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Large-Scale Simulation of Avalanche Cluster Distribution in Sand Pile Model

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The avalanche cluster distribution of the sand pile model of self-organized criticality is studied on the square lattice. A vectorized multispin coding algorithm is developed for this study with three bits per site. The exponents characterizing the size and the lifetime of the avalanches are slightly different from the previous estimates.

KEY WORDS: Self-organized criticality; BTW model; avalanche cluster distribution; multispin coding method.

1. INTRODUCTION

A wide variety of systems in nature behave in the following two ways. Objects with fractal geometry(1) obey long-range spatial correlations, whereas systems which follow long-range temporal correlations are characterized by $1/f$-like power spectra.(2) Bak *et al.*(3) (BTW) proposed to unify these two phenomena. In the opinion of Bak and Chen,(3) "those two phenomena are often two sides of the same coin: they are the spatial and temporal manifestations of a self-organized critical state."

BTW introduced a cellular automaton model which under time evolution goes to a stationary state that lacks characteristic time or length scales; then it is called a critical state. Here I describe one version of the model for the square lattice. At each site of this lattice a variable $z(i, j)$ is associated which can take positive integer values. Starting from an initial empty lattice [all $z(i, j) = 0$], the value of z is increased at randomly chosen sites of the lattice in steps of unity as

$$z(i, j) = z(i, j) + 1 \tag{1}$$

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When the value of z at any site reaches a maximum z_m, its value is decreased by four units and each of the four nearest neighbors gains one unit of z as follows:

$$z(i, j) = z(i, j) - 4 \tag{2}$$

$$z(i \pm 1, j \pm 1) = z(i \pm 1, j \pm 1) + 1 \tag{3}$$

when $z(i, j) \geqslant z_m$. At the boundary sites $z = 0$.

The variable z can be thought of as the local slope of a sand pile. Equation (1) represents the addition of slope to a site and Eqs. (2) and (3) represent the toppling of a particle when the slope is too high.[3]

Initially the system starts with slopes growing at different sites. After a long time it will reach a stationary state in which the average slope will not increase any more because the slope which is coming into the system with Eq. (1) will flow out through the periphery to maintain $z = 0$ on the boundary. At this stage, the addition of unity to a site with $z(i, j) = z_m - 1$ generally results in a series of topples in a chain reaction, forming an avalanche. When the addition of slope is followed by at least one topple, it is called an avalanche. The size (precise definition given below) and the lifetime of the avalanche vary over large scales. The probability distribution of the size and lifetime of the avalanche follow power laws: $D(s) \sim s^{1-\tau}$ and $D'(t) \sim t^{-b'}$ with $\tau = 2.0$ and $b' = 0.43$ according to ref. 3.

Recently different version of this model have been introduced and studied. In a continuum version of this model in which z can vary continuously Zhang[4] suggested that $\tau = 3 - 2/d$. Other models introducing anisotropy[5] and directedness[6,7] have also been studied where the direct BTW model is found to be exactly solvable.[7]

Here I report results of large-scale computer simulations on the isotropic BTW model for $z_m = 4$. I estimate the critical exponent $\tau = 2.22$ slightly different from both the theory[4] and earlier simulation[3] and obtain $b' = 0.85$, different from earlier simulation.[3] In Section 2 I describe the multispin coding algorithm for studying sand pile automaton on the square lattice. In Sections 3 and 4, I report results on critical points and critical exponents, and finally conclude in Section 5.

2. SIMULATION METHOD

The above-mentioned BTW model is studied on the square lattice using a multispin coding technique. For the square lattice any site having z values 0, 1, 2, and 3 does not topple, while for $z = 4$ it topples. This algorithm uses 3 bits for each lattice site and in a 64-bit integer word of the Cray computer, 21 lattice sites can be stored. Each set of these 3 bits together is called a cell. The whole lattice is updatted simultaneously. Out of the 2^3 possibilities of a cell, I choose 000, 001, 010, 011, and 100 to represent $z=0, 1, 2, 3$, and 4 respectively. I use an array IS(0: $L+1$, 0: $B+1$) with $B=L/21$ to simulate an $L \times L$ lattice. Any Kth cell of the (I, J)th elements of IS array has nearest neighbors at the Kth cells of the $(I-1, J)$, $(I, J+1)$, $(I+1, J)$, and $(I, J-1)$ elements.

One starts with an initial empty lattice having all elements of IS array set equal to zero. To choose a cell randomly, one calls three random numbers I, J, and K, where $(1 \leqslant I \leqslant L)$, $(1 \leqslant J \leqslant B)$, and $(1 \leqslant K \leqslant 21)$. Then to add unity at this cell as in Eq. (1) one adds 8^{K-1} to the (I, J)th element of the array IS, which increases the value of the Kth cell of IS(I, J) by 1. Next, to vectorize Eq. (3) one uses a second array IN(0: $L+1$, 0: $B+1$) to store the contributions due to the topplings. A cell has the third bit equal to 1 when its value is greater than 3. The first DO loop uses this fact to calculate the contributions by using a mask M100 (21 repetitions of the three bits ...100100100), a logical AND, and a right SHIFT as

$$\mathrm{IN}(I, J) = \mathrm{RSHIFT}((\mathrm{IS}(I, J).\mathrm{AND.M100}), 2)$$

In a second DO loop the IS array is updated by adding the contributions from the nearest neighbors and decreasing the values for topplings by using another mask M011 (21 repetitions of the three bits ...011011011) as

$$\mathrm{IS}(I, J) = \mathrm{IN}(I-1, J) + \mathrm{IN}(I, J+1) + \mathrm{IN}(I+1, J) + \mathrm{IN}(I, J-1)$$
$$+ (\mathrm{IS}(I, J).\mathrm{AND.M011})$$

To take into account the boundary condition $z=0$, one adds to the first and last J columns as

$$\mathrm{IS}(I, 1) = \mathrm{IS}(I, 1) + \mathrm{RSHIFT}(\mathrm{IN}(I, B), 3)$$

$$\mathrm{IS}(I, B) = \mathrm{IS}(I, B) + \mathrm{LSHIFT}(\mathrm{IN}(I, 1), 3)$$

This algorithm is fully vectorized and one has a speed of 322 cell updates/μsec for a lattice size $L=168$, increasing to 400 cell updates/μsec for a lattice size $L=672$ (without analysis, in one processor of Cray YMP). With this high-speed program I simulated lattices of linear lengths which are multiples of 21 and reached up to $L=672$. Note that at any instant of time during the avalanche only a fraction of the lattice sites topples, i.e., need to be updated, but one has to update the whole lattice to maintain the vectorization condition.

3. CRITICAL POINT

One uses Eq. (1) when at all sites of the lattice, z values are below $z_m = 4$. If the z value of a site is less than 3, this addition results in no toppling. Therefore, sites having z values 0, 1, and 2 are inactive sites. However, sites with $z = 3$ are active sites, as they initiate a series of topplings when one unit is added. After some time all sites of the lattice have z values less than $z_m = 4$ again and no longer change; then the next unit is added. At the critical state the average z per lattice site attains a steady value $\langle z \rangle = z_c$. Tang and Bak[3] have drawn an analogy between this BTW model and ordinary critical phenomena by recorgnizing z_c as the critical point, because for $\langle z \rangle$ above and below z_c there will be the presence and absence of spontaneous flow of slope in the system. In his continuum version (z values vary continuously) of the BTW model, Zhang[4] has estimated numerically the distribution of z values. He observed that the probability distribution $P(z)$ has four sharp peaks and the fractions of sites belong to these peaks are 0.10, 0.16, 0.32, and 0.42 in ascending order of z values. He recognized that z values for these peaks correspond to $z = 0$, 1, 2, and 3 of the discrete BTW model.

I estimate the value of z_c for the above-mentioned BTW model. I also estimate the fractions of sites having z values 0, 1, 2, and 3. I start

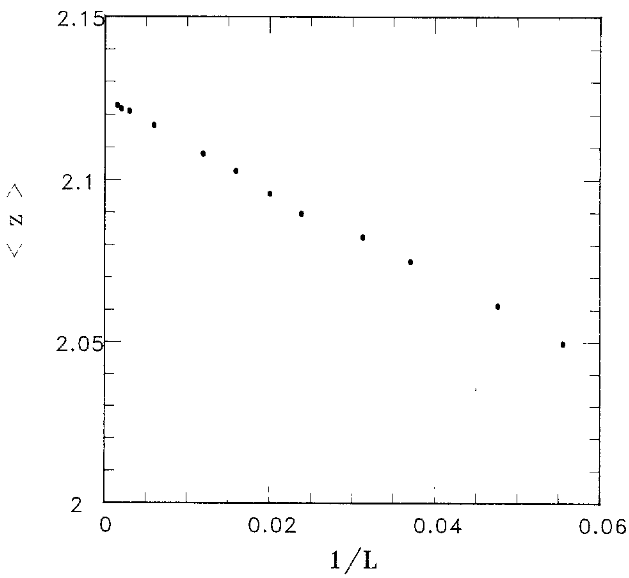

Fig. 1. Plot of $\langle z \rangle$ for different lattice sizes L with $1/L$. Extraploation to the $L \to \infty$ limit gives $z_c = 2.124$ for the system.

from an initial empty lattice and then add slopes one after another by Eq. (1). Some additions result in avalanches and some do not. I count the avalanches only and wait a long time to reach the critical state. The average z increases from zero and reaches a steady value. At this stage I start data collection by estimating the cumulative average of z at the interval of 100 avalanches. I follow a similar procedure to estimate the fractions of sites f_0, f_1, f_2, and f_3. I vary the lattice size L from 21 to 672. For $L = 21$ and 672, I wait for 10,000 and 130,000 avalanches and average over 2 million and 50,000 avalanches, respectively. I plot the data of z_c with respect to $1/L$, which fits quite good to a straight line (see Fig. 1). Extrapolation of the $L \to \infty$ limit gives $z_c = 2.124$. In a similar way one gets $f_0 = 0.073$, $f_1 = 0.174$, $f_2 = 0.307$, and $f_3 = 0.446$. Typical errors in these estimates are of the order of 0.003. The value of f_0 is very consistent with the exact value[12] $f_0 = (2 - 4/\pi)/\pi^2 \approx 0.0736$.

4. CRITICAL EXPONENTS

I study three different definitions of the cluster size s. They are as follows.

Definition 1. The cluster size s is the total number of distinct sites which have at least one toppling during the avalanche process.

To get the cluster size s, use a third array ID(0: $L+1$, 0: $B+1$) (set this equal to zero before addition of each unit of slope) and introduce

$$\mathrm{ID}(I, J) = (\mathrm{IS}(I, J).\mathrm{OR}.\mathrm{IN}(I, J))$$

in the first DO loop. After the avalanche stops, I count the number of non-zero cells in ID using a POPCNT in the Cray . This gives the cluster size s, and the sth element of an array NS(1:L*L) is increased by 1. Here, I have considered all clusters which reach the boundary or remain within the lattice. For finite lattice size and finite number of clusters, the number of s clusters fluctuates rapidly. I reduce the fluctuation by the standard coarse-graining method. I sum up the data of NS together for bins of sizes 1, 2–3, 4–7, 8–15, 16–31, and so on, bin size increasing exponentially. I plot this integrated distribution on a log-log scale. For a particular bin I take the geometric mean of the two border s values as the s value of that bin. Now, if the value of the size exponent τ were equal to 2, then this integrated plot should fit to a straight line parallel to the s axis (as the value of the right bin edge is almost double that of the left edge). However, instead of that one gets a monotonically falling curve which fits reasonably well to a straight line when very small and very large values of s are not considered. The slope of this straight line gives an estimate of $2 - \tau$. One

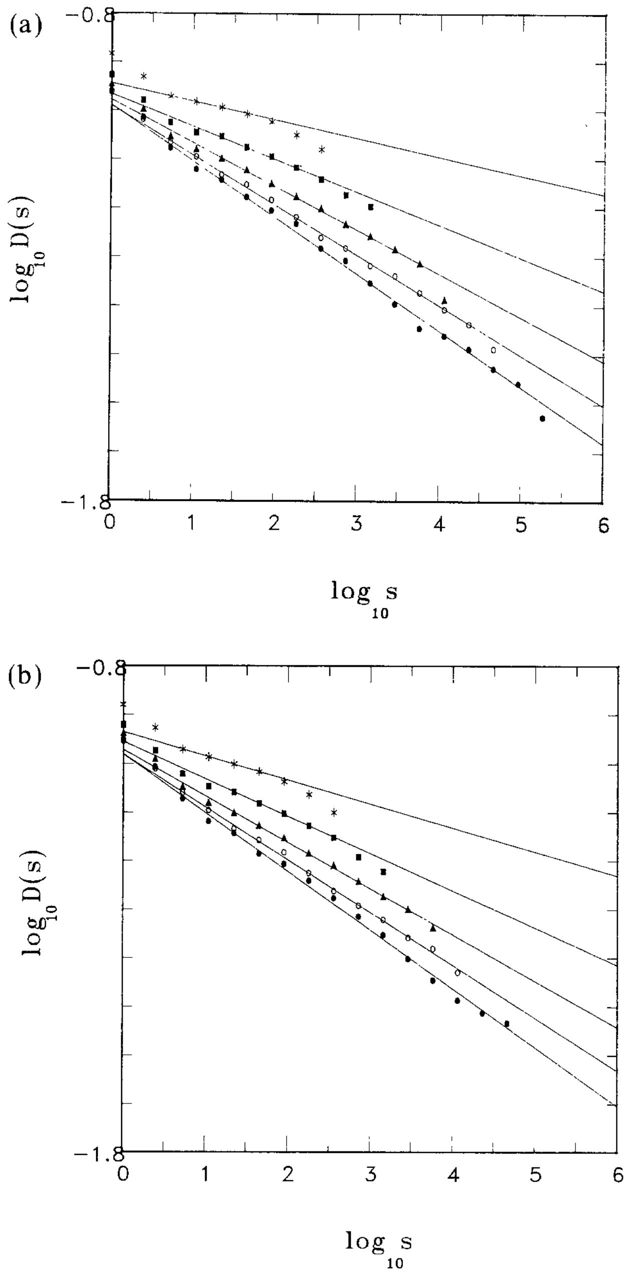

Fig. 2. Log-log plot of cluster size distribution $D(s)$ integrated over bins with respect to s, for cluster size definitions (a) 1, (b) 2, and (c) 3. Lattice size $L = (*)$ 42, (■) 84, (▲) 168, (○) 336, (●) 672.

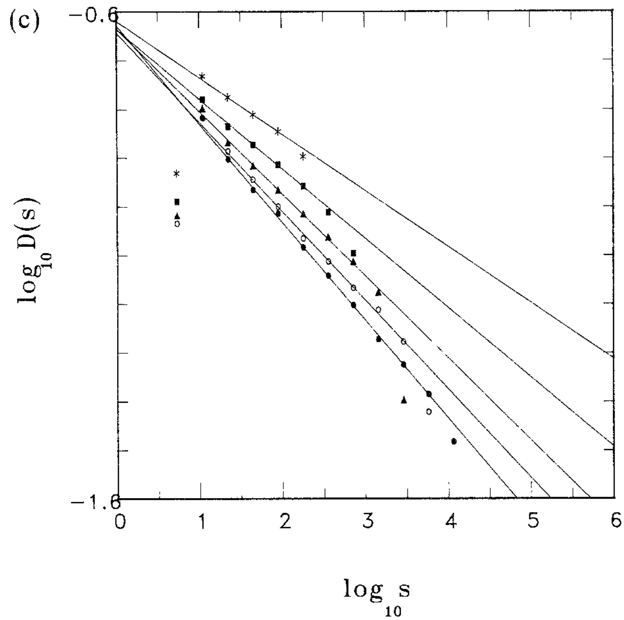

Fig. 2. (*Continued*)

estimates the slope by a standard least square fit and finds that the slope systematically changes with the lattice size (see Fig. 2a). One then tries to estimate the asymptotic value of the slope for the $L \rightarrow \infty$ limit by extrapolation. Without having any prior knowledge of the extrapolation procedure, I tried $1/\log L$, which seems to work well (see Fig. 3). This analysis gives the value of $\tau = 2.22$, which is somewhat larger than 2, as in the percolation theory.[8] This value of τ somewhat higher than 2 is also obtained by Kertész.[9] Duarte also studied the BTW model on the cyclic triangular lattice and obtained a τ value greater than 2 and increasing with lattice size.[10]

I have studied two more cluster size definitions, as follows.

Definition 2. The cluster size s is total number of topplings in an avalanche process. This number is in general greater than the s value in Definition 1 because one site may topple more than once.

Definition 3. In a comparison between two lattice configurations before and after the avalanche, the cluster size is the number of sites which differ in z values.

I estimated the exponent τ for cluster size definitions 2 and 3 following the above method. The cluster size distribution function integrated over

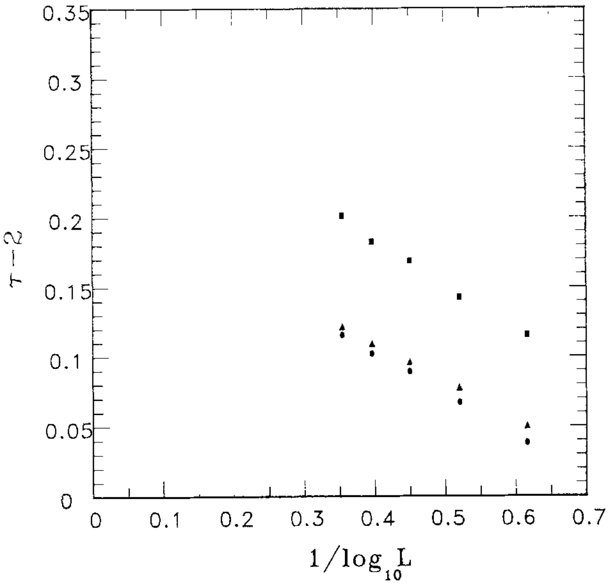

Fig. 3. Plot of exponent $(\tau - 2)$ for different lattice sizes L with $1/\log L$, for cluster size definitions (●) 1, (▲) 2, and (■) 3.

different bins is plotted with cluster size s in a log-log scale (see Figs. 2b and 2c). Here also I find a systematic variation of the slopes of these curves with lattice size L. I extrapolated these slopes with $1/\log L$ and obtain $\tau = 2.22$ and 2.31 for cluster size definitions 2 and 3, respectively (see Fig. 3). It seems that the exponent τ for Definition 2 is the same as that in Definition 1, but both are different from Definition 3.

Now I investigate the variation of the average cluster size (defined by $\langle s^2 \rangle = \sum s^2 n_s / \sum n_s$ and $\langle s \rangle = \sum s n_s / \sum n_s$) with lattice size $\langle s^2 \rangle \sim L^y$ and $\langle s \rangle \sim L^x$. Here n_s is the number of clusters of size s within the total number of avalanche clusters considered after reaching the critical state. I plot the average cluster size $\langle s^2 \rangle / L^4$ and $\langle s \rangle$ with L in a log-log plot, varying the lattice size from 21 to 672 (see Figs. 4 and 5). I estimate the values of y as 3.66 and 2.96 and the values of x as 1.64 and 1.13 for Definitions 1 and 3, respectively. For Definition 2 there is curvature in both $\langle s \rangle$ and $\langle s^2 \rangle / L^4$ plots. For $\langle s \rangle$ the slope varies as 1.82, 1.91, 1.96, 1.98, and 1.99 when the lattice size is increased by a factor of 2 starting from 21. Extrapolation of slopes with $1/L$ goes nicely to the value 2. The estimate for the exponent x for Definition 2 is 2.00 ± 0.05. For the exponent y I take the largest value of the slope between lattice size 336 and 672 and estimate $y = 4.79$ for Definition 2. Dhar[13] has exactly calculated the $\langle s \rangle$ for

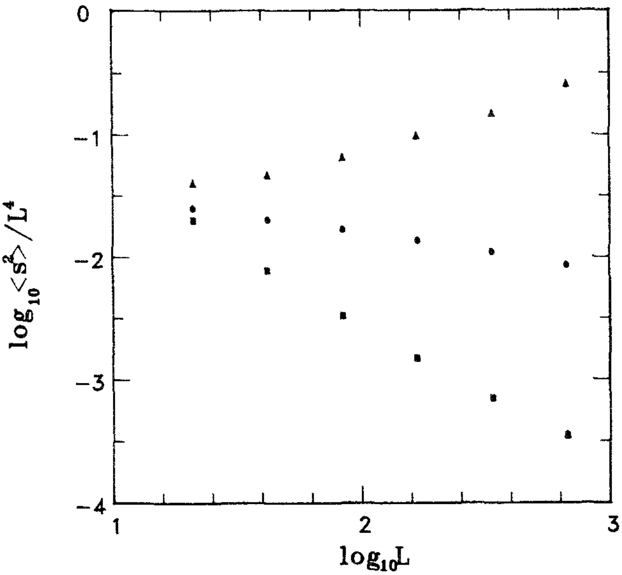

Fig. 4. Log-log plot of $\langle s^2 \rangle / L^4$ with L, for cluster size definitions (●) 1, (▲) 2, and (■) 3.

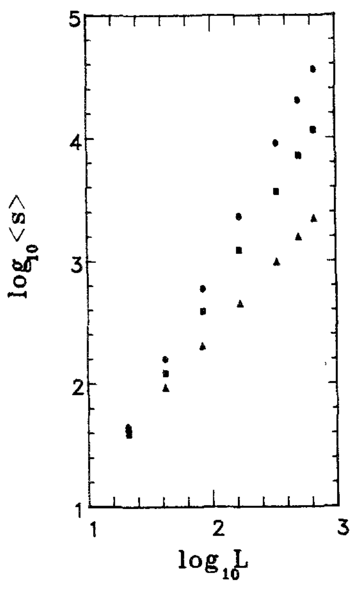

Fig. 5. Log-log plot of $\langle s \rangle$ with L, for cluster size definitions (●) 1, (▲) 2, and (■) 3.

Definition 2. The present data are in very good agreement with his formula for $\langle s \rangle$ for all L values and the exponent $x = 2$.

Next I study the distribution of lifetimes $D(t) \sim t^{-b}$. For that I count the number of sweeps t of the lattice needed for the avalanche to become quiet and the number of avalanches $D(t)$ which have same t. At the end of the simulation this distribution is integrated over bins and divided by the bin length. The resulting distribution is plotted with t in a log-log scale, and the slope of this straight line gives the value of b (see Fig. 6). Here also one finds a finite-lattice-size effect on the slope and extrapolates to obtain $b = 1.38$ for the infinite limit (see Fig. 7). I also study, following BTW,[3] the distribution of lifetimes $D'(t) \sim t^{-b'}$ weighted by the average response s/t. Different avalanche clusters can have the same value of t corresponding to different s values. I equate $D'(t)$ to the sum over all s values for a fixed t and then divide this sum by t. Here I use Definition 1 for the cluster size s. I plot these data again by integrating $D'(t)$ over different bin sizes and then dividing by the bin size (see Fig. 8). Extrapolation of the slopes for different lattice sizes gives the value of $b' = 0.85$ (see Fig. 9).

Finally, I estimate the correlation exponent ν using the formula[8]

$$\Delta = (\langle z^2 \rangle - \langle z \rangle^2)^{1/2} \sim L^{-1/\nu} \tag{4}$$

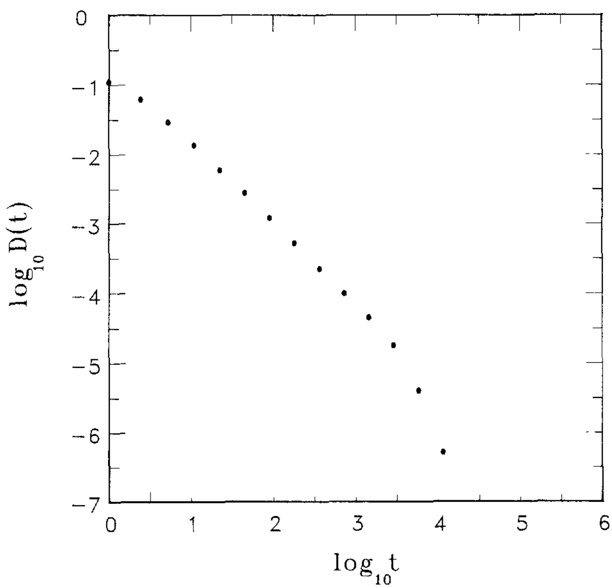

Fig. 6. Log-log plot of lifetime distribution $D(t)$ integrated over bins and divided by bin length with time t for lattice size $L = 672$.

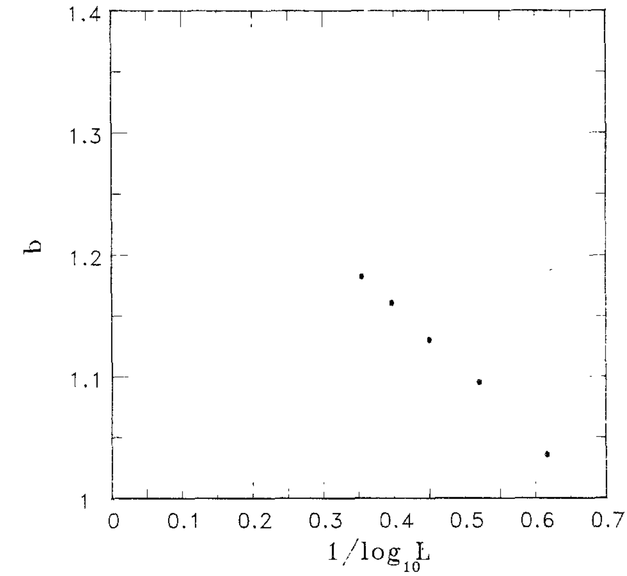

Fig. 7. Lifetime distribution exponent b plotted versus $1/\log L$ for different lattice sizes L.

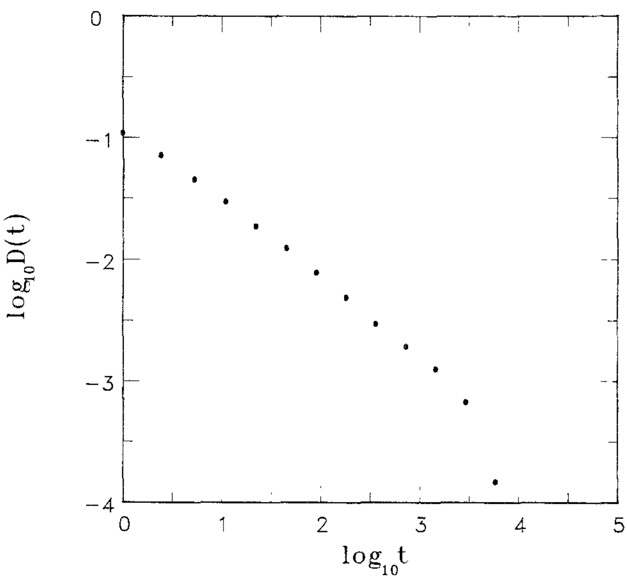

Fig. 8. Log-log plot of weighted lifetime distribution $D'(t)$ integrated over bins and divided by bin length with time t for lattice size $L = 672$.

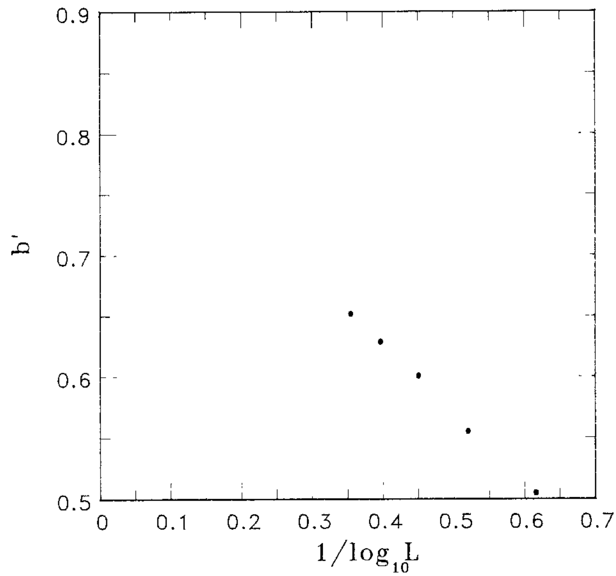

Fig. 9. Weighted lifetime distribution exponent b' plotted for different lattice sizes L as $1/\log L$.

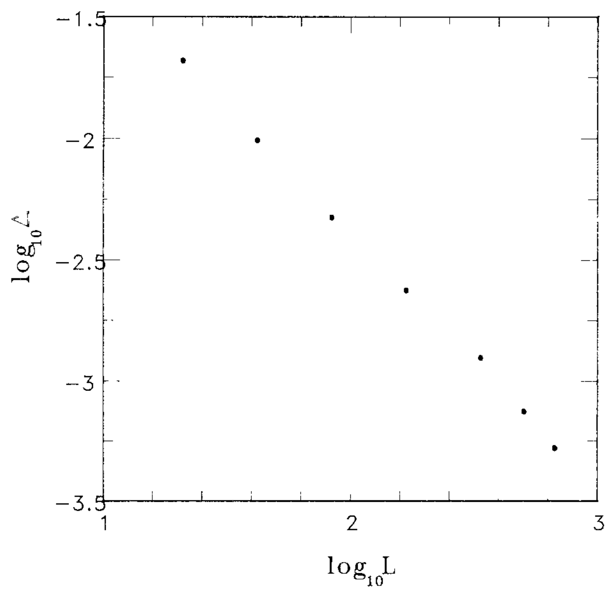

Fig. 10. Log-log plot of standard deviation Δ for different lattice size L.

I calculate the value of Δ for different lattice sizes after the critical state is reached and plot Δ with L in a log-log scale (see Fig. 10). From the slope of this curve I estimate $\nu = 0.98$. A similar value of ν close to 1 is also obtained by Burke.[11]

From the ordinary critical phenomena one recognizes our $x = \gamma/\nu$ and $y = (2\gamma + \beta)/\nu$. For cluster size Definition 1 one has $x = 1.64$ and $y = 3.66$, which gives $\beta/\nu = 0.38$. Using these values gives $(\gamma + 2\beta)/\nu = 2.4$, which shows that the hyperscaling relation may not be valid in this model.

5. CONCLUSION

I study the avalanche cluster size distribution in the sand pile model of BTW[3] on the square lattice with better computer simulations using 25 hr of Cray YMP time. I estimate the exponents τ and b' characterizing the cluster size and the weighted lifetime of the avalanches. I obtain $\tau = 2.22$ and $b' = 0.85$, somewhat different from the theory[4] and previous numerical study.[3]

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