considered by Dobrushin and Tirozzi [4] in the case of many-particle potentials with infinite interaction range.

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"HIDDEN SYMMETRY" OF ASKEY–WILSON POLYNOMIALS

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A new q-commutator Lie algebra with three generators, AW(3), is considered, and its finite-dimensional representations are investigated. The overlap functions between the two dual bases in this algebra are expressed in terms of Askey–Wilson polynomials of general form of a discrete argument: To the four parameters of the polynomials there correspond four independent structure parameters of the algebra. Special and degenerate cases of the algebra AW(3) that generate all the classical polynomials of discrete arguments — Racah, Hahn, etc. — are considered. Examples of realization of the algebra AW(3) in terms of the generators of the quantum algebras of SU(2) and the q-oscillator are given. It is conjectured that the algebra AW(3) is a dynamical symmetry algebra in all problems in which q-polynomials arise as eigenfunctions.

Introduction

In [1,2], Askey and Wilson constructed a remarkable system of orthogonal polynomials. In many respects, they can be regarded as the "most general" classical polynomials. Indeed, they possess all the properties of classical orthogonal polynomials, namely, they can be explicitly expressed in terms of a generalized hypergeometric function, the weight function and coefficients of a three-term recursion relation for them are known, etc. In addition, by limiting processes one can obtain from the Askey–Wilson polynomials all the classical polynomials of both discrete and continuous argument: Racah, Hahn, Jacobi, etc.

In view of the numerous remarkable properties of the Askey–Wilson polynomials, attempts were made to interpret these polynomials from a group-theoretical or algebraic point of view after the manner that Vilenkin interpreted the special functions of mathematical physics [3].

We cannot here give any exhaustive survey of the large number of studies on this theme. We shall merely mention some of the most characteristic ones.

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In the approach of the monograph [4], these polynomials arise as solution of the difference analog of the hypergeometric equation on nonuniform grids. Such an approach is convenient for the classification of these polynomials and the obtaining of a number of their characteristics, but it does not clarify their algebraic origin.

It was found comparatively recently that the Askey–Wilson polynomials of special form are intimately related to representations of the so-called quantum groups, in particular, the group $SU_q(2)$ ([5,6] and numerous publications of other authors). Also interesting is the connection between these polynomials and the wave functions of the q-analog of the harmonic oscillator [7].

However, in all the listed cases only Askey–Wilson polynomials of special form with restrictions on the parameters are obtained. In addition, the mathematical formalism of the quantum groups cannot explain why the same polynomials arise in problems that at the first glance are completely different.

In the present paper, we propose a simple scheme that permits construction of Askey–Wilson polynomials of general form for arbitrary values of the parameters as overlap function of dual bases on a Lie algebra with three generators.

In this sense, the present paper can be regarded as a development of [8–10], in which classical polynomials were constructed on the basis of ordinary Lie algebras. We note that [10] contains first attempts to construct three-term recursion relations for q-polynomials on the basis of Lie q-algebras; however, the resulting polynomials correspond only to some special cases of the Askey–Wilson polynomials. The representations of the new q-algebra, which we call the algebra AW(3) in honor of Askey and Wilson, makes it possible to explain the algebraic origin of polynomials of general form. In addition, the algebra AW(3) is also important by itself as a dynamical symmetry algebra in problems in which the Askey–Wilson polynomials arise as eigenfunctions (for example, in the approach of [4] these polynomials arise as eigenfunctions for a difference operator of second order; we show that this difference operator belongs to the algebra AW(3)). It is in this sense that we understand the expression "hidden symmetry" in the title of the paper.

This paper develops the approach proposed in the preprint [11], in which the Askey–Wilson polynomials were generated by a biquadratic commutator algebra of a somewhat more cumbersome structure than the one proposed here.

1. The Algebra AW(3) and Its Ladder Representation

In this section, we describe the algebra that generates the Askey–Wilson polynomials and construct its ladder representation.

Let K_0, K_1, K_2 be three operators that satisfy the commutation relations

$$[K_0, K_1]_\omega=K_2, \tag{1.1a}$$

$$[K_2, K_0]_\omega=BK_0+C_1K_1+D_1, \tag{1.1b}$$

$$[K_1, K_2]_\omega=BK_1+C_0K_0+D_0, \tag{1.1c}$$

where B, C_0, C_1, D_0, D_1 are the structure constants of the algebra, which we shall assume are real, and $[\dots]_\omega$ denotes the so-called q-commutator:

$$[L, M]_\omega=e^\omega LM-e^{-\omega}ML, \tag{1.2}$$

where ω is an arbitrary real parameter.

The concept of the q-commutator was proposed in [10,12,13], although the first example of a q-commutator algebra was considered in [14]. We shall call the algebra with the commutation relations (1.1) the algebra AW(3) (i.e., Askey–Wilson algebra with three generators).

In the limit $\omega \to 0$, the algebra AW(3) becomes an ordinary Lie algebra with three generators (we include D_0 and D_1 among the structure constants of the algebra in order to take into account algebras of Heisenberg–Weyl type).

The Casimir operator of AW(3), which commutes with all three generators, has the form

$$Q=(e^{-\omega}-e^{3\omega})K_0K_1K_2+e^{2\omega}K_2^2+B(K_0K_1+K_1K_0)+C_0e^{2\omega}K_0^2+C_1e^{-2\omega}K_1^2+$$

$$D_0(1+e^{2\omega})K_0+D_1(1+e^{-2\omega})K_1. \tag{1.3}$$

A remarkable fact about the algebra (1.1) is its ladder property, which (as in the case of ordinary Lie algebras) makes it possible to construct an entire ladder of eigenstates from one fixed state.

This property is as follows. Let ψ_p be an eigenfunction of the operator K_0 corresponding to eigenvalue λ_p,

$$K_0\psi_p=\lambda_p\psi_p. \tag{1.4}$$

We form the linear combination

$$\psi_s=(\alpha(p)K_0+\beta(p)K_1+\gamma(p)K_2)\psi_p \tag{1.5}$$

in such a way that ψ_s is an eigenfunction for K_0 but with different eigenvalue λ_s:

$$K_0\psi_s=\lambda_s\psi_s \tag{1.6}$$

(in contrast to Lie algebras, the coefficients α, β, γ are in general functions of the spectral parameter p).

Substituting (1.5) in (1.6), we see that ψ_s will be an eigenfunction only if

$$\lambda_p^2+\lambda_s^2-2\operatorname{ch}2\omega\lambda_p\lambda_s+C_1=0. \tag{1.7}$$

It follows from the condition (1.7) that for each state ψ_p there are two "neighboring" ψ_s, for which the eigenvalues λ are found as roots of the quadratic equation (1.7). By virtue of this condition, the operators K_1 and K_2 are tridiagonal in the ψ_p basis. Therefore, it is possible to choose the following ladder representation of the algebra:

$$K_1\psi_p=a_{p+1}\psi_{p+1}+a_p\psi_{p-1}+b_p\psi_p, \tag{1.8a}$$

$$K_2\psi_p=\xi_{p+1}\psi_{p+1}+\eta_p\psi_{p-1}+\zeta_p\psi_p, \tag{1.8b}$$

where

$$\xi_p=a_p(e^{\omega}\lambda_p-e^{-\omega}\lambda_{p-1}),\quad \eta_p=a_p(e^{\omega}\lambda_{p-1}-e^{-\omega}\lambda_p),\quad \zeta_p=2\operatorname{sh}\omega\lambda_p b_p.$$

We take the matrix coefficients a_p and b_p to be real. Such a choice means that both operators K_0 and K_1 are self-adjoint in the ψ_p basis (K_2 is obviously not self-adjoint in this basis).

As a result, Eq. (1.7) for the spectrum can be rewritten in the form

$$\lambda_{p+1}+\lambda_p^2-2\operatorname{ch}2\omega\lambda_p\lambda_{p+1}+C_1=0. \tag{1.7a}$$

This equation is invariant with respect to the change of sign $\lambda_p \to -\lambda_p$, i.e., there exist two branches of solutions for λ_p differing only in sign; we shall choose only the positive branch (the situation resembles the case of the Lie algebra SU(1,1), for which there also exist two equivalent discrete series differing in sign).

The form of the solution of Eq. (1.7a) depends strongly on the value of the constant C_1.

For $C_1 > 0$

$$\lambda_p=\frac{\sqrt{C_1}}{\operatorname{sh}2\omega}\operatorname{ch}\omega(2p+1). \tag{1.9a}$$

If $C_1 < 0$ then

$$\lambda_p=\frac{\sqrt{-C_1}}{\operatorname{sh}2\omega}\operatorname{sh}\omega(2p+1). \tag{1.9b}$$

Finally, for $C_1 = 0$

$$\lambda_p=e^{\pm\omega(2p+1)}. \tag{1.9c}$$

Here, p is a discrete variable with unit step and arbitrary origin, i.e., the substitution $p \to p + \text{const}$ does not take us out of the framework of Eqs. (1.9). In all that follows, we shall assume that the spectrum is everywhere nondegenerate: $\lambda_p\neq\lambda_{p+1}$, $\lambda_{p-1}\neq\lambda_{p+1}$. For this, it is sufficient to restrict the origin of the variable: $p > \frac{1}{2}$.

Thus, the spectrum of the operator K_0 in the ladder representation has hyperbolic or exponential form depending on the sign of the constant C_1. Such a form of the spectrum relates the algebra to quantum algebras of the type $SU_q(2)$.

We now find the form of the matrix coefficients a_p and b_p of the operator K_1. It is simplest to find b_p. Indeed, substituting (1.8) in the commutation relation (1.1b) and equating the coefficients of the diagonal terms ψ_p, we obtain

$$b_p=\frac{B\lambda_p+D_1}{g_p g_{p+1}}, \tag{1.10}$$

where

$$g_p=\lambda_p-\lambda_{p-1}. \tag{1.11}$$

To find the coefficients a_p, we substitute (1.8) in (1.1c) and again equate the coefficients of the diagonal terms ψ_p. As a result, we obtain a difference equation for a_p:

$$\Omega_{p+1}a_{p+1}^2-\Omega_{p-1}a_p^2=-4\,\mathrm{sh}^2\,\omega\lambda_p b_p^2+Bb_p+C_0\lambda_p+D_0, \tag{1.12}$$

where $\Omega_p=\lambda_{p+1}-\lambda_{p-1}$.

Similarly, substituting (1.8) in the expression (1.3) for the Casimir operator and equating the coefficients of ψ_p, we obtain

$$-{}^1/_2(\Omega_{p+1}\Omega_p a_{p+1}^2+\Omega_p\Omega_{p-1}a_p^2)=Q+4\,\mathrm{sh}^2\,\omega\lambda_p^2 b_p^2+$$
$$\mathrm{ch}\,2\omega\,(C_0\lambda_p^2+C_1 b_p^2)-2B\lambda_p b_p-2\,\mathrm{ch}^2\,\omega\,(D_0\lambda_p+D_1 b_p), \tag{1.13}$$

where Q = const is the value of the Casimir operator for the given ladder representation.

Solving (1.12) and (1.13) as a system for a_p^2 and a_{p+1}^2, we find the explicit form of the coefficient a_p:

$$\Omega_p\Omega_{p-1}a_p^2=\frac{(B\lambda_p+D_1)(B\lambda_{p-1}+D_1)}{g_p^2}+C_0\lambda_p\lambda_{p-1}+D_0(\lambda_p+\lambda_{p-1})-Q. \tag{1.14}$$

It is easy to prove the self-consistency of the system (1.12) and (1.13), i.e., the solution for a_{p+1} obtained from this system can also be obtained from (1.14) by shifting p by unity.

Note also that the balance of the terms in (1.1) of nondiagonal form, i.e., $\psi_{p\pm2}$, $\psi_{p\pm1}$, does not lead to new restrictions on the obtained expressions (1.10) and (1.14) for the matrix coefficients of the ladder representation.

Using Eq. (1.7a), we can rewrite the expression (1.14) in the compact form

$$a_p^2=\mathscr{P}(\Lambda_p)/g_p^2\Omega_p\Omega_{p-1}, \tag{1.15}$$

where $\mathscr{P}(\Lambda_p)$ is a polynomial of not higher than fourth degree in the argument $\Lambda_p=\lambda_p+\lambda_{p-1}$. We shall call $\mathscr{P}(\Lambda_p)$ the characteristic polynomial of the algebra AW(3). We shall show that its form determines the characteristics of a finite-dimensional representation.

Thus, we have found the explicit form of all the matrix coefficients λ_p, a_p, b_p of the ladder representation of the algebra AW(3).

Note that, in contrast to the case of Lie algebras such as SU(2), SU(3), SU(1,1), etc., for which the structure of the ladder representation is specified solely by the value of their Casimir operators, in the case of the algebra AW(3) the form of the matrix coefficients a_p, b_p also depends strongly on the structure constants. In particular, as will be shown in the following section, to fix the dimension N of a finite-dimensional representation of AW(3) it is necessary to specify an equation containing the constants B, D_0, D_1, Q.

2. Finite-Dimensional Representations of AW(3).

Dual Representation

In the previous section, we obtained the spectrum and matrix elements of the ladder representation of AW(3). We investigate the conditions under which this representation will be finite dimensional (infinite-dimensional representations will be investigated separately).

We first note that by a scale transformation of the generators K_0 and K_1 the constants

C_0 and C_1 can be reduced to preassigned numbers with the sign of each of these constants kept the same. Depending on the signs of C_0 and C_1, there can be nine forms of the algebra AW(3). Among these, four are nondegenerate (i.e., $C_0 \cdot C_1 \neq 0$) and five degenerate, when one or both of the constants C_0 and C_1 vanish.

For definiteness, we consider the case C_0, $C_1 > 0$. The remaining cases can be obtained similarly. By a scale transformation, we reduce our algebra to the canonical form

$$C_0 = C_1 = \mathrm{sh}^2 2\omega. \tag{2.1}$$

In this case, the spectrum of the operator K_0 takes its simplest form:

$$\lambda_p = \mathrm{ch}\, \omega (2p+1). \tag{2.2}$$

The expression (1.15) for the matrix element a_p can be written in the form

$$a_p{}^2 = \frac{\prod_{k=0}^{3} (\mathrm{ch}\, 2\omega p - \mathrm{ch}\, 2\omega p_k)}{4\, \mathrm{sh}^2 2\omega\, \mathrm{sh}^2 2\omega p\, \mathrm{sh}\, \omega(2p-1) \mathrm{sh}\, \omega (2p+1)}, \tag{2.3}$$

where $\cosh 2\omega p_k$ are the roots of the characteristic polynomial $\mathcal{P}$ of fourth degree in the argument $\cosh 2\omega p$. The connection between these roots and the structure parameters (by structure parameters we understand the values of the structure constants B, D_0, D_1 and the Casimir operator Q of AW(3)) is given by

$$B = 4\, \mathrm{th}\, \omega \cdot \mathrm{sh}\, 2\omega \left[\prod_{k=0}^{3} \mathrm{ch}\, \omega p_k + \prod_{k=0}^{3} \mathrm{sh}\, \omega p_k \right], \tag{2.4a}$$

$$D_1 = 4\, \mathrm{sh}\, \omega \cdot \mathrm{sh}\, 2\omega \left[\prod_{k=0}^{3} \mathrm{sh}\, \omega p_k - \prod_{k=0}^{3} \mathrm{ch}\, \omega p_k \right], \tag{2.4b}$$

$$D_0 = -2 \frac{\mathrm{sh}^2 2\omega}{\mathrm{ch}\, \omega} \sum_{k=0}^{3} \mathrm{ch}\, 2\omega p_k, \tag{2.4c}$$

$$Q = \mathrm{sh}^2 2\omega \left[\prod_{k=0}^{3} \mathrm{ch}\, 2\omega p_k + \mathrm{sh}^2 \omega - \frac{B^2 \mathrm{sh}^2 \omega + D_1{}^2}{4\, \mathrm{sh}^2 \omega \cdot \mathrm{sh}^2 2\omega} \right]. \tag{2.4d}$$

As can be seen from (2.3), the parameters p_k are much more convenient for analysis of the representations of AW(3) than the original structure parameters. We note in this connection that the parameters p_k possess a certain "excessiveness." Indeed, there are two operations on p_k that do not change the values of the structure parameters of the algebra: 1) transposition of any two parameters p_k; 2) change of sign simultaneously of an even number (two or four) of the parameters p_k. Without going into the details of these symmetry properties, we merely mention that they explain the remarkable symmetry properties of the Askey–Wilson polynomials with respect to transformation of their parameters [1,2] and are analogs of the famous Regge transformations in the theory of 6j symbols (for more details, see [16]).

For arbitrary real values of the structure parameters of the algebra AW(3), the values of the characteristic parameters p_k are, in general, complex. However, we require the ladder representation to be finite dimensional. For this, we require fulfillment of the condition

$$a_{p_0} = a_{p_0+N} = 0, \tag{2.5}$$

where N = 1, 2, ... is the dimension of the representation.

The condition (2.5) means that at least two characteristic parameters, say, p_0 and p_1, must be real, and there must be fulfillment of the "quantization condition"

$$p_1 - p_0 = N. \tag{2.6}$$

The variable p now ceases to be arbitrary:

$$p = p_0 + n, \quad n = 0, 1, \ldots, N-1. \tag{2.7}$$

As can be seen from (2.4), the parameters p_2 and p_3 must also be real. In addition, from the condition $a_p^2 > 0$, and also from the requirement of nondegeneracy of the spectrum λ_p, there follow the inequalities

$$1/2 < p_2 - 1 < p_0 < p_1 - 1 < p_3. \tag{2.8}$$

Thus, we have completely described the finite-dimensional ladder representation of the algebra AW(3) in the ψ_p basis.

The algebra AW(3) possesses a remarkable duality property, namely, there exists a dual basis φ_s such that the operator K_1 in this basis is diagonal, while K_0 is tridiagonal and self-adjoint:

$$K_1\varphi_s = \mu_s\varphi_s, \quad K_0\varphi_s = d_{s+1}\varphi_{s+1} + d_s\varphi_{s-1} + h_s\varphi_s. \tag{2.9}$$

Expressions for the matrix coefficients μ_s, h_s, d_s in the dual basis φ_s can be obtained from the corresponding expressions (1.9), (1.10), (1.14) by making in them the substitutions $p \to s$, $C_0 \rightleftarrows C_1$, $D_0 \rightleftarrows D_1$.

Indeed, we make the following transformation of the generators:

$$\tilde{K}_0 = K_1, \quad \tilde{K}_1 = K_0, \quad \tilde{K}_2 = [K_1, K_0]_\omega.$$

Then, as is readily verified, the commutation relations (1.1) remain as before if we also interchange the structure constants: $C_0 \rightleftarrows C_1$, $D_0 \rightleftarrows D_1$. Therefore, the basis $\tilde{\psi}_p$ for the operator $\tilde{K}_0$ is identical to φ_s, and we can use the previous expressions.

For our specific choice (2.1) of the constants C_0 and C_1,

$$\mu_s = \operatorname{ch}\omega(2s+1), \tag{2.10a}$$

$$h_s = \frac{B\mu_s + D_0}{4\operatorname{sh}^2\omega\operatorname{sh}2\omega s\cdot\operatorname{sh}2\omega(s+1)}, \tag{2.10b}$$

$$d_s^2 = \frac{\prod_{k=0}^{3}(\operatorname{ch}2\omega s - \operatorname{ch}2\omega s_k)}{4\operatorname{sh}^2 2\omega\cdot\operatorname{sh}^2 2\omega s\cdot\operatorname{sh}\omega(2s-1)\cdot\operatorname{sh}\omega(2s+1)}. \tag{2.10c}$$

In the expression (2.10c), the characteristic parameters s_k are related to p_k by the simple linear transformation

$$s_0 = \sigma - p_1, \quad s_1 = \sigma - p_0, \quad s_2 = \sigma - p_3, \quad s_3 = \sigma - p_2, \quad \sigma = \frac{1}{2}\sum_{k=0}^{3} p_k. \tag{2.11}$$

This transformation has an obvious geometrical meaning — it preserves the lengths of the intervals $[p_0, p_1]$ and $[p_2, p_3]$ on the transition to the "s representation" and also the mutual disposition:

$$1/2 < s_2 - 1 < s_0 < s_1 - 1 < s_3. \tag{2.12}$$

Since the dimension of the "s representation" is equal to that of the "p representation,"

$$s_1 - s_0 = p_1 - p_0 = N, \tag{2.13}$$

the basis φ_s is related to the basis ψ_p by a certain linear transformation

$$\varphi_s = \sum_{p=p_0}^{p_0+N-1} \langle s|p\rangle\psi_p, \tag{2.14}$$

where by $\langle s|p\rangle$ we denote the matrix elements of the overlap of the two dual bases. In the following section, we find the explicit form of these elements.

3. Overlap Functions and Askey-Wilson Polynomials

We show that the overlap functions of the dual bases ψ_p, φ_s of the algebra AW(3) can be explicitly expressed in terms of Askey-Wilson polynomials of general form.

We separate in $\langle s|p\rangle$ the "vacuum amplitude" $\psi_0(s)$:

$$\langle s|p\rangle=\langle s|p_0\rangle P_{p-p_0}(\mu_s)\equiv\psi_0(s)P_n(\mu_s), \tag{3.1}$$

where $P_n(\mu_s)$ are certain functions that are to be determined. Applying to the state ψ_p the operator K_1 and using (1.8a) and (2.9), we obtain for the functions $P_n(\mu_s)$ the relation

$$\tilde{a}_{n+1}P_{n+1}(\mu_s)+\tilde{a}_nP_{n-1}(\mu_s)+\tilde{b}_nP_n(\mu_s)=\mu_sP_n(\mu_s), \tag{3.2}$$

where $\tilde{a}_n\equiv a_{p_0+n}$, $\tilde{b}_n=b_{p_0+n}$.

It is clear from the definition (3.1) that the "initial" conditions

$$\tilde{a}_0=0, \quad P_0(\mu_s)\equiv 1, \tag{3.3}$$

hold. It follows from (3.2) and (3.3) that $P_n(\mu_s)$ is a system of orthogonal polynomials of n-th order of the argument μ_s. Indeed, the three-term recursion relation (3.2) in conjunction with the conditions (3.3) uniquely determines the first polynomial

$$P_1(\mu_s)=\frac{\mu_s-\tilde{b}_0}{\tilde{a}_1} \tag{3.4}$$

and all the following ones. It is obvious that all coefficients of these polynomials are real.

The condition of orthogonality of these polynomials follows from the completeness of the basis φ_s:

$$\delta_{mn}=\sum_s\langle p_0+m|s\rangle\langle s|p_0+n\rangle=\sum_{s=s_0}^{s_0+N-1} w(s)P_n(\mu_s)P_m(\mu_s), \tag{3.5}$$

and the weight function w(s) is

$$w(s)=|\langle s|p_0\rangle|^2. \tag{3.6}$$

Thus, we have obtained a system of polynomials $P_n(\mu_s)$ of the discrete argument μ_s that are orthogonal on the system of N points on the interval of the real axis $s_0 \leqq s \leqq s_1 - 1$.

To find the explicit form of these polynomials, it is sufficient to compare our formulas (1.10) and (2.3) for b_p and a_p with the recursion coefficients that determine the Askey–Wilson polynomials [1,2] and establish their identity.

We recall that in the notation of Askey and Wilson these polynomials depend on four parameters α, β, γ, δ and on an argument

$$\mu(x)=q^{-x}+q^{x+1+\gamma+\delta}. \tag{3.7}$$

These polynomials can be expressed explicitly in terms of the basis hypergeometric function (these polynomials are determined up to a normalization factor that is not important for our purposes)

$$P_n(\mu(x))={}_4\Phi_3\left[\begin{matrix}-n, n+1+\alpha+\beta, -x, x+1+\gamma+\delta \\ 1+\alpha, \quad 1+\beta+\delta, \quad 1+\gamma\end{matrix}; q\,\middle|\, q\right]. \tag{3.8}$$

We use the following definition of the basis hypergeometric function:

$${}_4\Phi_3\left[\begin{matrix}a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3\end{matrix}; q\,\middle|\, x\right]=\sum_{k=0}^{\infty}\frac{[a_1]_k[a_2]_k[a_3]_k[a_4]_k}{[1]_k[b_1]_k[b_2]_k[b_3]_k}x^k, \tag{3.9}$$

where the symbol $[a]_k$ is the so-called Pochhammer q-symbol

$$[a]_k=(1-q^a)(1-q^{a+1})\dots(1-q^{a+k-1}).$$

Direct comparison gives the following correspondence of the parameters:

$$q=e^{2\omega}, \quad \alpha=p_0+p_2, \quad \beta=p_0-p_2, \quad \gamma=p_0-p_1, \quad \delta=p_3+p_2,$$
$$x=s-s_0=s-\frac{\gamma+\delta}{2}=0, 1, \dots, N-1. \tag{3.10}$$

The condition for the finite-dimensional case is $\gamma = -N$, and this is identical to the condition for completeness of the Askey–Wilson polynomials on a finite interval of the real axis [1].

Thus, to the four parameters of the Askey-Wilson polynomials there correspond the four characteristic parameters p_k of the algebra AW(3). In view of the arbitrary choice of these parameters, we obtain the most general Askey-Wilson polynomials without any restrictions (we recall that in this paper we consider only the finite-dimensional case, when the Askey-Wilson polynomials are orthogonal only with discrete weight on the real axis; the infinite-dimensional representations, which lead to polynomials orthogonal with continuous weight [2], could be treated similarly).

4. Classification of Grids and Polynomials

We now relate our results to the approach of the monograph [4], in which the Askey-Wilson polynomials are classified in their dependence on the form of the grid μ_s on which the difference equation for these polynomials is defined.

In our approach, this equation itself is a consequence of the duality of AW(3). Indeed, from the relation

$$\langle s|K_0|p\rangle = d_{s+1}\langle s+1|p\rangle + d_s\langle s-1|p\rangle + h_s\langle s|p\rangle = \lambda_p\langle s|p\rangle \tag{4.1}$$

and the definition (3.1) we obtain a difference equation of second order for the polynomials $P_n(\mu_s)$:

$$t_1(s)P_n(\mu_{s+1}) + t_2(s)P_n(\mu_{s-1}) + h_sP_n(\mu_s) = \lambda_pP_n(\mu_s), \tag{4.2}$$

where

$$t_1(s) = d_{s+1}\frac{\psi_0(s+1)}{\psi_0(s)}, \qquad t_2(s) = d_s\frac{\psi_0(s-1)}{\psi_0(s)}. \tag{4.3}$$

Applying the operators K_0, K_1, K_2 to the vacuum amplitude $\psi_0(s)$, we obtain for $\psi_0(s)$ the recursion relation

$$\frac{\psi_0(s)}{\psi_0(s-1)} = e^{\omega\varkappa}\left[\frac{1-e^{2\omega(s-s_1)}}{1-e^{2\omega(s+s_1)}}\right]\sqrt{\frac{\Pi_+(s)}{\Pi_-(s)}\left[\frac{1-e^{\omega(2s+1)}}{1-e^{\omega(2s-1)}}\right]}, \tag{4.4}$$

where

$$\Pi_\pm(s) = \prod_{k=0}^{3}(1-e^{2\omega(s\pm s_k)}), \qquad \varkappa = s_1 - s_0 - s_2 - s_3 = -2p_0.$$

Substituting (4.4) in (4.2), we obtain an explicit expression for $t_i(s)$:

$$\begin{aligned} t_1(s) &= \frac{e^{\omega\varkappa}}{2\,\mathrm{sh}\,2\omega}\frac{\Pi_+(s+1)(1-e^{2\omega(s-s_1+1)})}{(1-e^{2\omega(s+s_1+1)})(1-e^{4\omega(s+1)})(1-e^{2\omega(2s+1)})} \\ t_2(s) &= \frac{e^{-\omega\varkappa}}{2\,\mathrm{sh}\,2\omega}\frac{\Pi_-(s)(1-e^{2\omega(s+s_1)})}{(1-e^{2\omega(s-s_1)})(1-e^{4\omega s})(1-e^{2\omega(2s+1)})} \end{aligned} \tag{4.5}$$

As was to be expected, Eq. (4.2) is identical to the difference equation that determines the hypergeometric polynomials $P_n(\mu_s)$ on the grid μ_s [4].

The forms of the grids μ_s are obtained for different choices of the constant C_0.

The case $C_0 > 0$ corresponds to

$$\mu_s = \frac{\sqrt{C_0}}{\mathrm{sh}\,2\omega}\,\mathrm{ch}\,\omega(2s+1). \tag{4.6a}$$

The case $C_0 < 0$ to

$$\mu_s = \frac{\sqrt{-C_0}}{\mathrm{sh}\,2\omega}\,\mathrm{sh}\,\omega(2s+1). \tag{4.6b}$$

Finally, the degenerate case $C_0 = 0$ corresponds to the exponential grid

$$\mu_s = e^{\pm\omega(2s+1)}. \tag{4.6c}$$

In turn, each grid corresponds to three choices of the constant C_1, i.e., three forms of the spectrum λ_p of the polynomials. Thus, for given value of ω we have nine types of Askey-Wilson polynomials corresponding to the nine choices of the constants C_0 and C_1 in the algebra AW(3).

The explicit form of the Askey–Wilson polynomials, i.e., their expression in terms of the hypergeometric function, depends strongly on the degree of the characteristic polynomial $\mathscr{P}(\Lambda_p)$ in (1.15).

Suppose $C_1 \neq 0$. Then for $C_0 \neq 0$, as can be seen from (1.14), we have a characteristic polynomial of fourth degree; the corresponding polynomials can be called q-analogs of the Racah polynomials, and they can be expressed in terms of ${}_4\Phi_3$.

If $C_0 = 0$, $D_0 \neq 0$, then the characteristic polynomial $\mathscr{P}(\Lambda_p)$ has third degree; the corresponding polynomials are q-analogs of the Hahn polynomials, and they can be expressed in terms of ${}_3\Phi_2$.

Finally, if $C_0 = D_0 = 0$, then the characteristic polynomial has second degree; the corresponding polynomials are q-analogs of the classical orthogonal polynomials of Kravchuk, Meixner, Charliea, and they can be expressed in the form ${}_2\Phi_2$ (or ${}_2\Phi_1$ for $C_1 = 0$).

Thus, we have constructed a classification of all q-polynomials of discrete argument on the basis of an analysis of the ladder representation of AW(3). The form of the grid is identical to the spectrum of the operator K_1. The explicit form of the polynomial $P_n(\mu_s)$ is determined by the degree of the characteristic polynomial $\mathscr{P}(\Lambda_p)$ of the algebra AW(3). We note that such an algebraic interpretation makes it possible to simplify significantly the classification of Askey–Wilson polynomials proposed earlier in [4].

5. The Algebra QAW(3) and Passage to the Limit of the Classical Polynomials

If in the algebra AW(3) we go to the limit $\omega \to 0$, we obtain a Lie algebra with three generators isomorphic to one of the algebras SU(2), SU(1,1), or the oscillator algebra. To these Lie algebras there correspond the classical orthogonal polynomials of Meixner, Pollaczek, Laguerre, Kravchuk, Charliea, Hermite [8–10]. However, there still remain "outside" the polynomials with quadratic spectrum (Racah, Hahn, Jacobi), since in a Lie algebra the spectrum of a generator cannot be quadratic.

To obtain these polynomials, we must somewhat modify the original algebra AW(3). We make shifts of the generators by constants:

$$K_0 \to K_0 + \nu_0, \quad K_1 \to K_1 + \nu_1. \tag{5.1}$$

The transformation (5.1) does not conserve the commutation relations (1.1). However, if we define a quadratic q-algebra with commutation relations

$$\begin{gathered}[K_0, K_1]_\omega = K_2, \quad [K_2, K_0]_\omega = A_0 K_0^2 + A_1 \{K_0, K_1\} + BK_0 + C_1 K_1 + D_1, \\ [K_1, K_2]_\omega = A_0 \{K_0, K_1\} + A_1 K_1^2 + BK_1 + C_0 K_0 + D_0,\end{gathered} \tag{5.2}$$

where A_0 and A_1 are arbitrary real constants, and {...} denotes the anticommutator, then the transformation (5.1) leads merely to renormalization of the structure constants A_0, A_1, B, ..., without changing the form of the algebra (5.2) itself.

We shall call the algebra with the commutation relations (5.2) the quadratic Askey–Wilson algebra QAW(3), since it differs from the algebra AW(3) only by the presence of the quadratic terms in the commutators.

The algebra QAW(3) is effectively equivalent to the algebra AW(3), since the ladder representations of these algebras differ only by shifts of the diagonal elements by a constant.

If, however, we now go to the limit $\omega \to 0$ for fixed values of the structure constants A_0, A_1, B, ..., we obtain a quadratic commutator algebra with three generators. As mathematical objects, quadratic algebras were discovered by Sklyanin [15], who considered in detail representations of one quadratic algebra of special form with four generators. A quadratic algebra of form (5.2) (for $\omega = 0$) was proposed in [11], and in [16] its ladder representations were considered. We call this algebra the quadratic Racah algebra QR(3), since the corresponding overlap functions $\langle s|p\rangle$ can be expressed in terms of Racah–Wilson polynomials (for $A_0 \cdot A_1 \neq 0$) and Hahn and Jacobi polynomials (for $A_0 \cdot A_1 = 0$).

Recently, the algebra QR(3) and special cases of it (Hahn and Jacobi algebras) have found applications in various problems of quantum physics: symmetries of the 6j and 3jm symbols [16], exactly solvable one-dimensional potentials of the Schrödinger equation [17], hidden symmetry algebra of the Hartmann potential [18], and other examples.

Thus, all the known classical orthogonal polynomials can be obtained from the algebra QAW(3) by limiting processes.

6. Some Realizations of the Algebra AW(3)

Since the algebra AW(3) possesses a number of remarkable properties (duality, explicit form of the spectrum and matrix elements, explicit expression for the overlap functions), it is natural to expect this algebra to play the part of a dynamical symmetry in all problems in which q-polynomials arise.

We shall consider only two examples, which demonstrate the connection between the AW(3) algebra, on the one hand, and the $SU_q(2)$ algebra and q-oscillator, on the other. These algebras are currently attracting the interest of physicists as possible candidates for the role of angular momentum and oscillator at very small (Planck) scales. Such a suggestion was made, for example, by Biedenharn [19].

We recall that the algebra $SU_q(2)$ is the algebra whose generators N_0, $N_\pm$ satisfy the commutation relations

$$[N_0, N_\pm]=\pm N_\pm, \quad [N_+, N_-]=\frac{\operatorname{sh} 2\omega N_0}{\operatorname{sh}\omega}; \tag{6.1}$$

in the limit $\omega \to 0$, this algebra becomes the SU(2) algebra.

As in the "classical" case, a finite-dimensional representation of the algebra has dimension $2\ell + 1$, where ℓ (the "q-angular momentum") takes integer and half-integer values: $\ell = 0, \frac{1}{2}, 1, \ldots$.

We define the operators

$$K_0=e^{-2\omega N_0}, \quad K_1=\{N_1, e^{\omega N_0}\}, \tag{6.2}$$

where $N_1={}^1/_2(N_++N_-)$.

It is readily verified that for given ℓ these operators do indeed form a representation of the algebra AW(3) with parameters

$$B=D_1=C_1=0, \quad C_0=-4\operatorname{ch}^2\omega/2\operatorname{ch}^2\omega \quad D_0=-C_0\operatorname{ch}\omega(2l+1)/\operatorname{ch}\omega, \quad Q=-C_0. \tag{6.3}$$

It follows from (6.3) that the realization (6.2) is one of the degenerate forms of AW(3) (note also that the structure parameter D_0 depends on the weight of ℓ, i.e., the form of the algebra AW(3) depends essentially on the dimension of the $SU_q(2)$ representation). For the spectra of K_0 and K_1 we have

$$\lambda_p=e^{-2\omega p}, \quad -l\leqslant p\leqslant l, \quad \mu_s=\operatorname{sh} 2\omega s\Big/2\operatorname{sh}\frac{\omega}{2}, \quad -l\leqslant s\leqslant l. \tag{6.4}$$

The corresponding overlap functions $\langle s|p\rangle$ can be expressed in terms of the q-analogs of the Kravchuk polynomials.

Note that in the algebra $SU_q(2)$ itself the spectrum of the operator N_1 is unknown and apparently has no explicit expression as analytic function of the number s. In this sense, the algebra AW(3) "improves" the spectral properties of the operators $SU_q(2)$.

We now consider the q-oscillator algebra

$$[N_0, N_\pm]=\pm N_\pm, \quad [N_+, N_-]=-e^{-2\omega N_0}. \tag{6.5}$$

This algebra is constructed from the so-called creation and annihilation operators of the q-oscillator (see [7,19]), which satisfy the relations

$$e^\omega N_-N_+-e^{-\omega}N_+N_-=e^\omega, \quad N_+N_-=\frac{e^\omega}{2\operatorname{sh}\omega}(1-e^{-2\omega N_0}). \tag{6.6}$$

It is easy to show that the operators

$$K_0=e^{\omega N_0}, \quad K_1=N_++N_- \tag{6.7}$$

also realize a representation of the algebra AW(3) in terms of the q-oscillator. The structure parameters have the form

$$B=C_1=D_1=D_0=0, \quad C_0=e^{2\omega}-1, \quad Q=2\,\mathrm{sh}\,\omega. \tag{6.8}$$

We are here dealing with an infinite-dimensional representation, the spectrum of the operator K_1 is continuous, and the overlap functions can be expressed in terms of the q-analog of the Hermite polynomials, this being natural, since the operator K_0 in the limit becomes the Hamiltonian of the harmonic oscillator, while the operator K_1 becomes the coordinate operator; the corresponding overlap functions are then ordinary wave functions of the oscillator in the coordinate representation, i.e., Hermite functions. In this sense, the operator K_1 can be treated as the q-analog of the coordinate for the q-oscillator.

These examples by no means exhaust the list of possible realizations of the algebra AW(3). Other examples will be considered separately.

Conclusions

We have shown that the Askey–Wilson polynomials of general form are generated by the algebra AW(3), which has a fairly simple structure and is the q-analog of a Lie algebra with three generators. The main properties of these polynomials (weight function, recursion relation, etc.) can be obtained directly from analysis of the representations of the algebra.

In this paper, we have considered finite-dimensional representations of the algebra AW(3) and the Aksey–Wilson polynomials of discrete argument corresponding to these representations. A separate analysis is required for the infinite-dimensional representations, which generate polynomials of a continuous argument (these polynomials were investigated in detail in the review [2]). Also of interest is investigation of representations of the algebra AW(3) for complex values of the basic parameter ω and of the structure parameters.

In our view, the algebra AW(3) by itself warrants careful study on account of several remarkable properties (in the first place, the duality with respect to the operators K_0, K_1) not present in the currently very popular quantum algebras of the type $SU_q(2)$.

We assume that the algebra AW(3) is an algebra of dynamical or "hidden" symmetry in all problems in which exponential or hyperbolic spectra and the corresponding q-polynomials arise. We hope that in time the algebra AW(3) will come to play the same role in "q-problems" as Lie algebras play in exactly solvable problems of quantum mechanics.

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VACUUM ENERGY IN THE THEORY OF A NONCRITICAL COMPACTIFIED BOSONIC STRING

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The single-loop vacuum energy of a free noncritical bosonic string compactified onto a torus is calculated. The modular-invariant nature of the result for any dimension D is discussed. In the case of a one-dimensional torus, the Hagedorn temperature is obtained. Inclusion of a Wilson loop in the theory of an open noncritical compactified string is also discussed.

1. Introduction

In recent studies [1–4], attempts have been made to construct a BRST-invariant theory of a bosonic string in a noncritical dimension ($D \neq 26$). If the Weyl mode is regarded as a dynamical variable, it is possible to recover background Weyl invariance [1–4], at least for free strings.

In [5], a calculation was made of the vacuum energy of a noncritical bosonic string in D-dimensional flat space at nonzero temperature. It was demonstrated that the result can be expressed in a manifestly modular-invariant form for any $D < 26$ (see also [6]).

A modular-invariant formulation of the critical bosonic string compactified onto a torus was given in [7,8]. It is now well known that at least at the single-loop level the vacuum energy of a bosonic string at nonzero temperature is equal to the vacuum energy of the bosonic string compactified onto a 1-torus [7,8]. However, the temperature formulation is inconvenient in the proof of modular invariance.

In this paper, the vacuum energy of a free noncritical closed bosonic string compactified onto a torus is calculated. In the case of a one-dimensional torus, an expression for the Hagedorn temperature is given. The modular invariance of the vacuum energy is obvious in the given formulation. The vacuum energy of an open noncritical string compactified onto a torus is also obtained, and the possibility of including Wilson loops in the theory is discussed.

2. Vacuum Energy in a Closed Noncritical String

We find the vacuum energy of a free noncritical bosonic string on a flat background $R_{D-1} \times T_1$. The mass operator and constraint are obtained, for example, in [4,5]:

$$M = -\frac{(D-1)}{12} + N + \tilde{N} + \frac{l^2R^2}{2} + \frac{m^2}{2R^2} + d_0^2, \tag{1}$$

$$N - \tilde{N} = ml, \tag{2}$$

where

$$N = \sum_{n=1}^{\infty} (a_{-n}^{\mu} a_n^{\mu} + d_{-n} d_n) + \sum_{n=1}^{\infty} n (c_{-n} b_n + b_{-n} c_n)$$

[4], $\tilde{N}$ is expressed in terms of the adjoint operators, R is the radius of the torus, m and ℓ are integers, m/R is the discrete momentum, ℓ is the topological quantum number, and the Weyl mode is compactified (see [5,6]) in a box with period of length L: $d_0 = \sqrt{\pi}k/L$ with

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