma=\begin{pmatrix}0 & \operatorname{sign}\varkappa\\ 1 & 0\end{pmatrix}. \tag{9}$$

The relation (8) for the newly defined function $T(x, y, \lambda)$ leads to Eq. (1) for $\psi(x)$ when $x \neq 0$ and to the equation

$$\partial_t L(\lambda)=V_+(\lambda)L(\lambda)-L(\lambda)V_-(\lambda), \tag{10}$$

which is equivalent to the following equations for the vector $\mathbf{S}$:

$$\begin{gathered} S_t^3=-i\sqrt{\varkappa}(\overline{\psi}_0'S^+ + S^-\psi_0') \quad (S^\pm=S^1\pm iS^2), \qquad S_t^+=2i\sqrt{\varkappa}(S^3\psi_0'-\sqrt{\varkappa}|\psi|_0^2S^+), \\ S_t^-=2i\sqrt{\varkappa}(\overline{\psi}_0'S^3+\sqrt{\varkappa}S^-|\psi|_0^2) \end{gathered} \tag{11}$$

and the constraints

$$C_1\equiv\Delta\psi-\frac{1}{\sqrt{\varkappa}}S^+=0, \quad \overline{C}_1\equiv\Delta\overline{\psi}+\frac{1}{\sqrt{\varkappa}}S^-=0, \quad C_2\equiv\Delta\psi'-2S^3\psi_0=0. \tag{12}$$

We call Eqs. (1) and (11) with the constraints (12) the generalized boundary-value problem or the NSE with spin impurity.*

The generalized boundary-value problem admits the symmetric reduction $\psi(x) = \psi(-x)$, $S^1 = S^2 = 0$, $S^3 = \pm\theta$, which leads to the boundary-value problem (1)–(3). In terms of the auxiliary linear problem the symmetric reduction is equivalent to the equation

$$T^t(x, y, \lambda)=\operatorname{sign}(xy)\cdot\sigma_1 T(-y, -x, -\lambda)\sigma_1.$$

The following arguments are traditional for the inverse scattering scheme, and we shall not dwell on the details. We introduce the Jost functions

$$T_\pm(x,\lambda)=\lim_{y\to\pm\infty}\left[T(x,y,\lambda)\exp\left(-\frac{i\lambda y}{2}\sigma_3\right)\right],$$

which for real λ are connected by the transition matrix $\mathbf{T}(\lambda)$:

$$T_-(x,\lambda)=T_+(x,\lambda)\mathbf{T}(\lambda), \quad \det\mathbf{T}(\lambda)=\lambda^2+\theta^2. \tag{13}$$

It follows from (9) that

$$\overline{T_\pm(x,\overline{\lambda})}=\sigma T_\pm(x,\lambda)\sigma^{-1}, \quad \mathbf{T}(\lambda)=\begin{pmatrix} a(\lambda) & \operatorname{sign}\varkappa\cdot\overline{b}(\lambda) \\ b(\lambda) & \overline{a}(\lambda)\end{pmatrix}, \quad |a(\lambda)|^2-\operatorname{sign}\varkappa\cdot|b(\lambda)|^2=\lambda^2+\theta^2. \tag{14}$$

The columns $T_-^{(1)}(x,\lambda)$, $T_+^{(2)}(x,\lambda)$ and $T_-^{(2)}(x,\lambda)$, $T_+^{(1)}(x,\lambda)$, respectively, are analytic in the upper and lower planes of the spectral parameter λ except, perhaps, for simple poles of the columns of the matrix $T_+(x, \lambda)$ for $x < 0$ at the points $\lambda = \pm i\theta$ inherited from $L^{-1}(\lambda)$. The function $a(\lambda)$ admits analytic continuation to the upper half-plane λ and has there, in general, zeros, which, as a rule, we shall assume are simple and not real but finite in number (other possibilities can be achieved by means of a limiting process). In addition, we assume that $a(i\theta) \neq 0$; the case when $a(i\theta) = 0$ will be considered in the Appendix. We shall discuss the zeros of the function $a(\lambda)$ in more detail in the following sections. If $a(\mu) = 0$, $\operatorname{Im}\mu > 0$, then the columns $T_-^{(1)}(x,\mu)$ and $T_+^{(2)}(x,\mu)$ are proportional:

$$T_-^{(1)}(x,\mu)=\gamma(\mu)T_+^{(2)}(x,\mu), \quad 0<|\gamma(\mu)|<\infty. \tag{15}$$

In the case of a symmetric reduction, we have

$$\begin{gathered} T_+^t(x,\lambda)=\operatorname{sign}x\cdot\sigma_1T_-^{-1}(-x,-\lambda)\sigma_1, \quad \overline{a(-\overline{\lambda})}=-a(\lambda), \quad b(-\lambda)=-b(\lambda), \\ \gamma(\mu)\overline{\gamma(-\overline{\mu})}=\operatorname{sign}\varkappa\cdot(\mu^2+\theta^2), \end{gathered} \tag{16}$$

and this restricts the positions of the purely imaginary zeros of $a(\lambda)$.

The Jost functions and elements of the transition matrix have the following behaviors as $\lambda \to \infty$:

$$\begin{aligned} T_+(x,\lambda)&=\exp\left(-\frac{i\lambda x}{2}\sigma_3\right)(I+o(1)), \quad \text{if} \quad x>0, \\ T_+(x,\lambda)&=\lambda^{-1}\exp\left(-\frac{i\lambda x}{2}\sigma_3\right)(I+o(1)), \quad \text{if} \quad x<0, \end{aligned} \tag{17}$$

*The transition to the generalized boundary-value problem and the subsequent symmetric reduction were proposed by E. K. Sklyanin.

$$T_-(x,\lambda)=\lambda\exp\left(-\frac{i\lambda x}{2}\sigma_3\right)(\mathrm{I}+o(1)),\quad \text{if}\quad x>0,$$

$$T_-(x,\lambda)=\exp\left(-\frac{i\lambda x}{2}\sigma_3\right)(\mathrm{I}+o(1)),\quad \text{if}\quad x<0.$$

A complete set of scattering data is provided by the functions $b(\lambda)$, the zeros of the function $a(\lambda)$: $\{\lambda_j$: $\mathrm{Im}\,\lambda_j > 0\}$, and the coefficients $\{\gamma(\lambda_j)\}$. The element $a(\lambda)$ can be recovered from its zeros and the element $b(\lambda)$:

$$a(\lambda)=(\lambda+i\theta)\prod_j\frac{\lambda-\lambda_j}{\lambda-\bar\lambda_j}\exp\left(\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\log(1+\mathrm{sign}\,\varkappa\,|b(\mu)|^2(\mu^2+\theta^2)^{-1})}{\mu-\lambda-i0}d\mu\right).$$

From Eq. (10) there follows the familiar dependence of the scattering data on the time:

$$a(t,\lambda)=a(0,\lambda),\quad b(t,\lambda)=b(0,\lambda)\exp(-i\lambda^2t),\quad \gamma(t,\mu)=\gamma(0,\mu)\exp(-i\mu^2t). \tag{19}$$

3. Equations of the Inverse Problem

The equations of the inverse problem can be conveniently represented in the form of a conjugation problem for meromorphic functions:

$$F(\lambda)+\frac{B(\lambda)}{A(\lambda)}\hat F(\lambda)=G(\lambda),\quad \hat F(\lambda)=\sigma\overline{F(\bar\lambda)},\quad F(\lambda)=\exp\left(-\frac{i\lambda x}{2}\right)\left[\begin{pmatrix}1\\0\end{pmatrix}+o(1)\right],$$

$$G(\lambda)=\exp\left(-\frac{i\lambda x}{2}\right)\left[\begin{pmatrix}1\\0\end{pmatrix}+o(1)\right]\quad\text{as}\quad\lambda\to\infty,\quad \operatorname*{res}_{\lambda=\lambda_j}G(\lambda)=\frac{\Gamma(\lambda_j)}{\dot A(\lambda_j)}\hat F(\lambda_j),\quad \dot A(\lambda)=\frac{d}{d\lambda}A(\lambda). \tag{20}$$

Here, $A(\lambda)$ is a function analytic in the upper half-plane of λ with simple zeros at the points λ_j, $\mathrm{Im}\,\lambda_j > 0$, and unit asymptotic behavior as $\lambda\to\infty$ with $|A(\lambda)|^2-\mathrm{sign}\,\varkappa|B(\lambda)|^2=1$ for real λ; $F(\lambda)$ is a vector-valued function that is analytic in the lower half-plane of λ and satisfies $\hat F(\lambda_j)\neq 0$, and $G(\lambda)$ is a meromorphic function in the upper half-plane of λ with simple poles at $\lambda=\lambda_j$. It is precisely the same problem that is the equation of the inverse problem for the NSE on the complete axis [1].

Specifically, the relations (13)-(15) and asymptotic behaviors (17) lead to several different conjugation problems for the positive and negative half-axes x, namely

$$F^{(+)}(\lambda)=T_+^{(1)}(x,\lambda),\quad F^{(-)}(\lambda)=(\lambda+i\theta)T_+^{(1)}(x,\lambda),$$

$$G^{(+)}(\lambda)=\frac{1}{a(\lambda)}T_-^{(1)}(x,\lambda),\quad G^{(-)}(\lambda)=\frac{\lambda+i\theta}{a(\lambda)}T_-^{(1)}(x,\lambda), \tag{21($\pm$)}$$

$$A^{(\pm)}(\lambda)=\frac{a(\lambda)}{\lambda+i\theta},\quad B^{(\pm)}(\lambda)=\frac{b(\lambda)}{\lambda\pm i\theta},\quad \Gamma(\mu)=\frac{\gamma(\mu)}{\mu\pm i\theta}\quad (x\gtrless 0).$$

We denote by $\psi^{(\pm)}(x)$ the potentials for the NSE on the complete axis corresponding to the scattering data $\{A^{(\pm)}(\lambda), B^{(\pm)}(\lambda), \{\Gamma^{(\pm)}(\lambda_j)\}\}$:

$$A^{(+)}(\lambda)=A^{(-)}(\lambda),\quad B^{(+)}(\lambda)=\frac{\lambda-i\theta}{\lambda+i\theta}B^{(-)}(\lambda),\quad \Gamma^{(+)}(\mu)=\frac{\mu-i\theta}{\mu+i\theta}\Gamma^{(-)}(\mu). \tag{22}$$

Then for the potential $\psi(x)$ in the generalized boundary-value problem we shall have $\psi(x)=\psi^{(+)}(x)$ if $x > 0$ and $\psi(x)=\psi^{(-)}(x)$ if $x < 0$. If $\varkappa<0$, then the described conjugation problems can be uniquely solved for all x [1]. For $\varkappa>0$ solvability can break down at not more than n points (n is the number of zeros of the function $a(\lambda)$), and the potentials $\psi^{(\pm)}(x)$ will have n poles with allowance for multiplicity (as a rule, n simple poles) [9], but $\psi(x)$ may also be regular.

It remains to show that the ratio $T_+(+0,\lambda)T_+^{-1}(-0,\lambda)$ found from the conjugation problems is indeed the L operator of the Heisenberg magnet. For this we construct the vector $\mathbf{S}$ with regular properties such that $(\lambda+iS^a\sigma_a)T_+^{(1)}(-0,\lambda)$ and $(\lambda+iS^a\sigma_a)T_-^{(1)}(-0,\lambda)$ satisfy the problem (20), (21$^{(+)}$). The first of Eqs. (20) is satisfied immediately; the second together with the relation for the residues follows from the reality properties of the components of the vector $\mathbf{S}$; the properties of analyticity and the asymptotic behavior are obvious; it is sufficient to establish that there is no pole of $(\lambda+iS^a\sigma_a)T_+^{(1)}(-0,\lambda)$ at $\lambda=-i\theta$. This can be achieved by the choice of the vector $\mathbf{S}$: $-\theta+S^a\sigma_a$ is a degenerate matrix, and its root vector is $\operatorname{res}_{\lambda=-i\theta} T_+^{(1)}(-0, \lambda)$ (the possibility is verified by direct calculation). By virtue of the uniqueness of the solution of the conjugation problem we obtain the required equation

$$T_+(+0, \lambda) = (\lambda + iS^a\sigma_a) T_+(-0, \lambda). \tag{23}$$

4. Local Integrals of the Motion and Constraints

To calculate the local integrals of the motion we go over to the generalized boundary-value problem on the interval $[-\ell, \ell]$ with impurity at $x = 0$ and periodic boundary conditions $\psi^{(k)}(l) = \psi^{(k)}(-l)$, $k=0, 1, \ldots$. The generating functional for the integrals of the motion is $\operatorname{tr} T(\ell, -\ell, \lambda)$. We shall proceed as in the case without impurity, using the notation and results of [1].

We consider for the monodromy matrix $T_0(x, y, \lambda)$ the representation

$$T_0(x, y, \lambda) = (I + W(x, \lambda)) \exp(Z(x, y, \lambda)) (I + W(y, \lambda))^{-1}, \tag{24}$$

where $W(x, \lambda)$ is an antidiagonal matrix, and $Z(x, y, \lambda)$ is a diagonal matrix with the properties

$$W(x, \lambda) = o(1), \quad Z(x, y, \lambda) = -\frac{i\lambda(x-y)}{2}\sigma_3 + o(1), \quad \lambda \to \infty.$$

By virtue of (24),

$$\operatorname{tr} T(l, -l, \lambda) = \operatorname{tr}[(I + W_+(\lambda))^{-1} L(\lambda)(I + W_-(\lambda)) \exp(Z(l, +0, \lambda) + Z(-0, -l, \lambda))]$$

(see (3)). Using the well-known asymptotic expansions for $W(x, \lambda)$ and $Z(x, y, \lambda)$ [1] and the relation (9), we find that

$$\operatorname{tr} T(l, -l, \lambda) = \sqrt{\lambda^2 + \theta^2}\left[\exp\left(-i\lambda l + i\varkappa \sum_{n=1}^{\infty} J_n \lambda^{-n}\right) + \exp\left(i\lambda l - i\varkappa \sum_{n=1}^{\infty} \bar{J}_n \lambda^{-n}\right)\right].$$

The functionals $I_n = \operatorname{Re} J_n$ generalize naturally the conservation laws of the NSE without impurity, and the functionals $K_n = \operatorname{Im} J_n$ depend only on $\psi_\pm^{(k)}$ and $\mathbf{S}$. The simplest integrals of the motion have the form

$$\mathbf{N} = I_1 = \int |\psi(x)|^2 dx + \frac{1}{\varkappa} S^3, \quad K_1 = 0,$$

$$\mathbf{P} = I_2 = -\frac{i}{2}\int (\bar{\psi}'(x)\psi(x) - \bar{\psi}(x)\psi'(x)) dx + \frac{i}{2}[(2\bar{C}_1 - \Delta\bar{\psi})\psi_0 - \bar{\psi}_0(2C_1 - \Delta\psi)], \quad K_2 = -\frac{1}{2}\bar{C}_1 C_1,$$

$$\mathbf{H} = I_3 = \int (|\psi'(x)|^2 + \varkappa|\psi(x)|^4) dx + \frac{1}{2\varkappa} S^3\left(4\varkappa|\psi_0|^2 + \frac{1}{3}S^{3^2} - \theta^2\right) + \bar{C}_1\psi_0' + \bar{\psi}_0' C_1 - \frac{1}{2}\bar{C}_1 C_1,$$

$$K_3 = \frac{i}{2}(\bar{C}_1 C_2 - \bar{C}_2 C_1) \qquad \left(\int = \text{v. p.} \int_{-l}^{l}\right).$$

The higher NSEs with impurity are obtained from Eq. (8) by replacing $V(x, \lambda)$ by a suitable matrix $V_m(x, \lambda)$ $(V_1(x, \lambda) = (i/2)\sigma_3,\ V_2(x, \lambda) = -Q(x, \lambda),\ V_3(x, \lambda) = V(x, \lambda))$. For them the constraints are written as follows:

$$\frac{\partial}{\partial\lambda}[V_{m+}(\lambda) L(\lambda) - L(\lambda) V_{m-}(\lambda)] = 0. \tag{25}$$

By means of the generating functional for $V_m(x, \lambda)$ the constraints (25) can be transformed to

$$[(I + W_+(\mu))^{-1} L(\mu)(I + W_-(\mu)), \sigma_3] = O(\mu^{-m+1}), \tag{26}$$

i.e.,

$$(I + W_+(\mu))^{-1} L(\mu)(I + W_-(\mu)) = \sqrt{\mu^2 + \theta^2}[\exp(\mathscr{K}(\mu)) + O(\mu^{-m})], \tag{27}$$

where $\mathscr{K}(\mu)$ is a diagonal matrix.

Since $\det T(\ell, -\ell, \lambda) = \lambda^2 + \theta^2$,

$$\det\exp(\mathscr{H}(\lambda)+Z(\lambda))=1+O(\lambda^{-2m}) \tag{28}$$

and

$$\operatorname{tr}(\mathscr{H}(\lambda)+Z(\lambda))=-2\varkappa\sum_{n=1}^{\infty}K_n\lambda^{-n}=O(\lambda^{-2m}).$$

Therefore

$$K_n=0,\quad n=1,\ldots,2m-1. \tag{29}$$

All the arguments, except for the transition from (27) to (28), can be readily generalized. To eliminate this gap, we can use the relation

$$\overline{(\mathrm{I}+W_+(\lambda))^{-1}L(\lambda)(\mathrm{I}+W_-(\lambda))}=\sigma(\mathrm{I}+W_+(\bar\lambda))^{-1}L(\bar\lambda)(\mathrm{I}+W_-(\bar\lambda))\sigma^{-1},$$

which follows from Eq. (9). Thus, we have established that the constraints for the m-th NSE with impurity are equivalent to the vanishing of the $2m-1$ lowest integrals of the motion K_n.

To go to the limit $\ell\to\infty$ we use the equations

$$\log\operatorname{tr}(e^{i\lambda l}T(l,-l,\lambda))=\frac{1}{2}\log(\lambda^2+\theta^2)+i\varkappa\sum_{n=1}^{\infty}J_n\lambda^{-n}+O(|\lambda|^{-\infty}),\quad \lambda\to i\infty,$$

$$\lim_{l\to\infty}\operatorname{tr}(e^{i\lambda l}T(l,-l,\lambda))=a(\lambda),\quad \text{if}\quad \operatorname{Im}\lambda>0,$$

which together lead to the relation

$$\log a(\lambda)=\frac{1}{2}(\lambda^2+\theta^2)+i\varkappa\sum_{n=1}^{\infty}J_n\lambda^{-n}+O(|\lambda|^{-\infty}),\quad \lambda\to\infty,$$

for the generalized boundary-value problem on the complete axis. With allowance for (14), the constraints (29) can now be represented in the form

$$|a(\lambda)|^2=(\lambda^2+\theta^2)(1+O(\lambda^{-2m})),\quad b(\lambda)=O(\lambda^{1-m}). \tag{30}$$

If in the scattering data it is assumed that $b(\lambda)=O(\lambda^{1-m})$, then, solving the inverse problem, we obtain $m-1$ times differentiable potentials $\psi^{(\pm)}(x)$. The smoothness of $\psi(x)$ will be not worse (almost always the same) than that of $\psi^{(\pm)}(x)$. This allows us to consider the m lowest equations and the constraints for them. In addition, it can be seen from comparison with (30) that all the allowed constraints are satisfied.

5. Soliton Solutions ($\varkappa<0$)

We consider in more detail the case $\varkappa<0$. The vector $\mathbf{S}$ lies on a sphere of radius θ. The function $a(\lambda)$ can have arbitrarily situated zeros in the upper half-plane of λ. The conjugation problems (20) and (21) can be uniquely solved for all x and make it possible to recover the regular potentials $\psi^{(\pm)}(x)$ and $\psi(x)$.

After symmetric reduction in the phase space there are distinguished two unconnected components, which differ in the sign of S^3. We describe their separation in terms of scattering data. Transforming the relation (13) into $T(\lambda)=T_{\mp}^{-1}(+0,\lambda)L(\lambda)T_-(-0,\lambda)$ and using (16), we can readily see that $-ia(i\theta)S^3>0$. By virtue of (16) the zeros of $a(\lambda)$ are situated either in pairs symmetric with respect to the imaginary axis or on the imaginary axis above the point $i\theta$. It can be seen from the representation (18) that the factors corresponding to the function $b(\mu)$ and pairs of symmetric zeros are positive, while the factors corresponding to purely imaginary zeros are negative for $\lambda=i\theta$. Thus, the sign of $-ia(i\theta)$ is determined by the parity of n, the number of zeros of the function $a(\lambda)$, and the preceding inequality is transformed to the convenient form $(-1)^nS^3>0$, which is also valid in the case when $a(i\theta)=0$.

We give some of the simplest soliton solutions ($b(\lambda)=0$), parametrizing them by the values of the scattering data at the initial time $t=0$ (the time dependence is given by the relations (19)):

$$a_{\mathrm{I}}(\lambda)=(\lambda+i\theta)\frac{\lambda-\lambda_0}{\lambda-\bar\lambda_0},\quad \lambda_0=\frac{1}{2}(v+iu),\quad \lambda_0\neq i\theta,\quad \gamma_{\mathrm{I}}(\lambda_0)=\gamma=(\lambda_0+i\theta)\exp\left(\frac{u}{2}x_\pm+i\varphi_\pm\right),$$

$$\psi_{\mathrm{I}}^{(\pm)}(x)=\frac{u}{2\sqrt{\varkappa}}\frac{\exp\left[i\left(\varphi_{\pm}+\frac{vx}{2}+\frac{u^2-v^2}{4}t\right)\right]}{\mathrm{ch}\left[\frac{u}{2}(x-vt-x_{\pm})\right]},\quad \overline{S_{\mathrm{I}}^{+}}=S_{\mathrm{I}}^{-}=\frac{2\theta\bar{\gamma}(\lambda_0-\bar{\lambda}_0)}{\mathscr{D}}(\lambda_0^2+\theta^2+|\gamma|^2),$$

$$S_{\mathrm{I}}^3=\frac{1}{\mathscr{D}}[(\lambda_0^2+\theta^2+|\gamma|^2)(\bar{\lambda}_0^2+\theta^2+|\gamma|^2)+(\lambda_0-\bar{\lambda}_0)|\gamma|^2],$$

$$\mathscr{D}=(|\lambda_0+i\theta|^2+|\gamma|^2)(|\lambda_0-i\theta|^2+|\gamma|^2),$$

$$a_{\mathrm{II}}(\lambda)=(\lambda-i\theta)\frac{\lambda-\lambda_0}{\lambda-\bar{\lambda}_0},\quad \gamma_{\mathrm{II}}(\lambda_0)=\gamma\frac{\lambda_0-i\theta}{\lambda_0+i\theta},\quad \gamma^{(+)}=0,\quad \gamma^{(-)}=\infty,\quad \psi_{\mathrm{II}}^{(\pm)}(x)=\psi_{\mathrm{I}}^{(\mp)}(x),\quad \mathbf{S}_{\mathrm{II}}=-\mathbf{S}_{\mathrm{I}}.$$

The solutions $\psi_{\mathrm{I}}(x, t)$ and $\psi_{\mathrm{II}}(x, t)$ describe the propagation of a soliton. Interaction with the impurity leads to a shift of the center of the soliton and a jump of the phase:

$$\Delta x_{\mathrm{I}}=-\Delta x_{\mathrm{II}}=x_+-x_-=\frac{2}{u}\log\left|\frac{\lambda_0-i\theta}{\lambda_0+i\theta}\right|,\quad \Delta\varphi_{\mathrm{I}}=-\Delta\varphi_{\mathrm{II}}=\varphi_+-\varphi_-=\arg\left(\frac{\lambda_0-i\theta}{\lambda_0+i\theta}\right).$$

The two following solutions are interesting in that they are continually localized on one half-axis:

$$a_{\mathrm{III}}(\lambda)=\lambda-i\theta,\quad \overset{(+)}{\gamma_{\mathrm{III}}}=\gamma=\exp(\theta x_0+i\varphi),\quad \overset{(-)}{\gamma_{\mathrm{III}}}=\infty,$$

$$\overset{(+)}{\psi_{\mathrm{III}}}(x)=\frac{\theta}{\sqrt{\varkappa}}\frac{\exp(i\varphi+i\theta^2t)}{\mathrm{ch}\,\theta(x-x_0)},\quad \overset{(-)}{\psi_{\mathrm{III}}}(x)=0,\quad \overline{S^{+}_{\mathrm{III}}}=S^{-}_{\mathrm{III}}=\frac{4i\theta^2\bar{\gamma}}{|\gamma|^2+4\theta^2},\quad S^3_{\mathrm{III}}=\theta\frac{|\gamma|^2-4\theta^2}{|\gamma|^2+4\theta^2};$$

$$a_{\mathrm{IV}}(\lambda)=\lambda-i\theta,\quad \overset{(+)}{\gamma_{\mathrm{IV}}}=0,\quad \overset{(-)}{\gamma_{\mathrm{IV}}}=\gamma,\quad \overset{(\pm)}{\psi_{\mathrm{IV}}}(x)=\overset{(\mp)}{\psi_{\mathrm{III}}}(x),\quad \mathbf{S}_{\mathrm{IV}}=-\mathbf{S}_{\mathrm{III}}.$$

The symmetric solutions corresponding to the scattering data

$$a_{\mathrm{V}}(\lambda)=(\lambda+i\theta)\frac{(\lambda-\lambda_0)(\lambda+\bar{\lambda}_0)}{(\lambda-\bar{\lambda}_0)(\lambda+\lambda_0)},\quad \gamma_{\mathrm{V}}(\lambda_0)\overline{\gamma_{\mathrm{V}}(-\bar{\lambda}_0)}=-(\lambda_0^2+\theta^2);$$

$$a_{\mathrm{VI}}(\lambda)=(\lambda-i\theta)\frac{(\lambda-\lambda_0)(\lambda+\bar{\lambda}_0)}{(\lambda-\bar{\lambda}_0)(\lambda+\lambda_0)},\quad \gamma_{\mathrm{VI}}(\lambda_0)\overline{\gamma_{\mathrm{VI}}(-\bar{\lambda}_0)}=-(\lambda_0^2+\theta^2),\quad \overset{(+)}{\gamma_{\mathrm{VI}}}=0,\quad \overset{(-)}{\gamma_{\mathrm{VI}}}=\infty,$$

describe reflection of a soliton incident from infinity on the boundary in the boundary-value problem (1)–(2), and $S^3_{\mathrm{V}} = - S^3_{\mathrm{VI}} = \theta$.

6. Soliton Solutions ($\varkappa>0$)

We turn to the case $\varkappa>0$. We recall that now only the component S^3 of the vector $\mathbf{S}$ is real, while the remaining two are purely imaginary. Therefore, the vector $\mathbf{S}$ lies on a two-sheeted hyperboloid, the sheets of which differ in the value of S^3.

We show that the inequality $S^3 > 0$ is equivalent to the function $a(\lambda)$ having no zeros. In the proof of the direct assertion we shall base ourselves on the formal self-adjointness of the operator $\mathscr{L}=i\sigma_3\partial_x+i\sqrt{\varkappa}(\psi\sigma_--\bar{\psi}\sigma_+)$. We denote by $\langle,\rangle$ the natural scalar product in $\mathbb{C}^2$, which is linear in the second factor. The Jost functions satisfy the differential equation $\mathscr{L}T_\pm(x, \lambda)=\lambda T_\pm(x, \lambda)$ for $x \neq 0$, whence, using the asymptotic behaviors

$$T_-^{(1)}(x,\lambda)=O(\exp(x\,\mathrm{Im}\,\lambda)),\quad x\to-\infty,\quad T_+^{(2)}(x,\lambda)=O(\exp(-x\,\mathrm{Im}\,\lambda)),\quad x\to+\infty,$$

and integrating by parts, we obtain for $\mathrm{Im}\ \lambda > 0$

$$\begin{aligned}&\int_{-\infty}^{0}\langle T_-^{(1)}(x,\lambda),\mathscr{L}(x)T_-^{(1)}(x,\lambda)\rangle dx=\lambda\int_{-\infty}^{0}|T_-^{(1)}(x,\lambda)|^2dx=\\&\bar{\lambda}\int_{-\infty}^{0}|T_-^{(1)}(x,\lambda)|^2dx+2i\langle T_-^{(1)}(-0,\lambda),\sigma_3T_-^{(1)}(-0,\lambda)\rangle,\\&\int_{0}^{\infty}\langle T_+^{(2)}(x,\lambda),\mathscr{L}(x)T_+^{(2)}(x,\lambda)\rangle dx=\lambda\int_{0}^{\infty}|T_+^{(2)}(x,\lambda)|^2dx=\\&\bar{\lambda}\int_{0}^{\infty}|T_+^{(2)}(x,\lambda)|^2-2i\langle T_+^{(2)}(+0,\lambda),\sigma_3T_+^{(2)}(+0,\lambda)\rangle.\end{aligned}\tag{31}$$

If $a(\lambda)=0$, then $L(\lambda)T_-^{(1)}(-0,\lambda)=\gamma(\lambda)T_+^{(2)}(+0,\lambda)$ and

$$\langle T_+^{(2)}(+0,\lambda),\sigma_3 T_+^{(2)}(+0,\lambda)\rangle=\frac{1}{|\gamma(\lambda)|^2}\langle T_-^{(1)}(-0,\lambda),\sigma_3(|\lambda|^2+\theta^2+i(\bar\lambda-\lambda)S^a\sigma_a)T_-^{(1)}(-0,\lambda)\rangle. \tag{32}$$

Substituting (32) in (31) and making some simple manipulations, we obtain

$$(\lambda-\bar\lambda)\left[\frac{|\lambda|^2+\theta^2}{|\gamma(\lambda)|^2}\int_{-\infty}^{0}|T_-^{(1)}(x,\lambda)|^2dx+\int_0^\infty|T_+^{(2)}(x,\lambda)|^2dx+\right.$$

$$\left.\frac{2}{|\gamma(\lambda)|^2}\langle T_-^{(1)}(-0,\lambda),\sigma_3 S^a\alpha_0 T_-^{(1)}(-0,\lambda)\rangle\right]=0.$$

We can verify that by virtue of the inequality $S^3>|S^{1^2}+S^{2^2}|^{1/2}$ the integrated term is always positive and, therefore, $\lambda=\bar\lambda$. However, on the real axis the function $a(\lambda)$ has no zeros, since $|a(\lambda)|^2=\lambda^2+\theta^2+|b(\lambda)|^2\geqslant\theta^2>0$.

To prove the inverse assertion, we consider the function $b_\varepsilon(\lambda)=\varepsilon b(\lambda)$. The conjugation problems (20) and (21) are uniquely solvable for all ε and x, and S^3 depends continuously on ε. Since for $\varepsilon=0$ we have $S^3=\theta>0$ and $|S^3|>\theta$ always, $S^3>0$ when $\varepsilon=1$ also. If in the conjugation problems (20) and (21) we take a function $a(\lambda)$ with zeros, then by what has been proven the potential $\psi(x)$ certainly inherits at least one pole from the potentials $\psi^{(\pm)}(x)$. Therefore, in the case $\varkappa>0, S^3>0$ regular soliton solutions are absent.

In the case $S^3<0$ there may exist regular soliton solutions that are absent for the NSE without impurity on the complete axis. However, such solutions cannot be continued globally with respect to the time and at some instant acquire a singularity. On the basis of the results of [9–11], in which solutions of the NSE with repulsion possessing pole singularities were obtained, we can introduce in a sensible way scattering data for singular potentials in the generalized boundary-value problem, and the opposite connection will, as before, be given by the conjugation problems (20) and (21).

A large class of globally regular solutions of the generalized boundary-value problem with $S^3<0$ are given by solitonless solutions; in the scattering data the function $a(\lambda)$ has a single simple pole at $\lambda=i\theta$ and the coefficients of proportionality are completely degenerate, $\gamma^{(+)}=0$, $\gamma^{(-)}=\infty$ (see the Appendix).

We consider the single-soliton solution

$$a(\lambda)=(\lambda+i\theta)\frac{\lambda-\lambda_0}{\lambda-\bar\lambda_0},\quad \lambda_0=-\frac{1}{2}(v+iu),\quad v>0,\quad \gamma(\lambda_0)=\gamma=(\lambda_0\pm i\theta)\exp\left(\frac{u}{2}x_\pm+i\varphi_\pm\right),$$

$$\psi^{(\pm)}(x)=\frac{u}{2\sqrt{\varkappa}}\frac{\exp\left[i\left(\varphi_\pm+\frac{vx}{2}+\frac{u^2-v^2}{4}t\right)\right]}{\mathrm{sh}\left[\frac{u}{2}(x-vt-x_\pm)\right]}\qquad -\overline{S^+}=S^-=\frac{2\bar\gamma\theta(\bar\lambda_0-\lambda_0)}{\mathscr{D}}(\lambda_0^2+\theta^2-|\gamma|^2),$$

$$S^3=\frac{1}{\mathscr{D}}[(\lambda_0^2+\theta^2-|\gamma|^2)(\bar\lambda_0^2+\theta^2-|\gamma|^2)-|\gamma|^2(\lambda_0-\bar\lambda_0)^2],\qquad \mathscr{D}=(|\lambda_0+i\theta|^2-|\gamma|^2)(|\lambda_0-i\theta|^2-|\gamma|^2).$$

The function $\psi(x)$ is regular for $t_-<t<t_+$, $t_\pm=-\frac{x_\pm}{v}$ and $S^3<0$; for $t<t_-$ and $t>t_+$, respectively, it has a pole on the negative and positive half-axis, and in both cases $S^3>0$. Thus, we encounter a collapse phenomenon. In the symmetric case the potential $\psi(x)$ is regular, and $S^3<0$ if and only if the function $a(\lambda)$ in the symmetric scattering data has precisely one zero $\lambda_0=i\eta$, $0<\eta\leqq\theta$; in particular, collapse is absent. To prove this, we note that under a continuous deformation of the scattering data that preserves the number of poles of the function $\psi^{(+)}(x)$ and the finite integral of the motion $\mathbf{N}$ the potential $\psi(x)$ corresponding to the scattering data $\{b(\lambda),\{\lambda_j,\gamma(\lambda_j)\}\}$ is regular simultaneously with the purely soliton potential $\psi_0(x)$ corresponding to the scattering data $\{b_0(\lambda)=0,\{\lambda_j,\gamma(\lambda_j)\}\}$.

For purely soliton solutions it is not difficult to obtain algebraic relations that are an analytic continuation of the analogous relations that hold in the case $\varkappa<0$ (see [1]): if

$$a(\lambda)=(\lambda+i\theta)\prod_{k=1}^{n}\frac{\lambda-\lambda_k}{\lambda-\bar\lambda_k},\text{ then }\psi^{(+)}(x)=\psi^{(-)}(-x)=\frac{i}{\sqrt{\varkappa}}\frac{\det\tilde M}{\det M},$$

where M and $\hat{M}$ are n × n and (n + 1) × (n + 1) matrices, respectively,

$$M_{jk}=\hat{M}_{jk}=\frac{1-\gamma_j\bar{\gamma}_k}{\lambda_j-\bar{\lambda}_k},\quad 1\leq j,\ k\leq n,\quad \hat{M}_{j,\,n+1}=\gamma_j,\quad M_{n+1,\,j}=1,\quad M_{n+1,\,n+1}=0,\quad \gamma_j=\frac{\gamma(\lambda_j)}{\lambda_j+i\theta}e^{i\lambda_j x}.$$

A simple but fairly lengthy analysis of these relations with allowance for the symmetric reduction makes it possible to establish the assertion in the purely soliton case and, hence, in the general case too.

We should note the possibility of going to the limit $\lambda_0 \to 0$ in all the relations (33) in the symmetric case. We then obtain the slowly decreasing solution

$$\psi^{(+)}(x)=\psi^{(-)}(-x)=\frac{\theta e^{i\varphi}}{\sqrt{\varkappa}(\theta x+1)},$$

which cannot be obtained directly in the framework of the scheme proposed in the paper.

Conclusions

It is well known that the inverse scattering method is a nonlinear analog of the Fourier method for the solution of linear evolution equations on the complete axis [1]. To solve the boundary-value problem in the linear case the boundary conditions on the potential in the Fourier method are satisfied by virtue of certain properties of evenness and decrease of its Fourier transform, these being identical in a suitable formulation to the relations (22) and (30). On the other hand, these relations do not contain an explicit dependence on the coupling constant and ensure fulfillment of the boundary conditions on the potential in the boundary-value problem for the NSE. Thus, we have found that the analogy between the inverse scattering method and the Fourier method can be extended naturally to boundary-value problems.

Appendix

Suppose the function $a(\lambda)$ has a simple zero at $\lambda = i\theta$. It is easy to show that three cases are then possible:

$$T_-^{(1)}(x,i\theta)=0,\quad \operatorname*{res}_{\lambda=i\theta} T_+^{(2)}(y,\lambda)\neq 0;\qquad T_-^{(1)}(x,i\theta)\neq 0,\quad \operatorname*{res}_{\lambda=i\theta} T_+^{(2)}(y,\lambda)=0;$$

$$T_-^{(1)}(x,i\theta)=0,\quad \operatorname*{res}_{\lambda=i\theta} T_+^{(2)}(y,\lambda)=0\quad (x>0,\ y<0).$$

We introduce coefficients of proportionality,

$$T_-^{(1)}(x,i\theta)=2i\theta\gamma^{(+)}T_+^{(2)}(x,i\theta),\quad x>0,\quad T_-^{(1)}(y,i\theta)=\gamma^{(-)}\operatorname*{res}_{\lambda=i\theta} T_+^{(2)}(y,\lambda),\quad y<0, \tag{A.1}$$

in terms of which these cases take the form

$$\gamma^{(+)}=0;\quad 0<|\gamma^{(-)}|<\infty;\quad 0<|\gamma^{(+)}|<\infty,\quad \gamma^{(-)}=\infty;\quad \gamma^{(+)}=0,\quad \gamma^{(-)}=\infty.$$

If $\gamma^{(+)}=0$ or $\gamma^{(-)}=\infty$, then we shall speak of degenerate coefficients of proportionality. It is obvious that the definitions (A.1) are consistent with Eqs. (21), and it can be assumed that $\gamma^{(\pm)}=\Gamma^{(\pm)}(i\theta)$ in nondegenerate cases. The relative independence of $\gamma^{(\pm)}$ is due to the fact that det $L(i\theta) = 0$.

The equations of the inverse problem are, as before, obtained from the relations (13)-(15) and the asymptotic behaviors (17) and reduce to the conjugation problem (20). For nondegenerate coefficients of proportionality the relations (21) remain, while for degenerate coefficients they are modified as follows:

$$F^{(+)}(\lambda)=T_+^{(1)}(x,\lambda),\quad F^{(-)}(\lambda)=(\lambda-i\theta)T_+^{(1)}(x,\lambda),$$

$$G^{(+)}(\lambda)=\frac{1}{a(\lambda)}T_-^{(1)}(x,\lambda),\quad G^{(-)}(\lambda)=\frac{\lambda-i\theta}{a(\lambda)}T_-^{(1)}(x,\lambda),$$

$$A^{(\pm)}(\lambda)=\frac{a(\lambda)}{\lambda-i\theta},\quad B^{(\pm)}(\lambda)=\frac{b(\lambda)}{\lambda\mp i\theta},\quad \Gamma^{(\pm)}(\mu)=\frac{\gamma(\mu)}{\mu\mp i\theta}\quad (\mu\neq i\theta,\ x\gtrless 0),$$

where $A^{(\pm)}(i\theta)\neq 0$ and there is no pole at the point $i\theta$.

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UNITARITY CONDITION IN COVARIANT QUANTUM FIELD THEORY WITH INDEFINITE METRIC

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Conditions that ensure the existence of a unitarity S matrix acting on the subspace of states with positive norm are formulated. A study is made of BRST quantization.

1. Introduction

The covariant description of the overwhelming majority of physically interesting field theories requires the introduction of a space with indefinite metric. The most important examples are quantum electrodynamics and Yang–Mills theory, quantum gravity, and models of relativistic strings. In this connection the problem of the physical interpretation of the theory arises. Gauge invariance usually ensures that there are no negative probabilities of observable processes. Using the gauge freedom, one can pass to a manifestly unitary gauge of Coulomb type in which unphysical excitations are absent and the state space has a positive norm. The procedure for passing from the covariant gauge to the unitary gauge is well developed for the case of gauge theories with closed algebra and independent constraints (for a detailed discussion, see [1]). However, in the general case in which the constraints can be dependent and the gauge algebra open the corresponding transition entails great complexities. For example, already in the theory of an antisymmetric tensor field, when there is a finite number of dependent constraints (finite degree of reducibility), direct application of the Faddeev–Popov procedure is impossible and it is necessary to use special devices [2]. For theories with an infinite degree of reducibility, which include, for example, string models, a similar procedure in the general case is as yet unknown.

There exists an alternative approach in which the theory is from the very beginning formulated in a space with indefinite metric and the absence of negative probabilities is ensured by imposing on the space of allowed states an additional condition that separates the subspace of physical vectors. This subspace must possess a non-negative norm and be invariant with respect to the dynamics (the operator of any observable must not carry the state vector out of this subspace).

In quantum electrodynamics this construction is realized in the Gupta–Bleuler formalism, in which all the components of the electromagnetic field are regarded as independent and the physical subspace is distinguished by the condition

$$\partial_\mu A_\mu^-(x)|\Phi\rangle=0. \tag{1}$$

The Gupta–Bleuler formalism cannot be directly generalized to non-Abelian gauge

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