4. B. S. DeWitt, The Spacetime Approach to Quantum Field Theory. Les Houches, Session XL, Oxford (1984), p. 383.
5. I. Jack and H. Osborn, Nucl. Phys. B, 234, 331 (1984).
6. G. A. Vilkovisky, in: Quantum Gravity (ed. S. Christensen), Adam Hilger, Bristol (1983), p. 169.
7. I. G. Avramidi, The Background Field Method in Quantum Theory [in Russian], Moscow State University (1984). Deposited at VINITI, No. 1512-85 Dep., VINITI, Moscow (1985).
8. I. G. Avramidi, "Covariant methods for calculating the effective action in quantum field theory and investigation of quantum gravity with higher derivatives," Author's Abstract of Candidate's Dissertation [in Russian], Moscow State University (1987).
9. J. S. Schwinger, Phys. Rev., 82, 664 (1951).
10. N. D. Birrell and P. C. W. Davies, Quantum Fields in Curved Space, C.U.P., Cambridge (1982).
11. V. P. Frolov, Tr. Fiz. Inst. Akad. Nauk, 169, 3 (1986).
12. I. G. Avramidi, in: Abstracts of Papers at Sixth Soviet Gravitational Conference [in Russian], MGPI, Moscow (1984), p. 199.
13. V. P. Frolov and A. I. Zel'nikov, Phys. Rev. D, 29, 1057 (1984).
14. A. I. Vainshtein, V. I. Zakharov, V. A. Novikov, and M. A. Shifman, Yad. Fiz., 39, 124 (1984).
15. V. A. Fock, Izv. Akad. Nauk SSSR, Ser. Fiz., No. 4, 5, 551 (1937).
16. P. B. Gilkey, J. Diff. Geom., 10, 601 (1975).
17. E. S. Fradkin and G. A. Vilkovisky, Phys. Lett. B, 77, 262 (1978).
18. C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation, W. H. Freeman, San Francisco (1973).
19. A. O. Barvinsky and G. A. Vilkovisky, Phys. Rep. C, 119, 1 (1985).
20. J. L. Synge, Relativity, The General Theory, Amsterdam (1960).

CALCULATION OF SCALAR PRODUCTS OF WAVE FUNCTIONS AND FORM FACTORS IN THE FRAMEWORK OF THE ALGEBRAIC BETHE ANSATZ

N. A. Slavnov

Explicit expressions are obtained for a special case of the scalar product of the wave functions and form factor of the particle number operator in a generalized two-dimensional model.

1. Introduction

The Bethe ansatz method is widely used to investigate two-dimensional completely integrable models [1-3]. In the framework of the quantum inverse scattering method [4,5] it has proved to be possible to construct an algebraic scheme of the Bethe ansatz, and this has been successfully applied to calculation of correlation functions [6,7]. One of the important questions of the method is that of the scalar products of the wave functions. In particular, knowledge of the properties of the scalar products is necessary for investigating the form factors and correlation functions.

In the present paper we consider a generalized model with R matrix of the model of the nonlinear Schrödinger equation [8]. The main formulas and notation are given in Sec. 2. In Sec. 3 we calculate the scalar product of an arbitrary function and an eigenfunction of the Hamiltonian. The generalized two-site model [6] is introduced in Sec. 4. In Sec. 5 we calculate the form factor of the particle number operator.

2. Generalized Model

In the framework of the quantum inverse scattering method the Hamiltonian of a physical system is constructed by means of the monodromy matrix $T(\lambda)$. We shall consider a 2×2 matrix

---

V. A. Steklov Mathematics Institute, USSR Academy of Sciences. Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 79, No. 2, pp. 232-240, May, 1989. Original article submitted June 8, 1988.

 0040-5779/89/7902-0502$12.50 © 1989 Plenum Publishing Corporation

$$T(\lambda)=\begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix},$$

whose elements are quantum operators that depend on the spectral parameter λ. The commutation relations between these operators are specified by the equation

$$R(\lambda,\mu)(T(\lambda)\otimes T(\mu))=(T(\mu)\otimes T(\lambda))R(\lambda,\mu), \tag{1}$$

where $R(\lambda, \mu)$ is a 4×4 matrix with c-number elements. For the model of the nonlinear Schrödinger equation the R matrix has the form

$$R(\lambda,\mu)=\begin{pmatrix} f(\mu,\lambda) & 0 & 0 & 0 \\ 0 & g(\mu,\lambda) & 1 & 0 \\ 0 & 1 & g(\mu,\lambda) & 0 \\ 0 & 0 & 0 & f(\mu,\lambda) \end{pmatrix}, \tag{2}$$

where $f(\lambda,\mu)=(\lambda-\mu+ic)/(\lambda-\mu)$, $g(\lambda,\mu)=ic/(\lambda-\mu)$, c is the coupling constant. In what follows, it will be convenient to use the function $h(\lambda,\mu)=f(\lambda,\mu)/g(\lambda,\mu)=(\lambda-\mu+ic)/ic$. We give the explicit form of some commutation relations from (1):

$$[B(\lambda),B(\mu)]=[C(\lambda),C(\mu)]=0,\quad [C(\mu),B(\lambda)]=g(\mu,\lambda)(A(\mu)D(\lambda)-A(\lambda)D(\mu)). \tag{3}$$

In addition, the operators $C(\lambda)$ and $B(\lambda)$ satisfy

$$C(\lambda)=B^{+}(\bar{\lambda}). \tag{4}$$

An important object in the quantum inverse scattering method is the pseudovacuum vector $|0\rangle$ and its dual vector $\langle 0|$. The elements of the monodromy matrix act on these vectors as follows:

$$\begin{aligned} &A(\lambda)|0\rangle=a(\lambda)|0\rangle,\quad D(\lambda)|0\rangle=d(\lambda)|0\rangle,\quad C(\lambda)|0\rangle=0,\\ &\langle 0|A(\lambda)=a(\lambda)\langle 0|,\quad \langle 0|D(\lambda)=d(\lambda)\langle 0|,\quad \langle 0|B(\lambda)=0, \end{aligned} \tag{5}$$

where $a(\lambda)$ and $d(\lambda)$ are complex-valued functions whose form depends on the particular model (for the nonlinear Schrödinger equation $a(\lambda)=\exp(-i\lambda L/2)$, $d(\lambda)=\exp(i\lambda L/2)$, where L is the interval over which the equation is considered). In the framework of the generalized model, these functions are not particularized and remain free functional parameters. It was shown in [9] that for arbitrary functions $a(\lambda)$ and $d(\lambda)$ there exists a monodromy matrix $T(\lambda)$ satisfying Eq. (1) with R matrix (2), and the action of the operators A, B, C, D on the pseudovacuum is given by Eqs. (5).

Thus, in what follows, if no restrictions are imposed, we shall regard λ and $r(\lambda) = a(\lambda)/d(\lambda)$ as independent variables.

The eigenfunctions of the Hamiltonian are constructed by means of the operators $B(\lambda)$:

$$\Psi_N(\{\lambda_j\})=\prod_{j=1}^{N} B(\lambda_j)|0\rangle,$$

and on the parameters λ_j we impose the conditions

$$r(\lambda_j)\prod_{\substack{k=1\\ k\neq j}} \frac{f(\lambda_j,\lambda_k)}{f(\lambda_k,\lambda_j)}=1,\quad j=1,\ldots,N. \tag{6}$$

In the specific model, (6) is a system of equations for the allowed values of λ_j. In the framework of the generalized model λ_j remain arbitrary, and the condition (6) can be regarded as a constraint between the variables $r_j = r(\lambda_j)$ and λ_j.

The dual eigenfunction is constructed similarly by means of the operators $C(\lambda)$:

$$\tilde{\Psi}_N(\{\lambda_j\})=\langle 0|\prod_{j=1}^{N} C(\lambda_j).$$

The set $\{\lambda\}$ also satisfies (6).

3. Scalar Products

We shall call the quantity S_N defined by

$$S_N=\langle 0|\prod_{j=1}^{N}\mathbf{C}(\lambda_j^C)\prod_{j=1}^{N}\mathbf{B}(\lambda_j^B)|0\rangle,$$

where $\mathbf{C}(\lambda)=C(\lambda)/d(\lambda)$, $\mathbf{B}(\lambda)=B(\lambda)/d(\lambda)$, the scalar product. From the general theory of scalar products [8] it is known that S_N is a function of 4N independent variables $S_N=S_N(\{\lambda_j^C\}, \{\lambda_j^B\}, \{r_j^C\}, \{r_j^B\})$, $r_j^{C,B}=r(\lambda_j^{C,B})$. The scalar product depends linearly and homogeneously on the variables r_j^C and r_j^B. The coefficients of products of $r_j^{C,B}$ are rational functions of λ_j^C and λ_j^B that are symmetric separately with respect to the arguments of the operators $\mathbf{C}$ and $\mathbf{B}$ (see [3]) and decrease as $\lambda_j^{C,B}\to\infty$ as $1/\lambda_j^{C,B}$. The points $\lambda_k^C=\lambda_m^B$ (k, m = 1, ..., N) are the poles of these rational functions. The residues at these poles reduce to S_{N-1}:

$$S_N(\{\lambda_j^C\},\{\lambda_j^B\},\{r_j^C\},\{r_j^B\})\big|_{\lambda_k^C\to\lambda_m^B}=g(\lambda_k^C,\lambda_m^B)(r_k^C-r_m^B)\times$$
$$\prod_{\substack{j=1\\ j\neq k}}^{N} f(\lambda_k^C,\lambda_j^C)\prod_{\substack{j=1\\ j\neq m}}^{N} f(\lambda_m^B,\lambda_j^B)\,S_{N-1}(\{\lambda_j^C\}_{j\neq k},\{\lambda_j^B\}_{j\neq m},\{\tilde r_j^C\}_{j\neq k},\{\tilde r_j^B\}_{j\neq m}),\tag{7}$$

where

$$\tilde r_j^{C,B}=r_j^{C,B}f(\lambda_j^{C,B},\lambda_m^B)/f(\lambda_m^B,\lambda_j^{C,B}).\tag{8}$$

In this section we shall consider a special case of the scalar product when the vector $\langle 0|\prod_{j=1}^{N}\mathbf{C}(\lambda_j^C)$ is an eigenvector of the Hamiltonian

$$\tilde S_N=\langle 0|\prod_{j=1}^{N}\mathbf{C}(\lambda_j^C)\prod_{j=1}^{N}\mathbf{B}(\lambda_j^B)|0\rangle,$$

where

$$r_j^C\prod_{\substack{k=1\\ k\neq j}}^{N}\frac{f(\lambda_j^C,\lambda_k^C)}{f(\lambda_k^C,\lambda_j^C)}=1.\tag{9}$$

By virtue of (9), the scalar product $\tilde S_N$ depends on the 3N independent variables $\{\lambda_j^C\}, \{\lambda_j^B\}, \{r_j^B\}$.

<u>THEOREM.</u> The explicit form of the scalar product $\tilde S_N$ is given by

$$\tilde S_N=G_N(\{\lambda_j^C\},\{\lambda_j^B\})\det{}_N M_{lk}(\{r_j^B\},\{\lambda_j^C\},\{\lambda_j^B\}),\tag{10}$$

where

$$G_N(\{\lambda_j^C\},\{\lambda_j^B\})=\prod_{j>k}^{N}g(\lambda_j^B,\lambda_k^B)g(\lambda_k^C,\lambda_j^C)\prod_{j=1}^{N}\prod_{k=1}^{N}h(\lambda_j^C,\lambda_k^B),$$

$$M_{lk}(\{r_j^B\},\{\lambda_j^C\},\{\lambda_j^B\})=\frac{g(\lambda_k^C,\lambda_l^B)}{h(\lambda_k^C,\lambda_l^B)}-r_l^B\frac{g(\lambda_l^B,\lambda_k^C)}{h(\lambda_l^B,\lambda_k^C)}\prod_{m=1}^{N}\frac{f(\lambda_l^B,\lambda_m^C)}{f(\lambda_m^C,\lambda_l^B)}.$$

We <u>prove</u> the relation (10) by induction. We denote $G_N \det_N M$ by Θ_N. For N = 1, $\tilde S_N$ can be calculated by means of (3). With allowance for (9) it is readily seen that $\tilde S_1=\Theta_1=g(\lambda^C,\lambda^B)(1-r^B)$. In the next stage of the proof we use the formula for the residue of the scalar product (7) and the property of decrease $\tilde S_N\to 1/\lambda_k^C$ as $\lambda_k^C\to\infty$.

Let $\Theta_{N-1}=S_{N-1}$. We consider Θ_N as the function of λ_N^C. It is obvious that Θ_N is a rational function of λ_N^C that decreases as $\lambda_N^C\to\infty$ as $1/\lambda_N^C$ and has simple poles at the points $\lambda_N^C=\lambda_j^B$ (j = 1, ..., N). Note that although G_N has additional poles at $\lambda_N^C=\lambda_j^C$ the corresponding residues are zero, since for $\lambda_N^C=\lambda_j^C$ two columns in the matrix $M_{\ell k}$ are identical.

We consider the residues of the function Θ_N at the points $\lambda_N^C=\lambda_m^B$. We have

$$G_N(\{\lambda_j^C\},\{\lambda_j^B\})\big|_{\lambda_N^C\to\lambda_m^B}=(-1)^{N-m}h(\lambda_N^C,\lambda_m^B)\prod_{\substack{j=1\\j\neq m}}^{N}f(\lambda_m^B,\lambda_j^B)\times$$
$$\prod_{j=1}^{N-1}f(\lambda_j^C,\lambda_N^C)G_{N-1}(\{\lambda_j^C\}_{j\neq N},\{\lambda_j^B\}_{j\neq m}), \tag{11}$$

$$\det{}_N M_{lk}(\{r_j^B\},\{\lambda_j^C\},\{\lambda_j^B\})\big|_{\lambda_N^C\to\lambda_m^B}=\det{}_{N-1}M_{lk}(\{\tilde r_j^B\}_{j\neq m},\{\lambda_j^C\}_{j\neq N},\{\lambda_j^B\}_{j\neq m})\times$$
$$(-1)^{N+m}\left\{\frac{g(\lambda_N^C,\lambda_m^B)}{h(\lambda_N^C,\lambda_m^B)}-r_m{}^B\frac{g(\lambda_m^B,\lambda_N^C)}{h(\lambda_m^B,\lambda_N^C)}\prod_{j=1}^{N}\frac{f(\lambda_m^B,\lambda_j^C)}{f(\lambda_j^C,\lambda_m^B)}\right\}. \tag{12}$$

Combining (11) and (12), we obtain

$$\Theta_N\big|_{\lambda_N^C\to\lambda_m^B}=g(\lambda_N^C,\lambda_m^B)\left[\prod_{j=1}^{N}\frac{f(\lambda_j^C,\lambda_N^C)}{f(\lambda_N^C,\lambda_j^C)}-r_m{}^B\right]\times$$
$$\prod_{j=1}^{N-1}f(\lambda_N^C,\lambda_j^C)\prod_{\substack{j=1\\j\neq m}}^{N}f(\lambda_m^B,\lambda_j^B)\,\Theta_{N-1}(\{\lambda_j^C\}_{j\neq N},\{\lambda_j^B\}_{j\neq m},\{\tilde r_j^B\}_{j\neq m}). \tag{13}$$

Note that the parameters $\{\lambda_j^C\}_{j\neq N}$ again satisfy the condition (9) with function $\tilde r_j^C$:

$$\tilde r_j{}^C\prod_{\substack{k=1\\k\neq j}}^{N-1}\frac{f(\lambda_j^C,\lambda_k^C)}{f(\lambda_k^C,\lambda_j^C)}=1.$$

Since by the inductive hypothesis $\Theta_{N-1}=\tilde S_{N-1}$, we see, comparing (13) and (7), that the difference $\Delta=\tilde S_N-\Theta_N$ as a function of λ_N^C is bounded in the complete complex plane. Therefore, $\Delta\equiv 0$ since $\Delta\to 0$ as $\lambda_N^C\to\infty$, and this completes the proof.

We consider two special cases of the relation (10).

The partition function of the 6-vertex model is determined by the coefficient K_N of the product $\prod_{j=1}^{N}r_j^B$ in the scalar product $\tilde S_N$. It is readily seen that

$$K_N=\prod_{k<j}^{N}g(\lambda_j^B,\lambda_k^B)g(\lambda_k^C,\lambda_j^C)\prod_{j=1}^{N}\prod_{k=1}^{N}h(\lambda_j^B,\lambda_k^C)\det{}_N\frac{g(\lambda_l^B,\lambda_k^C)}{h(\lambda_l^B,\lambda_k^C)}.$$

It is in precisely such a form that this relation was first obtained in [10].

From the formula for the scalar product $\tilde S_N$ we can also readily obtain an explicit expression for the square of the norm of an eigenfunction of the Hamiltonian. We consider the limit $\lambda_j^B=\lambda_j^C+\varepsilon$, $\varepsilon\to 0$. Then

$$r_j^B=\prod_{\substack{k=1\\k\neq j}}^{N}\frac{f(\lambda_k^C,\lambda_j^C)}{f(\lambda_j^C,\lambda_k^C)}\left(1+\varepsilon\frac{\partial}{\partial\lambda_j}\ln r(\lambda_j)\right)+O(\varepsilon^2).$$

Substituting this expression in (10), we obtain the well-known expression for the square of the norm [8]:

$$\langle 0|\prod_{j=1}^{N}\mathrm{C}(\lambda_j)\prod_{j=1}^{N}\mathrm{B}(\lambda_j)|0\rangle=c^N\prod_{j\neq k}f(\lambda_j,\lambda_k)\det{}_N\frac{\partial\varphi_j}{\partial\lambda_k},$$

where

$$\varphi_j=i\ln\left\{r_j\prod_{\substack{k=1\\k\neq j}}\frac{f(\lambda_j,\lambda_k)}{f(\lambda_k,\lambda_j)}\right\}.$$

4. Two-Site Model

A generalized two-site model was introduced in [6,7]. In this model the total monodromy matrix on the interval [0, L] is represented as a product of two monodromy matrices:

$$T(\lambda)=T_2(\lambda)T_1(\lambda), \quad T_i(\lambda)=\begin{pmatrix} A_i(\lambda) & B_i(\lambda) \\ C_i(\lambda) & D_i(\lambda) \end{pmatrix} \quad i=1,2.$$

The matrices $T_1(\lambda)$ and $T_2(\lambda)$ are associated with the intervals [0, x] and [x, L], respectively; here, x is some fixed point. Each matrix $T_i(\lambda)$ satisfies Eq. (1) with the R matrix (2). The elements of different matrices commute. For each $T_i(\lambda)$ there exists a corresponding pseudovacuum $|0\rangle_i$ and dual vector $\langle 0|_i$. The complete pseudovacuum is $|0\rangle=|0\rangle_2\otimes|0\rangle_1$. The action of $T_i(\lambda)$ on $|0\rangle_i$ and $\langle 0|_i$ is given by the relations (5) with functions $a_i(\lambda)$ and $d_i(\lambda)$, and $a(\lambda)=a_1(\lambda)a_2(\lambda)$, $d(\lambda)=d_1(\lambda)d_2(\lambda)$.

We consider the particle number operator Q_1 on the interval [0, x]. We give the commutation relations of the operator Q_1 with the elements $T_1(\lambda)$:

$$[Q_1, B_1(\lambda)]=B_1(\lambda), \quad [C_1(\lambda), Q_1]=C_1(\lambda), \quad [Q_1, A_1(\lambda)]=[Q_1, D_1(\lambda)]=0, \quad Q_1|0\rangle_1=\langle 0|_1 Q_1=0.$$

The operator Q_1 commutes with the elements of $T_2(\lambda)$.

We shall call the matrix element

$$F_N=\langle 0|\prod_{j=1}^{N}\mathbf{C}(\lambda_j^C)Q_1\prod_{j=1}^{N}\mathbf{B}(\lambda_j^B)|0\rangle \tag{14}$$

the form factor of the operator Q_1. We assume here that $\prod_{j=1}^{N}\mathbf{B}(\lambda_j^B)|0\rangle$ and $\langle 0|\prod_{j=1}^{N}\mathbf{C}(\lambda_j^C)$ are eigenfunctions, in general different, of the Hamiltonian. It is easy to calculate F_N for small N:

$$F_0=0, \quad F_1=g(\lambda^C,\lambda^B)\left[\frac{a_1(\lambda^C)d_1(\lambda^B)}{d_1(\lambda^C)a_1(\lambda^B)}-1\right].$$

In the following section we give the explicit form of F_N for any N.

5. Form Factor of the Operator Q_1

The properties of the form factor of the particle number operator were investigated in detail in [6,7], in which the following representation was obtained for F_N:

$$F_N=\left[\prod_{j=1}^{N}\frac{a_1(\lambda_j^C)d_1(\lambda_j^B)}{d_1(\lambda_j^C)a_1(\lambda_j^B)}-1\right]\frac{\partial}{\partial\alpha}\sigma_N^\alpha(\{\lambda_j^C\},\{\lambda_j^B\})|_{\alpha=0}, \tag{15}$$

in which σ_N^α is a rational function of the 2N variables $\{\lambda_j^C\}$ and $\{\lambda_j^B\}$ and it suppresses the following properties:

1) σ_N^α is symmetric separately with respect to λ_j^C and λ_j^B;

2) as $\lambda_j^{C,B}\to\infty$ $\quad \sigma_N^\alpha\sim 1/\lambda_j^{C,B}$;

3) the points $\lambda_j^C = \lambda_k^B$ (j, k = 1, ..., N) are poles of σ_N^α. The residue at these poles reduces to σ_{N-1}^α:

$$\sigma_N^\alpha(\{\lambda_j^C\},\{\lambda_j^B\})|_{\lambda_N^C\to\lambda_N^B}=g(\lambda_N^C,\lambda_N^B)\left\{e^\alpha\prod_{j=1}^{N-1}f_{jN}^C f_{Nj}^B-\prod_{j=1}^{N-1}f_{Nj}^C f_{jN}^B\right\}\sigma_{N-1}^\alpha(\{\lambda_j^C\}_{j\neq N},\{\lambda_j^B\}_{j\neq N}),$$

where we have introduced the notation $f_{jk}^{C,B}=f(\lambda_j^{C,B},\lambda_k^{C,B})$. The remaining residues can be obtained by using the symmetry property 1;

4) $\sigma_1^\alpha=g(\lambda^C,\lambda^B)(e^\alpha-1)$.

These properties uniquely fix σ_N^α (see [6,7]) and in principle make it possible to construct σ_N^α recursively for arbitrary N. An explicit expression for σ_N^α in the model of the nonlinear Schrödinger equation was obtained for the first time in [11] in the form of the determinant of a 2N × 2N matrix. In this paper we propose a different representation for σ_N^α, which, in our view, is more convenient for applications and generalization to the

XXZ model.

We show that σ_N^α is a special case of scalar product. Suppose

$$S_N^\alpha=\langle 0|\prod_{j=1}^{N}C(\lambda_j^C)\prod_{j=1}^{N}B(\lambda_j^B)|0\rangle,$$

and

$$r_j^B\prod_{\substack{k=1\\k\neq j}}^{N}\frac{f_{jk}^B}{f_{kj}^B}=1,\qquad r_j^C\prod_{\substack{k=1\\k\neq j}}\frac{f_{jk}^C}{f_{kj}^C}=e^\alpha.$$

We prove that $\sigma_N^\alpha = S_N^\alpha$. For this it is obviously sufficient to establish that S_N^α has all the listed properties of σ_N^α.

Using (4) and (10), we obtain for S_N^α

$$S_N^\alpha=\prod_{j>k}^{N}g(\lambda_j^B,\lambda_k^B)g(\lambda_k^C,\lambda_j^C)\prod_{j=1}^{N}\prod_{k=1}^{N}h(\lambda_j^B,\lambda_k^C)\times$$

$$\det{}_N\left\{e^\alpha\frac{g(\lambda_k^C,\lambda_j^B)}{h(\lambda_k^C,\lambda_j^B)}\left[\prod_{m=1}^{N}\frac{h(\lambda_m^C,\lambda_j^B)h(\lambda_j^B,\lambda_m^B)}{h(\lambda_j^B,\lambda_m^C)h(\lambda_m^B,\lambda_j^B)}\right]+\frac{g(\lambda_j^B,\lambda_k^C)}{h(\lambda_j^B,\lambda_k^C)}\right\}. \tag{16}$$

Properties 2 and 4 are readily verified. To prove property 1, we note that replacement of λ_j^C by λ_k^C is equivalent to replacement of the j-th column by the k-th column in the determinant in (16) (when λ_j^B is replaced by λ_k^B the positions of two rows are interchanged) and to a change in sign of the factor in front of the determinant. As a result, S_N^α is unchanged. It also follows from this argument that S_N^α does not have poles at $\lambda_j^C = \lambda_k^C$ and $\lambda_j^B = \lambda_k^B$ (j, k = 1, ..., N).

Property 3 can be verified in the same way as the properties of the residues of the scalar products (see Sec. 3).

However, it follows from (16) that S_N^α may have additional poles at $h(\lambda_k^B, \lambda_j^B) = 0$. We show that the residues of these poles are zero. To prove this we extract from each row of the determinant (16) the factor $\left[\prod_{m=1}^{N}h(\lambda_j^B,\lambda_m^C)h(\lambda_m^B,\lambda_j^B)\right]^{-1}$. We denote the remaining matrix by $\tilde{M}_{jk}$:

$$\tilde{M}_{jk}=e^\alpha\frac{g(\lambda_k^C,\lambda_j^B)}{h(\lambda_k^C,\lambda_j^B)}\prod_{m=1}^{N}h(\lambda_m^C,\lambda_j^B)h(\lambda_j^B,\lambda_m^B)+\frac{g(\lambda_j^B,\lambda_k^C)}{h(\lambda_j^B,\lambda_k^C)}\prod_{m=1}^{N}h(\lambda_j^B,\lambda_m^C)h(\lambda_m^B,\lambda_j^B). \tag{17}$$

By virtue of the symmetry of S_N^α with respect to λ^B it is sufficient to consider the case $h(\lambda_2^B, \lambda_1^B) = 0$. Then

$$\tilde{M}_{1k}=e^\alpha\frac{g(\lambda_k^C,\lambda_1^B)}{h(\lambda_k^C,\lambda_1^B)}\prod_{m=1}^{N}h(\lambda_m^C,\lambda_1^B)h(\lambda_1^B,\lambda_m^B), \tag{18}$$

$$\tilde{M}_{2k}=\frac{g(\lambda_2^B,\lambda_k^C)}{h(\lambda_2^B,\lambda_k^C)}\prod_{m=1}^{N}h(\lambda_2^B,\lambda_m^C)h(\lambda_m^B,\lambda_2^B). \tag{19}$$

for $h(\lambda_2^B, \lambda_1^B) = 0$ we have

$$h(\lambda_2^B,\lambda_k^C)=-1/g(\lambda_k^C,\lambda_1^B),\quad g(\lambda_2^B,\lambda_k^C)=-1/h(\lambda_k^C,\lambda_1^B).$$

Substituting these expressions in (19), we obtain

$$\tilde{M}_{2k}=(-1)^N\frac{g(\lambda_k^C,\lambda_1^B)}{h(\lambda_k^C,\lambda_1^B)}\prod_{m=1}^{N}\frac{h(\lambda_m^B,\lambda_2^B)}{g(\lambda_m^C,\lambda_1^B)}. \tag{20}$$

Comparing (18) and (20), we see that $\tilde{M}_{2k}$ is proportional to $\tilde{M}_{1k}$ and, therefore, $\det_N \tilde{M}_k = 0$ for $h(\lambda_2^B, \lambda_1^B) = 0$, i.e., S_N^α does not have poles at the points $h(\lambda_j^B,\lambda_k^B)=0$.

Thus, we have shown that $\sigma_N^\alpha = S_N^\alpha$ and we have thus obtained an explicit expression for the form factor F_N (Eqs. (15) and (16)).

In conclusion we make a number of remarks that make it possible to generalize our results to the XXZ model. In the XXZ model, the R matrix also has the form (2), but the functions $f(\lambda, \mu)$ and $g(\lambda, \mu)$ are different:

$$f(\lambda, \mu) = \frac{\operatorname{sh}(\lambda-\mu+2i\eta)}{\operatorname{sh}(\lambda-\mu)}, \quad g(\lambda, \mu) = \frac{i\sin 2\eta}{\operatorname{sh}(\lambda-\mu)},$$

where $\cos 2\eta = \Delta$ is the anisotropy parameter. The properties of the functions f and g in the XXZ model are analogous to those of the same functions in the model of the nonlinear Schrödinger equation:

1) $g(\lambda, \mu) = -g(\mu, \lambda)$;

2) f and g have simple poles at $\lambda = \mu$;

3) f and g are rational functions of e^λ and e^μ;

4) as $\lambda \to \infty$ $g(\lambda, \mu) \sim 1/e^\lambda$, $f(\lambda, \mu) \sim 1$.

With allowance for these properties, the proofs of the expressions for the scalar product and form factor given in this paper can be applied almost unchanged to the XXZ model. In particular, the representations for $\tilde{S}_N$ (10) and F_N (15), (16) remain valid. In our view the obtained expressions are convenient for the passage to the thermodynamic limit and may be helpful in the investigation of the correlation functions.

I should like to thank V. E. Korepin for helpful discussions.

LITERATURE CITED

1. H. Bethe, Z. Phys., 71, 205 (1931).
2. C. N. Yang and C. P. Yang, Phys. Rev., 150, 321 (1966).
3. E. H. Lieb, Phys. Rev. Lett., 18, 692 (1966).
4. L. D. Faddeev, in: Problems of Quantum Field Theory. Proc. Fifth International Symposium on Nonlocal Field Theories. Alushta, 1979 [in Russian], JINR, Dubna (1979), p. 249.
5. L. A. Takhtadzhyan and L. D. Faddeev, Usp. Mat. Nauk, 34, 13 (1979).
6. A. G. Izergin and V. E. Korepin, Commun. Math. Phys., 94, 67 (1984).
7. A. G. Izergin and V. E. Korepin, Commun. Math. Phys., 99, 271 (1985).
8. V. E. Korepin, Commun. Math. Phys., 86, 391 (1982).
9. V. E. Korepin, Dokl. Akad. Nauk SSSR, 265, 1361 (1982).
10. A. G. Izergin, Dokl. Akad. Nauk SSSR, 297, 331 (1987).
11. A. N. Kirillov and F. A. Smirnov, Zap. Nauchn. Semin. LOMI, 164, 67 (1987).