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FURRY THEOREM FOR NON-ABELIAN GAUGE LAGRANGIANS

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It is shown that in a gauge-invariant Lagrangian (without γ^5 anomalies) based on an irreducible representation of simple compact Lie algebras it is always possible to define an operation of C parity with respect to which the Lagrangian is invariant. This imposes certain requirements on the algebraic structure of the vertex functions.

1. Introduction

In non-Abelian gauge theories, the question of the algebraic structure of the vertex functions frequently arises. For such an analysis, the most common procedure is to use the Ward–Slavnov identities [1-3] for the generating functional of the Green's functions. The analysis is most readily performed if the Ward identities are expressed in terms of single-particle-irreducible vertex Green's functions [4]. In [5, 6], it is shown that in the loop approximation the Ward identities determine gauge invariance of the generating functional of the vertex Green's functions with respect to a certain Lie group which is isomorphic in the general case to the original group. And it is only if consideration is restricted to the divergent parts of the corresponding functions, whose explicit form in the momenta we know, it is possible to show, on the basis of the Ward identities, that the divergent parts (i.e., the counterterms needed to eliminate the divergences) have the same algebraic and Lorentz structure as the corresponding terms in the Lagrangian [7]. Strictly speaking, the renormalizability of gauge theories is based on this last fact.

On the other hand, the study of the properties of charmonium in quantum chromodynamics (QCD) has led naturally to the concept of the C parity of charmonium. Under the assumption that the strong interactions are described by QCD, one can define the operation of C conjugation of a gluon. The Lagrangian of QCD is invariant with respect to this operation of C conjugation of quarks and gluons. The colorless states of gluons have C parity equal to +1 or −1. This immediately enables one to say how many gluons can result from the decay of a particular charmonium state [8, 9] or identify the channel through which particles with hidden charm are produced in hadron–hadron collisions [10].

It should be noted that C invariance in QCD leads to consequences different from those of the C invariance in electrodynamics, and the automatic transfer of results of electrodynamics to QCD leads to an incorrect answer [11].

In the present paper, it is shown that an analogous operation of C conjugation can be defined for any gauge-invariant Lagrangian based on an irreducible representation of a simple compact Lie group. It follows from the invariance of the Lagrangian with respect to a global group transformation that the algebraic structures which occur in the vertex functions must be invariant with respect to the corresponding transformations. It follows from the invariance of the Lagrangian with respect to the C parity that the corresponding vertex function must also be C even. This last condition makes it possible to determine a class of algebraic structures that certainly will not appear in the vertex functions. In particular, the total three-gluon

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vertex function and the total vertex for the interaction of the Faddeev–Popov ghosts with a gluon are proportional to the structure constants of the given Lie algebra.

The paper is arranged as follows. In Sec. 2, we formulate the operation of C parity for gauge-invariant Lagrangians without γ^5 anomalies. In Sec. 3, we prove the existence of the operation. In Sec. 4, we show that the total three-gluon vertex is necessarily proportional to the structure constants of the given Lie algebra.

2. C Parity

Let τ_a be the basis of a real simple compact Lie algebra L:

$$[\tau_a, \tau_b]=if_{abc}\tau_c.$$

Summation is understood throughout the paper over repeated indices. We assume that the τ_a are chosen such that the tensor f_{abc} is completely antisymmetric.

Let $\pi_a \equiv \pi(\tau_a)$ be an irreducible representation of L. For simplicity, we assume that the π_a are Hermitian. Then $[\pi_a, \pi_b]=if_{abc}\pi_c$. The matrices $\tilde{\pi}_a \equiv -\pi_a^T$ also form an irreducible representation of L: $[\tilde{\pi}_a, \tilde{\pi}_b]=if_{abc}\tilde{\pi}_c$.

In Sec. 3, it will be shown that there exist matrices U and A such that

$$U^+\tilde{\pi}_a U=A_{ab}\pi_b \tag{1}$$

with the properties

1) $U^+U=1$; $A_{ab}^*=A_{ab}$; $A^TA=1$;

2) $A_{aa'}A_{bb'}A_{cc'}f_{a'b'c'}=f_{abc}$;

3) the explicit form of A does not depend on the choice of the representation $\pi(\tau_a)$ and is determined solely by the algebra L;

4) suppose $\varphi^{ab\ldots l}$ has an odd number of indices $a, b, \ldots, l$, is completely symmetric with respect to them, and is invariant with respect to the adjoint representation:

$$f_{iak}\varphi^{kb\ldots l}+\ldots+f_{ilk}\varphi^{ab\ldots k}=0$$

(i.e., $\varphi^{ab\ldots l}\tau_a\tau_b\ldots\tau_l$ is a Casimir operator in the universal covering algebra of the algebra L). Then

$$A_{aa'}A_{bb'}\ldots A_{ll'}\varphi^{a'b'\ldots l'}=-\varphi^{ab\ldots l};$$

5) A is not a multiple of the unit matrix.

We consider the operation of C conjugation taking the example of the Lagrangian of gauge and spinor fields:

$$L=i\bar{\psi}_i^\alpha\gamma_{ij}^\mu(\partial_\mu\delta_{\alpha\beta}-ig\pi_{\alpha\beta}{}^a a_\mu{}^a)\psi_j^\beta-m\bar{\psi}_i^\alpha\psi_i^\alpha-\frac{1}{4}F^{a\mu\nu}F_{\mu\nu}{}^a+\frac{1}{2\lambda}(\partial^\mu a_\mu{}^a)^2+L_{FP}(\bar{\chi}, \chi, a),$$

where α, β, a are the indices of the group symmetry, i and j are the spinor Lorentz indices, and L_{FP} is the Lagrangian of the ghosts. It is readily verified that by virtue of (1) the Lagrangian remains invariant under the C conjugation operation

$$\tilde{a}_\mu{}^a=A_{ab}a_\mu{}^b, \quad \tilde{\psi}_i^\alpha=U_{\beta\alpha}{}^+C_{ij}\bar{\psi}_j^\beta, \quad \tilde{\bar{\psi}}_i^\alpha=U_{\alpha\beta}C_{ij}\psi_j^\beta, \quad \tilde{\chi}_a=A_{ab}\chi_b, \quad \tilde{\bar{\chi}}_a=A_{ab}\bar{\chi}_b, \quad C=\gamma^0\gamma^2, \tag{2}$$

i.e.,

$$L(\bar{\psi}, \psi, \bar{\chi}, \chi, a)=L(\tilde{\bar{\psi}}, \tilde{\psi}, \tilde{\bar{\chi}}, \tilde{\chi}, \tilde{a}). \tag{3}$$

Obviously, the Lagrangian of gauge and scalar fields has the same invariance. Spontaneous breaking of the symmetry does not break the C invariance if $\langle\Phi_\alpha\rangle_0^*=\langle\Phi_\alpha\rangle_0$; $U_{\alpha\beta}\langle\Phi_\beta\rangle_0=\langle\Phi_\alpha\rangle_0$. It is clear that such an operation applies not only to gauge Lagrangians but also to a number of Lagrangians possessing only a global internal symmetry.

The invariance of the Lagrangian (3) with respect to the C conjugation (2) means that the total vertex n-gluon functions must satisfy the conditions

$$f_{iak}\Gamma_{\mu\nu\ldots\xi}^{kb\ldots c}+\ldots+f_{ick}\Gamma_{\mu\nu\ldots\xi}^{ab\ldots k}=0, \tag{4}$$

$$A_{aa'}A_{bb'}\ldots A_{cc'}\Gamma_{\mu\nu\ldots\xi}^{a'b'\ldots c'}=\Gamma_{\mu\nu\ldots\xi}^{ab\ldots c}, \tag{5}$$

where $a, b, \ldots, c$ are group indices and $\mu, \nu, \ldots, \xi$ Lorentz indices. Similar relations can be written down for the vertices with spinors.

The conditions (4) and (5) make it possible to identify the algebraic structures that will occur in the vertex functions.

3. Construction and Properties of the Matrices A_{ab} and U

Let L be a real simple compact Lie algebra, and L_c be the complex extension of L. Let e_α, h_i be a Cartan–Weyl basis in L_c, h_α be the corresponding nonzero roots, h_i be the simple roots, and H be the Cartan subalgebra in L_c, $h \in H$. Then

$$[h, e_\alpha]=(h, h_\alpha)e_\alpha, \quad [e_\alpha, e_{-\alpha}]=-h_\alpha, \quad [e_\alpha, e_\beta]=N_{\alpha,\beta}e_{\alpha+\beta}, \quad (e_\alpha, e_{-\alpha})=-1; \quad N_{\alpha,\beta}=N_{-\alpha,-\beta}, \tag{6}$$

where (x, y) is the Killing form. Let θ be the involution defined by the real form L of the algebra L_c. Then in L_c there exists a Cartan–Weyl basis [12] such that

$$\theta(e_\alpha)=e_{-\alpha}; \quad \theta(h_\alpha)=-h_\alpha. \tag{7}$$

We shall work in this basis.

We define an automorphism $L_c \to L_c$ by the relations

$$\tilde{h}_i=-h_i, \quad \tilde{e}_\alpha=e_{-\alpha}. \tag{8}$$

Then $\tilde{e}_\alpha$, $\tilde{h}_i$ satisfy the relations (6). Let τ_a be a basis in L such that f_{abc} is completely antisymmetric. We have

$$\theta(\tau_a)=\tau_a. \tag{9}$$

Suppose

$$\tau_a=B_{ai}h_i+B_{a\alpha}e_\alpha; \quad h_i=B_{ia}^{-1}\tau_a; \quad e_\alpha=B_{\alpha a}^{-1}\tau_a; \quad \tilde{\tau}_a=B_{ai}\tilde{h}_i+B_{a\alpha}\tilde{e}_\alpha. \tag{10}$$

It follows from (6)-(10) that

$$\frac{1}{i}[\tilde{\tau}_a, \tilde{\tau}_b]=f_{abc}\tilde{\tau}_c, \quad \theta(\tilde{\tau}_a)=\tilde{\tau}_a\Rightarrow\tilde{\tau}_a\in L, \quad A_{ab}\equiv -B_{ai}B_{ib}^{-1}+B_{a\alpha}B_{-\alpha b}^{-1}, \quad \tilde{\tau}_a=A_{ab}\tau_b, \tag{11}$$

since $A_{ab}^*=A_{ab}$, because $\tau_a, \tilde{\tau}_a\in L$. It follows from (11) by virtue of the fact that the Killing form $f_{aij}f_{bji}\sim\delta_{ab}$ that

$$A_{aa'}A_{bb'}A_{cc'}f_{a'b'c'}=f_{abc}, \quad AA^T=1.$$

Property 4 is obtained from the following. Let $\varphi^{a\cdots c}\tau_a\ldots\tau_c$ be a Casimir operator. We consider its restriction to the Cartan subalgebra, i.e., we express τ_a in terms of h_i, e_α, and then set $e_\alpha = 0$. Then under the automorphism (8) this restriction changes sign. But by Chevalley's theorem [13] there is a one-to-one correspondence between Casimir operators and their restrictions to the Cartan subalgebra, from which Property 4 follows.

From (8) we also have $A \neq \lambda\cdot 1$. Let $\pi(\tau_a)$ be an irreducible representation of L. By linearity, we construct $\pi(h_i)$, $\pi(e_\alpha)$, an irreducible representation of L_c. It is readily verified that the matrices $\pi'(h_i)\equiv\pi^T(h_i)$, $\pi'(e_\alpha)\equiv-\pi^T(e_{-\alpha})$ also form a representation of L_c.

The character of the representation determines the representation itself uniquely up to equivalence. The character itself is uniquely determined by its restriction to the universal covering of the Cartan subalgebra [12]. It is obvious that these restrictions of the representations π and π' are the same. Therefore, there exists a matrix V such that

$$\pi(x)=V^{-1}\pi'(x)V, \quad x\in L_c. \tag{12}$$

It is easy to show that there exists a number $\varkappa$ such that

$$U=\varkappa V, \quad U^+=U^{-1}. \tag{13}$$

From (12) and (13), we obtain $A_{ab}\pi(\tau_b)=U^+\tilde{\pi}(\tau_a)U$.

By virtue of the relation

$$e^{i\pi_b\delta_b}\pi_a e^{-i\pi_b\delta_b}=(e^{F_b\delta_b})_{ac}\pi_c,$$

where $(F_b)_{ac}=f_{abc}$, it is readily seen that the matrices $U'=Ue^{i\pi_b\delta_b}$, $A'=Ae^{-F_b\delta_b}$ also satisfy the relation (1) and have the properties 1-4. If some algebraic tensor invariant with respect to the adjoint representation has definite transformation properties under the action of the matrix A, then it has the same transformation properties under the matrix A′. Therefore, in a number of cases it is convenient to use the matrices U′ and

A' rather than the matrices U and A obtained above. In particular, if π_a is the fundamental representation of SU(n) then the "angles" δ_a can always be chosen such that $U' = 1$. In such a case, the relation (1) takes the form $-\pi_a^T=A_{ab}'\pi_b$, which means that we can write down the matrix A'_{ab} directly without recourse to the Cartan–Weyl basis.

4. Algebraic Structure of Three-Gluon Vertices

The tensor φ_{abc} at the three-gluon vertex must satisfy the relations

$$f_{iak}\varphi_{kbc}+f_{ibk}\varphi_{akc}+f_{ick}\varphi_{abk}=0, \tag{14}$$

$$A_{aa'}A_{bb'}A_{cc'}\varphi_{a'b'c'}=\varphi_{abc}. \tag{15}$$

A tensor φ_{abc} that satisfies (14) and is antisymmetric with respect to at least one pair of indices is necessarily proportional to f_{abc} (see the Appendix). A completely symmetric φ_{abc} satisfying (14) cannot satisfy (15), since $\varphi_{abc}\tau_a\tau_b\tau_c$ is a Casimir operator. Only one possibility remains: $\varphi_{abc}\sim f_{abc}$. Obviously, the same is true for the vertex of the interaction of the ghosts with the gauge field.

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Appendix

We show that a tensor φ_{abc} satisfying (14) and antisymmetric with respect to two indices (say, the first and the second) is proportional to f_{abc}. We denote $(\Phi_a)_{bc}=\varphi_{bac}$; $(F_a)_{bc}=f_{bac}$. We rewrite (14) in the form

$$F_a\Phi_b-\Phi_bF_a=f_{abi}\Phi_i, \tag{16}$$

$$F_a\Phi_b-F_b\Phi_a=\varphi_{abi}F_i. \tag{17}$$

From (16) we readily conclude that

$$F_a\Phi_bF_a=-\frac{1}{2}F\Phi_b, \tag{18}$$

where $\mathrm{Sp}\,F_aF_b=-F\delta_{ab}$. From (16)-(18),

$$F_aF_a\Phi_b-F_a\Phi_bF_a=f_{abi}F_a\Phi_i={}^1/_2 f_{abi}\varphi_{aij}F_j, \quad F\Phi_a=f_{aij}\varphi_{ijk}F_k. \tag{19}$$

By virtue of (14), the matrix $\Lambda_{ab}=f_{aij}\varphi_{ijb}$ commutes with F_a. Since the algebra is simple, $\Lambda_{ab}\sim\delta_{ab}$. Indeed, Λ_{ab} is the sum of a symmetric and an antisymmetric matrix, each of which commutes with F_a. The symmetric matrix is a multiple of the identity, and it can be shown that the antisymmetric matrix belongs to the center of the adjoint representation of the algebra. But the center of a simple algebra is equal to zero. It follows from this and (19) that $\Phi_a \sim F_a$.

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